TWISTED ENDOSCOPIC CHARACTER RELATION FOR TORAL SUPERCUSPIDAL L-PACKETS OF CLASSICAL GROUPS

MASAO OI

Abstract. We prove that Kaletha's toral supercuspidal L-packets satisfy the twisted endoscopic character relation in some cases, including general linear groups equipped with an involution. Consequently, we show that Kaletha's construction of the local Langlands correspondence for toral supercuspidal representations coincides with Arthur's. The strategy is to emulate Kaletha's proof of the standard endoscopic character relation in the twisted setting by appealing to Waldspurger's framework "l'endoscopie tordue n'est pas si tordue".

Contents

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1. Introduction

One fundamental objective in representation theory of reductive groups over local fields is to establish the *local Langlands correspondence*. For a connected reductive group $\mathbf{G}$ defined over a local field F, the local Langlands correspondence is a natural map from the set $\Pi(\mathbf{G})$ of isomorphism classes of irreducible admissible representations of $G := \mathbf{G}(F)$ to the set $\Phi(\mathbf{G})$ of equivalence classes of L-parameters of $\mathbf{G}$. Here, each fiber of the map is expected to be finite; we let $\Pi_\phi^{\mathbf{G}}$ denote the fiber at $\phi \in \Phi(\mathbf{G})$ and call it an *L-packet*. Thus, we may think of the local Langlands correspondence as a natural partition of $\Pi(\mathbf{G})$ into finite sets labeled by L-parameters:

$$\Pi(\mathbf{G}) = \bigsqcup_{\phi \in \Phi(\mathbf{G})} \Pi_\phi^{\mathbf{G}}.$$

While the local Langlands correspondence was constructed by Langlands ([Lan89]) when F is archimedean, its existence is still conjectural in general when F is non-archimedean. However, numerous results have been obtained up to the present. Let us review some of them in the following by focusing only on the case where F is a p-adic field, i.e., a non-archimedean local field of characteristic zero.

Firstly, the local Langlands correspondence has been completely established for several specific groups. The particularly important examples include the results of Harris–Taylor for GL_n ([HT01]), Arthur for quasi-split special orthogonal or symplectic groups ([Art13]), and Mok for quasi-split unitary groups ([Mok15]).

On the other hand, one might also attempt to construct the local Langlands correspondence by restricting the class of representations instead of the class of groups. An important example of this direction is the work of DeBacker–Reeder ([DR09]), which established the local Langlands correspondence for supercuspidal representations which are of depth zero and regular of arbitrary unramified groups. After the work of DeBacker–Reeder, Kaletha investigated some particular cases of positive depth supercuspidal representations ([Kal13, Kal15]). Currently, all these constructions have been uniformly generalized by Kaletha himself to a considerably broad class of supercuspidal representations called *regular* (more generally, *non-singular/semisimple*) supercuspidal representations of tamely ramified connected reductive groups ([Kal19b, Kal19c]).

Taking into account these two possible approaches toward the local Langlands correspondence (i.e., the "vertical" direction which restricts the class of groups and the "horizontal" direction which restricts the class of representations), it is natural to ask whether two different constructions indeed give rise to the identical correspondence. This problem is not only interesting in itself but also technically important. For example, the above-mentioned constructions for specific groups have a favorable compatibility with the global classification theory of automorphic representations. On the other hand, Kaletha's construction is highly explicit since it is ultimately based on the local Langlands correspondence for tori (note that this is parallel to Langlands' construction [Lan89] in the archimedean case). If we get the coincidence of these different constructions, we can combine their advantages.

Based on this motivation, we first investigated the case of GL_n in a joint work of Kazuki Tokimoto. In [OT21], we proved that the local Langlands correspondences of Harris–Taylor and Kaletha coincide for any regular supercuspidal representation of $\mathrm{GL}_n(F)$ whenever $p \neq 2$ (note that now this result has been generalized by Tokimoto to all inner forms of GL_n in [Tok23]).

The aim of this paper is to establish a methodology for comparing Kaletha's construction with others for more general groups. Especially, we prove the following:

Theorem 1.1 (Theorems 15.11 and 15.12)**.** *Let* $\mathbf{H}$ *be a quasi-split special orthogonal or symplectic group over* F*. Suppose that* p *is sufficiently large. The Local Langlands correspondences of Arthur and Kaletha coincide for any "toral" supercuspidal representation of* $H := \mathbf{H}(F)$*.*

We briefly explain what "toral" supercuspidal representations are (see Section 4 for more details). Let $\mathbf{G}$ be a tamely ramified connected reductive group over F in the following. In [Yu01], Yu established an explicit method for producing a broad class of supercuspidal representations, which are called *tame supercuspidal representations*. Yu's construction associates a tame supercuspidal representation to each tuple $(\vec{\mathbf{G}}, \vec{\vartheta}, \vec{r}, \mathbf{x}, \rho_0)$ called a cuspidal $\mathbf{G}$-datum. Here, we only recall that $\vec{\mathbf{G}} = (\mathbf{G}^0 \subsetneq \cdots \subsetneq \mathbf{G}^d)$ is a sequence of tame Levi subgroups of $\mathbf{G}$ and ρ_0 is a depth zero cuspidal representation of an open compact-modulo-center subgroup of $\mathbf{G}^0$ (hence regarded as a representation of a finite reductive group). In [Kal19b], by invoking the Deligne–Lusztig theory [DL76], Kaletha introduced the notion of *regularity* for tame supercuspidal representations and discovered that regular supercuspidal representations can be re-parametrized by much simpler data $(\mathbf{S}, \vartheta)$ called *tame elliptic regular pairs*, which consist only of a tame elliptic maximal torus $\mathbf{S}$ of $\mathbf{G}$ contained in $\mathbf{G}^0$ and a character ϑ of $S := \mathbf{S}(F)$ satisfying a certain regularity condition. Based on this re-parametrization, he assigned an L-parameter to each regular supercuspidal representation and analyzed the internal structures of the resulting L-packets. *Toral supercuspidal representations* constitute a special class among regular supercuspidal representations; they are tame supercuspidal representations obtained from cuspidal $\mathbf{G}$-data whose $\vec{\mathbf{G}}$ are of the form $(\mathbf{S} \subsetneq \mathbf{G})$.

To obtain Theorem 1.1, what we have to do is verify that Kaletha's toral supercuspidal L-packets satisfy the *twisted endoscopic character relation*, which is the characterization of Arthur's correspondence. Thus we next review the general framework of twisted endoscopy. Suppose that $\mathbf{H}$ is an endoscopic group for $(\mathbf{G}, \theta)$ in the sense of Kottwitz–Shelstad, where θ is an F-rational pinning-preserving automorphism of $\mathbf{G}$ (see Section 8). In particular, $\mathbf{H}$ is equipped with an L-embedding $\hat{\xi}\colon {}^L\mathbf{H} \hookrightarrow {}^L\mathbf{G}$, which enables us to regard any L-parameter $\phi_{\mathbf{H}}$ of $\mathbf{H}$ as an L-parameter of $\mathbf{G}$ (write ϕ) by composing $\phi_{\mathbf{H}}$ with $\hat{\xi}$. Suppose that the local Langlands correspondence both for $\mathbf{G}$ and $\mathbf{H}$ are available, so that we may associate L-packets $\Pi^{\mathbf{H}}_{\phi_{\mathbf{H}}}$ and $\Pi^{\mathbf{G}}_{\phi}$ to both $\phi_{\mathbf{H}}$ and ϕ. Assume that $\phi_{\mathbf{H}}$ and ϕ are tempered.

$$
\begin{array}{ccccc}
\Pi(\mathbf{G}) \supset \Pi^{\mathbf{G}}_{\phi} & \xleftrightarrow{\text{LLC for } \mathbf{G}} & & & {}^L\mathbf{G} \\
 & & & \overset{\phi}{\nearrow} & \uparrow \hat{\xi} \\
\Pi(\mathbf{H}) \supset \Pi^{\mathbf{H}}_{\phi_{\mathbf{H}}} & \xleftrightarrow{\text{LLC for } \mathbf{H}} & W_F \times \mathrm{SL}_2(\mathbb{C}) & \xrightarrow[\phi_{\mathbf{H}}]{} & {}^L\mathbf{H}
\end{array}
$$

In this situation, it is expected that the following holds:

Expectation 1.2 (Twisted endoscopic character relation)**.** *For each* $\pi \in \Pi^{\mathbf{G}}_{\phi}$ *there exists a constant* $\Delta^{\mathrm{spec}}_{\phi,\pi} \in \mathbb{C}$ *such that the following identity holds for any strongly regular semisimple* $\delta \in \tilde{\mathbf{G}}(F)$*:*

$$
(1) \qquad \sum_{\pi \in \Pi^{\mathbf{G}}_{\phi}} \Delta^{\mathrm{spec}}_{\phi,\pi} \Phi_{\tilde{\pi}}(\delta) = \sum_{\gamma \in H/\mathrm{st}} \mathring{\Delta}(\gamma, \delta) \sum_{\pi_{\mathbf{H}} \in \Pi^{\mathbf{H}}_{\phi_{\mathbf{H}}}} \Phi_{\pi_{\mathbf{H}}}(\gamma).
$$

Here,

- $\tilde{\mathbf{G}}$ denotes the twisted space determined by $\mathbf{G}$ and θ, i.e., the non-identity component of the semi-direct product group $\mathbf{G} \rtimes \langle\theta\rangle$ (see Section 3.1);
- $\Phi_{\pi_{\mathbf{H}}}$ is the normalized (Harish-Chandra) character of $\pi_{\mathbf{H}} \in \Pi^{\mathbf{H}}_{\phi_{\mathbf{H}}}$ and $\Phi_{\tilde{\pi}}$ is the normalized twisted character of $\pi \in \Pi^{\mathbf{G}}_{\phi}$ (see Section 5.1), which can be defined only when π is θ-stable (thus the coefficient $\Delta^{\mathrm{spec}}_{\phi,\pi}$ is zero unless π is θ-stable);
- the sum on the right-hand side is over the stable conjugacy classes of norms of δ in H in the sense of twisted endoscopy (see Section 8.2);
- $\mathring{\Delta}$ on the right-hand side is the transfer factor of Kottwitz–Shelstad (see Section 8.3).

By linear independence of twisted characters, a family $\{\Delta^{\mathrm{spec}}_{\phi,\pi}\}_{\pi\in\Pi^{\mathbf{G}}_{\phi}}$ of constants as above is unique if exists.

In the untwisted case (i.e., θ is trivial), Kaletha proved that Expectation 1.2 is indeed true for toral supercuspidal L-packets under some assumptions on p ([Kal19b, Theorem 6.3.4]; see [FKS23, Section 4.4] for a more general result in the untwisted situation).

The point is that when $\mathbf{H}$ is a quasi-split special orthogonal or symplectic group, we can find a general linear group $\mathbf{G} = \mathrm{GL}_n$ with an involution θ such that $\mathbf{H}$ is an endoscopic group for $(\mathbf{G}, \theta)$. The expected identity (1) then tells us the sum of characters of representations in each L-packet $\Pi^{\mathbf{H}}_{\phi_{\mathbf{H}}}$ of $\mathbf{H}$ in terms of the twisted characters of representations of GL_n; this information is enough to characterize $\Pi^{\mathbf{H}}_{\phi_{\mathbf{H}}}$ as a finite set of representations by linear independence of characters. What Arthur did is to prove that there indeed exists a finite set $\Pi^{\mathbf{H}}_{\phi_{\mathbf{H}}}$ for each $\phi_{\mathbf{H}}$ satisfying the identity (1) with $\Pi^{\mathbf{G}}_{\phi}$ which is determined by Harris–Taylor's local Langlands correspondence for GL_n. Therefore, as we already know the coincidence of Kaletha's construction with Harris–Taylor's, it is enough to verify the twisted endoscopic character relation for Kaletha's L-packets in order to obtain the coincidence of Kaletha's and Arthur's constructions.

The main result of this paper is as follows:

Theorem 1.3. *Suppose that p is sufficiently large (compared to $\mathbf{G}$). Kaletha's toral supercuspidal L-packets satisfy Expectation 1.2 in the following cases:*

(1) $\mathbf{G} = \mathrm{GL}_n$ *or*
(2) $\mathbf{G}$ *is general, θ is involutive, and toral supercuspidal L-packets arise from a torus $\mathbf{S}$ splitting over a finite extension E/F with odd ramification index.*

We explain the outline of proof of Theorem 1.3 in the following. Our strategy is quite simple in some sense; we reproduce Kaletha's proof of the standard (untwisted) endoscopic character relation while taking into account the effect of the twist θ. Thus let us first review Kaletha's proof in the untwisted setting briefly.

The starting point of Kaletha's proof is an explicit formula of the characters of tame supercuspidal representations due to Adler–DeBacker–Spice ([AS09, DS18]). In the toral setting, it is as follows. Let $\pi_{(\mathbf{S},\vartheta)}$ be the toral supercuspidal representation of G arising from a tame elliptic regular pair $(\mathbf{S}, \vartheta)$. Let $r \in \mathbb{R}_{>0}$ be the depth of the character ϑ and $X^* \in (\mathrm{Lie}\,\mathbf{S})^*(F)$ be an element representing the restriction of ϑ to the "depth r part" S_r of S (see Section 4.2 for details). Let $\delta \in G$ be any elliptic regular semisimple element. Then, by the theory of Adler–Spice

[AS08], we can take a *normal r-approximation* $\delta = \delta_{<r} \cdot \delta_{\geq r}$. Roughly speaking, this is a nice product decomposition of δ into a part "p-adically shallower than r" and a part "p-adically deeper than or equal to r". One of its important properties is that two parts $\delta_{<r}$ and $\delta_{\geq r}$ commute; even more strongly, the deeper part $\delta_{\geq r}$ belongs to the connected centralizer $\mathbf{G}_{\delta_{<r}} := \mathbf{Z}_{\mathbf{G}}(\delta_{<r})^{\circ}$ of the shallower part. The Adler–DeBacker–Spice formula is expressed by using a normal r-approximation as follows (the symbol ${}^{g}(-)$ denotes the conjugate $g(-)g^{-1}$):

$$(2) \qquad \Phi_{\pi_{(\mathbf{S},\vartheta)}}(\delta) = \Delta_{\mathrm{IV}}^{\mathbf{G}}(\delta) \sum_{\substack{g \in S\backslash G/G_{\delta_{<r}} \\ {}^{g}\delta_{<r} \in S}} \varepsilon({}^{g}\delta_{<r}) \cdot \vartheta({}^{g}\delta_{<r}) \cdot \hat{\iota}^{\mathbf{G}_{\delta_{<r}}}_{{}^{g}X^{*}}(\log(\delta_{\geq r})),$$

where $\Delta_{\mathrm{IV}}^{\mathbf{G}}(-)$ is the fourth transfer factor of Kottwitz–Shelstad, $\varepsilon({}^{g}\delta_{<r})$ is a root of unity determined by ${}^{g}\delta_{<r}$, and $\hat{\iota}^{\mathbf{G}_{\delta_{<r}}}_{{}^{g}X^{*}}(-)$ denotes the normalized Fourier transform of the orbital integral with respect to ${}^{g}X^{*}$ taken in the (Lie algebra of) $\mathbf{G}_{\delta_{<r}}$ (see Section 6.7). The important observation here is that $\hat{\iota}^{\mathbf{G}_{\delta_{<r}}}_{{}^{g}X^{*}}(-)$, which is nothing but the Lie algebra analogue of the Harish-Chandra characters of representations, is used to express the contribution of the deeper part $\delta_{\geq r}$. In fact, by the theorem of Waldspurger and Ngô ([Wal06, Ngô10]; see Section 11.4), we can compare the Fourier transforms of Lie algebra orbital integrals between any group and its standard endoscopic group (this can be thought of as a Lie algebra analogue of the standard endoscopic character relation). The principal idea of Kaletha's strategy is to reduce the standard endoscopic character relation to the Lie algebra transfer theorem of Waldspurger–Ngô through the Adler–DeBacker–Spice character formula.

Now let $\mathbf{H}$ be an endoscopic group of $\mathbf{G}$ (with trivial θ) and suppose that both $\Pi^{\mathbf{H}}_{\phi_{\mathbf{H}}}$ and $\Pi^{\mathbf{G}}_{\phi}$ consist of toral supercuspidal representations (of depth $r \in \mathbb{R}_{>0}$). When $\gamma \in H$ is a norm of $\delta \in G$, we may transfer a normal r-approximation $\delta = \delta_{<r} \cdot \delta_{\geq r}$ to $\gamma = \gamma_{<r} \cdot \gamma_{\geq r}$. Therefore, by applying the Adler–DeBacker–Spice formula to all the characters of representations in $\Pi^{\mathbf{G}}_{\phi}$ and $\Pi^{\mathbf{H}}_{\phi_H}$ with respect to these r-approximations, the $\mathbf{G}$-side and the $\mathbf{H}$-side of (1) are rewritten as follows:

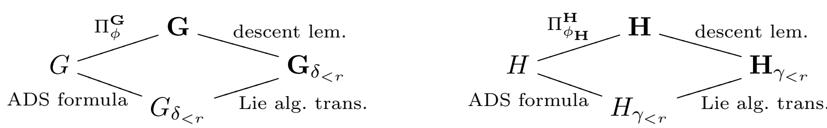
$$(3) \qquad \sum_{\pi_{(\mathbf{S},\vartheta)} \in \Pi^{\mathbf{G}}_{\phi}} \Delta^{\mathrm{spec}}_{\phi,\pi_{(\mathbf{S},\vartheta)}} \cdot \Delta_{\mathrm{IV}}^{\mathbf{G}}(\delta) \sum_{\substack{g \in S\backslash G/G_{\delta_{<r}} \\ {}^{g}\delta_{<r} \in S}} \varepsilon({}^{g}\delta_{<r}) \cdot \vartheta({}^{g}\delta_{<r}) \cdot \hat{\iota}^{\mathbf{G}_{\delta_{<r}}}_{{}^{g}X^{*}}(\log(\delta_{\geq r})),$$

$$(4) \qquad \sum_{\gamma \in H/\mathrm{st}} \mathring{\Delta}(\gamma,\delta) \sum_{\pi_{(\mathbf{S}_{\mathbf{H}},\vartheta_{\mathbf{H}})} \in \Pi^{\mathbf{H}}_{\phi_{\mathbf{H}}}} \Delta_{\mathrm{IV}}^{\mathbf{H}}(\gamma) \sum_{\substack{h \in S_{\mathbf{H}}\backslash H/H_{\gamma_{<r}} \\ {}^{h}\gamma_{<r} \in S_{\mathbf{H}}}} \varepsilon({}^{h}\gamma_{<r}) \cdot \vartheta_{\mathbf{H}}({}^{h}\gamma_{<r}) \cdot \hat{\iota}^{\mathbf{H}_{\gamma_{<r}}}_{{}^{h}X^{*}_{\mathbf{H}}}(\log(\gamma_{\geq r})).$$

Note that the Lie algebra orbital integrals are taken not in $\mathbf{G}$ and $\mathbf{H}$ but in the "descended" groups $\mathbf{G}_{\delta_{<r}}$ and $\mathbf{H}_{\gamma_{<r}}$. The crucially important ingredient here is the theory of *descent for standard endoscopy* due to Langlands–Shelstad [LS90], which guarantees that the group $\mathbf{H}_{\gamma_{<r}}$ again has a structure of an endoscopic group of $\mathbf{G}_{\delta_{<r}}$. Moreover, the transfer factor for the pair $(\mathbf{G},\mathbf{H})$ is related to that of the descended pair $(\mathbf{G}_{\delta_{<r}},\mathbf{H}_{\gamma_{<r}})$. Then the basic setup for utilizing the Lie algebra transfer is done.

$$\begin{array}{ccc} \mathbf{G} & \xrightarrow{\text{descent}} & \mathbf{G}_{\delta_{<r}} \\ \text{standard endoscopy} \Big| & & \Big| \text{standard endoscopy} \\ \mathbf{H} & \xrightarrow{\text{descent}} & \mathbf{H}_{\gamma_{<r}} \end{array}$$

However, there are still several subtle points remaining. Firstly, before thinking about comparing the summands of (3) and (4) via the Lie algebra transfer, we must investigate how the index sets of those sums can be compared. Another related task is to rewrite both sides (3) and (4) in a way such that the Fourier transforms of orbital integrals are summed up over rational conjugacy classes within a stable conjugacy class (in $\mathbf{G}_{\delta_{<r}}$ or $\mathbf{H}_{\gamma_{<r}}$), so that the Lie algebra transfer can be applied. Kaletha resolved these issues by an ingenious trick of rearranging the sums. By construction, the members of $\Pi_\phi^{\mathbf{G}}$ are labeled by the rational conjugacy classes within the stable conjugacy class of *admissible embeddings* (see Definition 7.7) of $\mathbf{S}$ into $\mathbf{G}$. The second index set of (3) can be thought of as a set measuring the difference between the rational conjugacy in $\mathbf{G}$ and that in $\mathbf{G}_{\delta_{<r}}$. Hence the double sums in (3) are combined into a single sum over $G_{\delta_{<r}}$-conjugacy classes within the stable $\mathbf{G}$-conjugacy class of admissible embeddings. If we again partition this sum based on the stable $\mathbf{G}_{\delta_{<r}}$-conjugacy, we can obtain a sum over the desired index set, i.e., $G_{\delta_{<r}}$-conjugacy classes within the stable $\mathbf{G}_{\delta_{<r}}$-conjugacy class. The same argument can be also applied to the $\mathbf{H}$-side (4). Then, by utilizing the "*descent lemma*", which was established in [Kal15, Section 5.4], we can relate the sum over stable $\mathbf{G}_{\delta_{<r}}$-conjugacy classes within the stable $\mathbf{G}$-conjugacy class to that over $\mathbf{H}_{\gamma_{<r}}$-conjugacy classes within the stable $\mathbf{H}$-conjugacy class.



Secondly, we also have to relate the roots of unity "ε" in the summands in (3) and (4). These factors are explicitly computed in [AS09, DS18]; they reflect the symmetry of the root system $\Phi(\mathbf{G}, \mathbf{S})$, which is a finite set equipped with a (typically, highly nontrivial) Galois action. Kaletha first showed that this part can be re-interpreted in terms of several invariants having a more "endoscopic" nature such as the second transfer factors Δ_{II} ([Kal19b, Corollary 4.8.2]). Then, by computing the transfer factor $\mathring{\Delta}(\gamma, \delta)$ explicitly and also by utilizing various nontrivial results on the arithmetic invariants such as local root numbers and Weil constants, he eventually proved that all of these subtle quantities perfectly fit together.

Now, let us move on to the twisted situation. In the following, we let θ be a nontrivial F-rational involution of $\mathbf{G}$ and $\mathbf{H}$ is an endoscopic group of $(\mathbf{G}, \theta)$. Our first task is to establish a twisted version of the Adler–DeBacker–Spice formula. For this, we have to start with investigating a twisted version of the notion of a normal r-approximation because the Adler–DeBacker–Spice formula is based on it. The key observation is the following. Let $\gamma = \gamma_{<r} \cdot \gamma_{\geq r}$ be a normal r-approximation to an elliptic regular semisimple element $\gamma \in G$ (in the usual, untwisted, sense). In fact, a normal r-approximation is a refinement of a topological Jordan decomposition; the shallower part $\gamma_{<r}$ is furthermore decomposed into a topologically semisimple part γ_0 and a topologically unipotent part $\gamma_{<r}^+$ such that $\gamma = \gamma_0 \cdot (\gamma_{<r}^+ \cdot \gamma_{\geq r})$ gives a topological Jordan decomposition of γ in the sense of [Spi08]. If we put $\gamma_+ := \gamma_{<r}^+ \cdot \gamma_{\geq r}$, then γ_+ belongs to $\mathbf{G}_{\gamma_0}$ and the decomposition $\gamma_+ = \gamma_{<r}^+ \cdot \gamma_{\geq r}$ is a normal r-approximation to γ_+ in $\mathbf{G}_{\gamma_0}$. Here, we note that the converse is not always true; even if we take a topological Jordan decomposition $\gamma = \gamma_0 \cdot \gamma_+$ and a normal r-approximation $\gamma_+ = \gamma_{<r}^+ \cdot \gamma_{\geq r}$ in $\mathbf{G}_{\gamma_0}$, the resulting decomposition

$\gamma = (\gamma_0 \cdot \gamma_{<r}^{+}) \cdot \gamma_{\geq r}$ might not be a normal r-approximation to γ in $\mathbf{G}$. The point is that, however, the property that $\gamma_+ = \gamma_{<r}^{+} \cdot \gamma_{\geq r}$ is a normal r-approximation in $\mathbf{G}_{\gamma_0}$ is enough so that the arguments in [AS08, AS09, DS18] work well. Based on this observation, we arrive at the following construction (rather than the "definition") of a normal r-approximation to an elliptic regular semisimple element $\delta \in \tilde{\mathbf{G}}(F)$:

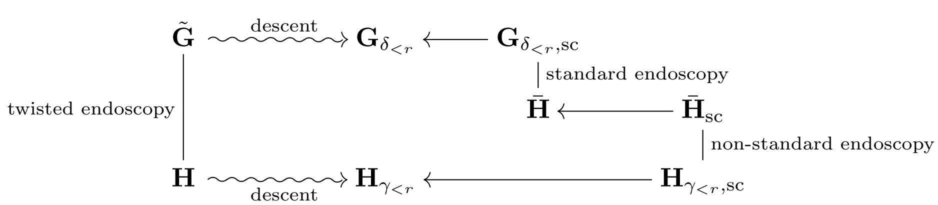
(1) Take a topological Jordan decomposition $\delta = \delta_0 \cdot \delta_+$ by [Spi08]. Here, while δ_0 lies in $\tilde{\mathbf{G}}(F)$, δ_+ lies in the "untwisted" part $\mathbf{G}(F)$, in fact, even $\mathbf{G}_{\delta_0}(F)$.
(2) Take a normal r-approximation $\delta_+ = \delta_{<r}^{+} \cdot \delta_{\geq r}$ in $\mathbf{G}_{\delta_0}$. Here, note that $\delta_+ \in \mathbf{G}_{\delta_0}$ is no longer "twisted", hence the work of Adler–Spice [AS08] is enough so that we can find a normal r-approximation to δ_+ in $\mathbf{G}_{\delta_0}$.

With the normal r-approximation to δ obtained in this way, we can reproduce all the arguments necessary for the Adler–DeBacker–Spice formula in the twisted setting. The resulting formula is expressed in the following way:

$$\Phi_{\tilde{\pi}_{(\mathbf{S},\vartheta)}}(\delta) = \Delta_{\mathrm{IV}}^{\tilde{\mathbf{G}}}(\delta) \sum_{\substack{g \in S\backslash G/G_{\delta_{<r}} \\ {}^{g}\delta_{<r} \in \tilde{S}}} \tilde{\varepsilon}({}^{g}\delta_{<r}) \cdot \tilde{\vartheta}({}^{g}\delta_{<r}) \cdot \hat{\iota}_{{}^{g}X^*}^{\mathbf{G}_{\delta_{<r}}}(\log(\delta_{\geq r})). \tag{5}$$

Here, we put "$\sim$" on the symbols to indicate that these are quantities determined in this twisted context. Although we wrote the above formula (5) in a way parallel to (2), the actual expression of $\tilde{\varepsilon}$ is much more complicated than in the untwisted case (see Proposition 6.11 for the details). One of the subtleties comes from the twisted character formula of the Weil representations of finite symplectic groups. As a toral supercuspidal representation is constructed by using the (Heisenberg–)Weil representation of a finite symplectic group, the proof of Adler–DeBacker–Spice formula is eventually reduced to computing the characters of Weil representations. In [AS09, DS18], it was done by appealing to an explicit formula of Gérardin [Gér77]. Gérardin's result can be also applied to compute the twisted characters of Weil representations, but we additionally need to handle a lot of case-by-case computation depending on the symmetry of $\Phi(\mathbf{G}, \mathbf{S})$ (see Sections 6.6).

By looking at the formula (5), we notice that the contribution of the deeper part $\delta_{\geq r}$ is expressed via the Fourier transform of a Lie algebra orbital integral with respect to the descended group $\mathbf{G}_{\delta_{<r}}$ as well as in the untwisted case. Therefore, one might expect that the same strategy again works in this twisted setting. Unfortunately, in the twisted setting, it is not always the case that $\mathbf{H}_{\gamma_{<r}}$ has a structure of an endoscopic group of $\mathbf{G}_{\delta_{<r}}$. Nevertheless, it is still possible to relate $\mathbf{H}_{\gamma_{<r}}$ to $\mathbf{G}_{\delta_{<r}}$ by introducing another variant of the notion of standard endoscopy called *non-standard endoscopy*. More precisely, there exists a group $\bar{\mathbf{H}}$ such that $\bar{\mathbf{H}}$ is a standard endoscopic group of the simply-connected cover of $\mathbf{G}_{\delta_{<r}}$ and also that the simply-connected covers of $\bar{\mathbf{H}}$ and $\mathbf{H}_{\gamma_{<r}}$ form a non-standard endoscopic pair. Furthermore, the Lie algebra transfer for Fourier transforms of orbital integrals is also available for the non-standard endoscopic pair. This is the framework "*l'endoscopie tordue n'est pas si tordue*" established by Waldspurger ([Wal08]).



Now we gained the right to attempt to mimic Kaletha's proof. Let us discuss the rearranging argument on the index sets of the sums in the twisted endoscopic character relation. The first difficulty is that only θ-stable members of $\Pi_{\phi}^{\mathbf{G}}$ contribute to the twisted endoscopic character relation. Hence we must clarify the θ-stability condition in terms of admissible embeddings, which parametrize the members of $\Pi_{\phi}^{\mathbf{G}}$. We deal with this issue by examining the notion of a *twisted maximal torus*. The second difficulty is that Kaletha's descent lemma, which is necessary for the index sets comparison, needs a major modification. The idea of the descent lemma in the untwisted case is to utilize *admissible isomorphisms*, which are F-rational isomorphisms between maximal tori of $\mathbf{G}$ and those of $\mathbf{H}$. For a given F-rational admissible embedding of a maximal torus $\mathbf{S}_{\mathbf{H}}$ into $\mathbf{H}$, by composing it with an admissible isomorphism between $\mathbf{S}_{\mathbf{H}}$ and a maximal torus (say $\mathbf{S}$) in $\mathbf{G}$, we may produce an F-rational admissible embedding of $\mathbf{S}$ into $\mathbf{G}$. However, this construction no longer works in the twisted setting because an admissible isomorphism in twisted endoscopy is an F-rational isomorphism between an F-rational maximal torus of $\mathbf{H}$ and the coinvariant (with respect to the "twist") of a maximal torus of $\mathbf{G}$. To resolve this issue, we utilize the notion of a *diagram* introduced by Waldspurger (see Definition 10.1). A diagram induces an admissible isomorphism, but also encapsulates more information. Hence it can be thought of as a "rigidification" of an admissible isomorphism. Using diagrams instead of admissible isomorphisms, we can reproduce Kaletha's descent lemma in the twisted setting.

We next discuss comparing the roots of unity appearing in the $\mathbf{G}$-side with those in the $\mathbf{H}$-side. Our basic strategy is to unravel various arithmetic or root-theoretic invariants using similar techniques as in the untwisted case. However, somehow our computation left us with a very complicated quantity as a ratio of a summand in the $\mathbf{G}$-side to that in the $\mathbf{H}$-side (see (37) and also (43)). What we do is, rather than trying to express this ratio more explicitly, just defining the coefficient "$\Delta_{\phi,\pi}^{\mathrm{spec}}$" in the endoscopic character relation to be exactly this ratio. Then the endoscopic character relation holds almost tautologically. Instead, the well-definedness of $\Delta_{\phi,\pi}^{\mathrm{spec}}$ becomes quite nontrivial; a priori $\Delta_{\phi,\pi}^{\mathrm{spec}}$ heavily depends on the elliptic regular semisimple element $\delta \in \tilde{G}$ taken at the beginning. Thus what we do next is to show that $\Delta_{\phi,\pi}^{\mathrm{spec}}$ is in fact independent of the choice of δ. By examining each factor involved in $\Delta_{\phi,\pi}^{\mathrm{spec}}$, we see that it is enough to check that the quantity (44), which is much simpler than (37), is independent of δ. In fact, every factor appearing in (37) is related to ramified symmetric roots contained in the restricted root system (in the sense of Kottwitz–Shelstad; see Section 3.3) of a maximal torus $\mathbf{S}$ of $\mathbf{G}$ associated to a θ-stable toral supercuspidal representation π. Therefore, if the restricted root system does not contain any ramified symmetric root, then (44) is trivial. This is how we obtained Theorem 1.3 (2). What we eventually verified is that (44) is trivial also when $\mathbf{G} = \mathrm{GL}_n$; this is achieved by explicitly classifying the possible Galois actions on (restricted) root systems of GL_n.

Let us finish this introduction by giving several concluding remarks. We believe that it is possible to generalize our results in various directions. The artificial assumption on $\mathbf{G}$ or $\mathbf{S}$ of Theorem 1.3 stems only from the last part of the proof, which is about the well-definedness of $\Delta_{\phi,\pi}^{\mathrm{spec}}$. Probably this part can be dealt with in general by a case-by-case computation based on a classification of twisted endoscopy; cf. [Wal08, Chapitre 14–18]. Also, the assumption that θ is involutive is in fact not necessary in most parts of our arguments (except only for the above-mentioned part on $\Delta^{\mathrm{spec}}_{\phi,\pi}$). Thus it should be also fairly possible to drop the assumption on θ. We expect that it is also possible to establish a depth-zero version of our result by replacing the Adler–DeBacker–Spice character formula with the one of DeBacker–Reeder [DR09]. It is a natural problem to extend our result to the case of general regular (or even non-singular) supercuspidal representations, but it should require twisting the recent work of Spice [Spi18, Spi21], which are quite deep.

We finally would like to emphasize that our arguments are also inspired by Mezo's proof of the twisted endoscopic character relation for discrete series L-packets of real reductive groups (cf. [Mez13]). We think that our constant $\Delta^{\mathrm{spec}}_{\phi,\pi}$ is nothing but the p-adic version of what is called the *spectral transfer factor* in the archimedean setting; see Chapter 1 and also (115)–(117) of [Mez13].

Organization of this paper. In Section 2, we list our fundamental notation. In Section 3, we establish a version of the theory of good product expansion by Adler–Spice in the twisted space setting. In Section 4, we review Yu's construction of tame supercuspidal representations with emphasis on the toral case. In Section 5, we establish a preliminary version of a twisted Adler–DeBacker–Spice character formula for toral supercuspidal representations. In the main theorem of this section (Theorem 5.16), the contribution of the shallow part of δ remains not to be computed. In Section 6, we compute the contribution of the shallow part by appealing to Gérardin's character formula. Some of the results needed in this section are summarized in Appendix A. In Section 7, we review Kaletha's construction of the local Langlands correspondence for regular supercuspidal representations. In Section 8, we review the framework of twisted endoscopy. In Section 9, we investigate the structure of a θ-stable regular supercuspidal L-packets. In Section 10, we examine the notion of a diagram and establish a twisted version of Kaletha's descent lemma. In Section 11, we briefly review Waldspurger's framework. In Sections 12 and 13, we prove some technical lemmas needed in the computation of the spectral transfer factor. In Section 14, we compare the $\mathbf{G}$-side and the $\mathbf{H}$-side of the endoscopic character relation. Especially, we introduce the spectral transfer factor. In Section 15, we prove that the spectral transfer factor is well-defined in the GL_n case, which implies that the twisted endoscopic character relation holds for toral supercuspidal L-packets of twisted GL_n.

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2. Notation and assumptions on p

2.1. **Notation.** Let us summarize the basic notation used in this paper here.

2.1.1. *p-adic fields.* We fix a p-adic field F, i.e., F is a finite extension of $\mathbb{Q}_p$. We also fix an algebraic closure $\overline{F}$ of F. For any extension E of F in $\overline{F}$, we write $\mathcal{O}_E$, $\mathfrak{p}_E$, k_E, and Γ_E for the ring of integers of E, the maximal ideal of $\mathcal{O}_E$, the residue field $\mathcal{O}_E/\mathfrak{p}_E$, and the absolute Galois group $\mathrm{Gal}(\overline{F}/E)$ of E, respectively. For any finite extension E of F in $\overline{F}$, we write W_E, I_E, and P_E for the Weil groups of E, its inertia subgroup, and its wild inertia subgroup, respectively. For any $r \in \mathbb{R}_{>0}$, let I_F^r denote the r-th upper ramification filtration of I_F. We fix a valuation val_F of F such that $\mathrm{val}_F(F) = \mathbb{Z}$. We extend it to $\overline{F}$ and again write val_F for it. We define an absolute value $|\cdot|_{\overline{F}}$ of $\overline{F}$ by $|\cdot|_{\overline{F}} := p^{-\mathrm{val}_F(\cdot)}$.

Because Γ_F appears so often in this paper, we simply write Γ for Γ_F. Similarly, we simply write k for k_F.

We fix an additive character $\psi_F\colon F \to \mathbb{C}^\times$ satisfying $\psi_F|_{\mathfrak{p}_F} \equiv \mathbb{1}$ but $\psi_F|_{\mathcal{O}_F} \not\equiv \mathbb{1}$.

2.1.2. *Algebraic varieties and algebraic groups.* In this paper, we use a bold letter for an algebraic variety and use an italic letter for the set of its F-valued points when it is defined over F. For example, if $\mathbf{X}$ is an algebraic variety defined over F, then $X := \mathbf{X}(F)$.

For any algebraic group $\mathbf{G}$, we write $X^*(\mathbf{G})$ and $X_*(\mathbf{G})$ for the group of characters and cocharacters of $\mathbf{G}$, respectively. We let $\mathbf{Z}_{\mathbf{G}}$ denote the center of $\mathbf{G}$. When $\mathbf{G}$ is defined over F, so is $\mathbf{Z}_{\mathbf{G}}$ and the set of its F-valued points is denoted by $Z_{\mathbf{G}}$.

For any torus $\mathbf{S}$ torus equipped with an automorphism $\theta_{\mathbf{S}}$, we let $\mathbf{S}^{\theta_{\mathbf{S}}}$ and $\mathbf{S}_{\theta_{\mathbf{S}}}$ denote the invariant and coinvariant of $\mathbf{S}$ with respect to $\theta_{\mathbf{S}}$, respectively.

2.1.3. *Centralizers and normalizers.* Suppose that $\mathbf{G}$ is an algebraic group and $\mathbf{X}$ is an algebraic variety having a left and right actions of $\mathbf{G}$, for which we write $\mathbf{G} \times \mathbf{X} \times \mathbf{G} \to \mathbf{X}\colon (g_1, x, g_2) \mapsto g_1 \cdot x \cdot g_2$. Then we define the conjugate action of $\mathbf{G}$ on $\mathbf{X}$ by $\mathbf{G} \times \mathbf{X} \to \mathbf{X}\colon (g, x) \mapsto g \cdot x \cdot g^{-1}$. We introduce the following notation:

- For $g \in \mathbf{G}$, let $[g]$ denote the conjugation automorphism $\mathbf{X} \to \mathbf{X}\colon x \mapsto g \cdot x \cdot g^{-1}$. We also often write ${}^g x := [g](x) = g \cdot x \cdot g^{-1}$.
- For $x \in \mathbf{X}$, let $\mathbf{G}^x$ denote the full stabilizer of x in $\mathbf{G}$ with respect to the conjugate action, i.e., $\mathbf{G}^x := \{g \in \mathbf{G} \mid [g](x) = x\}$.
- For $x \in \mathbf{X}$, let $\mathbf{G}_x$ denote the connected stabilizer of x in $\mathbf{G}$ with respect to the conjugate action, i.e., $\mathbf{G}_x := \mathbf{G}^{x,\circ}$.

Note that, when $\mathbf{G}$ and $\mathbf{X}$ are F-rational, $[g]$ is also F-rational if $g \in \mathbf{G}(F)$. Similarly, $\mathbf{G}^x$ and $\mathbf{G}_x$ are F-rational if $x \in \mathbf{X}(F)$.

For any subset $Y \subset \mathbf{X}$, we put

- $\mathbf{Z}_{\mathbf{G}}(Y) := \{g \in \mathbf{G} \mid [g](y) = y \text{ for any } y \in Y\}$ and
- $\mathbf{N}_{\mathbf{G}}(Y) := \{g \in \mathbf{G} \mid [g](Y) \subset Y\}$.

When Y is a singleton $\{y\}$, we simply write $\mathbf{Z}_{\mathbf{G}}(y) := \mathbf{Z}_{\mathbf{G}}(Y)$ $(= \mathbf{G}^y)$ and $\mathbf{N}_{\mathbf{G}}(y) := \mathbf{N}_{\mathbf{G}}(Y)$. If $\mathbf{G}$ and $\mathbf{X}$ are defined over F and Y is a subset of $\mathbf{X}(F)$, then $\mathbf{Z}_{\mathbf{G}}(Y)$ and $\mathbf{N}_{\mathbf{G}}(Y)$ are defined over F and the sets of their F-valued points are denoted by $Z_{\mathbf{G}}(Y)$ and $N_{\mathbf{G}}(Y)$, respectively.

2.1.4. *Reductive groups.* For any connected reductive group $\mathbf{G}$ and its maximal torus $\mathbf{S}$, we let $\Phi(\mathbf{G},\mathbf{S})$ and $\Phi^\vee(\mathbf{G},\mathbf{S})$ denote the set of roots and coroots of $\mathbf{S}$ in $\mathbf{G}$, respectively. Note that, when both $\mathbf{G}$ and $\mathbf{S}$ are defined over F, the sets $\Phi(\mathbf{G},\mathbf{S})$ and $\Phi^\vee(\mathbf{G},\mathbf{S})$ are equipped with an action of Γ. We let $\Omega_{\mathbf{G}}(\mathbf{S})$ be the Weyl group of $\mathbf{S}$ in $\mathbf{G}$, i.e., $\Omega_{\mathbf{G}}(\mathbf{S}) := \mathbf{N}_{\mathbf{G}}(\mathbf{S})/\mathbf{S}$. We sometimes loosely write $\Omega_{\mathbf{G}}$ for $\Omega_{\mathbf{G}}(\mathbf{S})$ when the choice of a maximal torus $\mathbf{S}$ is clear from the context (e.g., when $\mathbf{S}$ is a maximal torus belonging to a splitting of $\mathbf{G}$).

We write $\boldsymbol{\mathfrak{g}}$ for the Lie algebra of $\mathbf{G}$. When $\mathbf{G}$ is defined over F, $\boldsymbol{\mathfrak{g}}$ is an algebraic variety over F, hence we write $\mathfrak{g} := \boldsymbol{\mathfrak{g}}(F)$ as explained above.

2.1.5. *Bruhat–Tits theory.* Suppose that $\mathbf{G}$ is a connected reductive group over F. We follow the notation on Bruhat–Tits theory used by [AS08, AS09, DS18]. (See, for example, [AS08, Section 3.1] for details.) Especially, $\mathcal{B}(\mathbf{G},F)$ (resp. $\mathcal{B}^{\mathrm{red}}(\mathbf{G},F)$) denotes the enlarged (resp. reduced) Bruhat–Tits building of $\mathbf{G}$ over F. We define $\widetilde{\mathbb{R}}$ to be the set $\mathbb{R}\sqcup\{r+ \mid r\in\mathbb{R}\}\sqcup\{\infty\}$ with a natural order. Then, for any $r\in\widetilde{\mathbb{R}}_{\geq 0}$ and $\mathbf{x}\in\mathcal{B}^{\mathrm{red}}(\mathbf{G},F)$, we can consider the r-th Moy–Prasad filtration $G_{\mathbf{x},r}$ of G with respect to the point $\mathbf{x}$. For any $r,s\in\widetilde{\mathbb{R}}_{\geq 0}$ satisfying $r<s$, we write $G_{\mathbf{x},r:s}$ for the quotient $G_{\mathbf{x},r}/G_{\mathbf{x},s}$. We put $G_r := \bigcup_{\mathbf{x}\in\mathcal{B}^{\mathrm{red}}(\mathbf{G},F)} G_{\mathbf{x},r}$ for $r\in\widetilde{\mathbb{R}}_{\geq 0}$. Similarly, we have the Moy–Prasad filtration $\{\mathfrak{g}_{\mathbf{x},r}\}_r$ on the Lie algebra $\mathfrak{g}=\boldsymbol{\mathfrak{g}}(F)$, their quotients $\mathfrak{g}_{\mathbf{x},r:s}$, and the unions $\mathfrak{g}_r$. We also have the Moy–Prasad filtration on the dual Lie algebra $\mathfrak{g}^* := \mathrm{Hom}_F(\mathfrak{g},F)$ defined by

$$\mathfrak{g}^*_{\mathbf{x},r} := \{Y^*\in\mathfrak{g}^* \mid \langle\mathfrak{g}_{\mathbf{x},(-r)+},Y^*\rangle\subset\mathfrak{p}_F\}$$

for any $r\in\mathbb{R}_{\geq 0}$ and $\mathbf{x}\in\mathcal{B}^{\mathrm{red}}(\mathbf{G},F)$ ($\mathfrak{g}^*_{\mathbf{x},r+}$ is defined to be $\bigcup_{s>r}\mathfrak{g}^*_{\mathbf{x},s}$).

Suppose that $\mathbf{S}$ is an F-rational tamely ramified maximal torus of $\mathbf{G}$. By fixing an S-equivariant embedding of $\mathcal{B}(\mathbf{S},F)$ into $\mathcal{B}(\mathbf{G},F)$, we may regard $\mathcal{B}(\mathbf{S},F)$ as a subset of $\mathcal{B}(\mathbf{G},F)$. Then, for any point $\mathbf{x}\in\mathcal{B}(\mathbf{G},F)$, the property that "$\mathbf{x}$ belongs to the image of $\mathcal{B}(\mathbf{S},F)$" does not depend on the choice of such an embedding (see the second paragraph of [FKS23, Section 3] for details). For any point $\mathbf{x}\in\mathcal{B}(\mathbf{G},F)$ which belongs to $\mathcal{B}(\mathbf{S},F)$, we have $S_{\mathrm{b}}\subset G_{\mathbf{x}}$, where S_{b} denotes the maximal bounded subgroup of S. When $\mathbf{S}$ is elliptic in $\mathbf{G}$, the image of $\mathcal{B}(\mathbf{S},F)$ in $\mathcal{B}^{\mathrm{red}}(\mathbf{G},F)$ consists of only one point. If $\mathbf{x}\in\mathcal{B}(\mathbf{G},F)$ belongs to the image of $\mathcal{B}(\mathbf{S},F)$, we say that $\mathbf{x}$ is associated to $\mathbf{S}$.

We also fix a family of mock-exponential maps $\mathfrak{g}_{\mathbf{x},r}\to G_{\mathbf{x},r}$ for $x\in\mathcal{B}(\mathbf{G},F)$ and $r\in\tilde{\mathbb{R}}_{>0}$ and simply write "exp" for it (see [AS09, Appendix A]; cf. [Hak18, Section 3.4]). We write "log" for the inverse of exp. It is guaranteed that a mock exponential map in the sense of [AS09, Appendix A] always exists under the assumption that $p\nmid|\Omega_{\mathbf{G}}|$, which we will assume later.

2.1.6. *Finite sets with Galois actions.* We put $\Sigma := \Gamma\times\{\pm1\}$. Suppose that Φ is a finite set with an action of Σ, e.g., the set of roots of an F-rational maximal torus in a connected reductive group (-1 acts on Φ via $\alpha\mapsto-\alpha$ in this case). Following [AS09], we put $\dot{\Phi} := \Phi/\Gamma$ and $\ddot{\Phi} := \Phi/\Sigma$.

For each $\alpha\in\Phi$, we put Γ_α (resp. $\Gamma_{\pm\alpha}$) to be the stabilizer of α (resp. $\{\pm\alpha\}$) in Γ. Let F_α (resp. $F_{\pm\alpha}$) be the subfield of $\overline{F}$ fixed by Γ_α (resp. $\Gamma_{\pm\alpha}$). Hence we have $\Gamma_\alpha=\Gamma_{F_\alpha}$ and $\Gamma_{\pm\alpha}=\Gamma_{F_{\pm\alpha}}$:

$$F\subset F_{\pm\alpha}\subset F_\alpha \quad\longleftrightarrow\quad \Gamma\supset\Gamma_{\pm\alpha}\supset\Gamma_\alpha.$$

We abbreviate the residue field k_{F_α} of F_α (resp. $k_{F_{\pm\alpha}}$ of $F_{\pm\alpha}$) as k_α (resp. $k_{\pm\alpha}$).

We say that α is *asymmetric* if $F_\alpha = F_{\pm\alpha}$ and that α is *symmetric* if $F_\alpha \supsetneq F_{\pm\alpha}$. We remark that α is symmetric if and only if the Γ-orbit of α contains $-\alpha$. By noting that the extension $F_\alpha/F_{\pm\alpha}$ is necessarily quadratic if α is symmetric, we say that α is *(symmetric) unramified* (resp. *ramified*) if $F_\alpha/F_{\pm\alpha}$ is unramified (resp. ramified). We write Φ_{asym}, Φ_{ur}, Φ_{ram}, and Φ_{sym} for the set of asymmetric elements, symmetric unramified elements, symmetric ramified elements, and symmetric elements of Φ, respectively.

For $\alpha \in \Phi_{\mathrm{sym}}$, we let $\kappa_\alpha\colon F_{\pm\alpha}^\times \to \mathbb{C}^\times$ denote the quadratic character of $F_{\pm\alpha}^\times$ corresponding to the quadratic extension $F_\alpha/F_{\pm\alpha}$ under the local class field theory.

Note that, if α is symmetric, $\Gamma\alpha = \Sigma\alpha$. This implies that the sets $\dot{\Phi}_{\mathrm{sym}}$ and $\ddot{\Phi}_{\mathrm{sym}}$ can be naturally identified (and, of course, the same is true for Φ_{ur} or Φ_{ram}).

2.1.7. *Several arithmetic invariants.* For any finite extension $E_\pm$ of F and its quadratic extension E, we let $\lambda_{E/E_\pm} := \lambda_{E/E_\pm}(\psi_F \circ \mathrm{Tr}_{E_\pm/F})$ denote the Langlands constant with respect to the nontrivial additive character $\psi_F \circ \mathrm{Tr}_{E_\pm/F}$ of $E_\pm$ (see, e.g., [BH06, 30.4]). When the quadratic extension $E/E_\pm$ is given by $F_\alpha/F_{\pm\alpha}$ as in Section 2.1.6, we even write λ_α for $\lambda_{F_\alpha/F_{\pm\alpha}}$.

In this paper, we often consider the root number $\varepsilon(\frac{1}{2}, X^*(\mathbf{S})_\mathbb{C}, \psi_F)$ of the ε-factor of the Galois representation $X^*(\mathbf{S})_\mathbb{C}$ obtained from an F-rational torus (see [BH06, Section 30] or [Tat79, Section 3.6] for the definition of the ε-factor). We shortly write $\varepsilon(\mathbf{S}) := \varepsilon(\frac{1}{2}, X^*(\mathbf{S})_\mathbb{C}, \psi_F)$.

2.1.8. *Finite fields.* Suppose that $\underline{k}$ is a finite field of odd characteristic p. Then the multiplicative group $\underline{k}^\times$ is cyclic of even order, hence there exists a unique nontrivial sign character $\underline{k}^\times \to \{\pm 1\}$. We write $\mathrm{sgn}_{\underline{k}^\times}(-)$ for this character.

Next, we furthermore suppose that $[\underline{k} : \mathbb{F}_p]$ is even. Then, there uniquely exists a subextension $\underline{k}_\pm$ satisfying $[\underline{k} : \underline{k}_\pm] = 2$. We let $\underline{k}^1$ denote the kernel of the norm map $\mathrm{Nr}_{\underline{k}/\underline{k}_\pm} : \underline{k}^\times \to \underline{k}_\pm^\times$. By noting that $\underline{k}^1$ is also cyclic of even order, we write $\mathrm{sgn}_{\underline{k}^1}(-)$ for the unique nontrivial sign character of $\underline{k}^1$.

2.2. **Assumptions on p.** From Section 3.4, we assume that p is odd. In Section 4.1, we add the assumption that p does not divide the order of the absolute Weyl group of $\mathbf{G}$. From Section 11 to the end of this paper, we furthermore assume that p is greater than or equal to $(2 + e_F)n$, where n is the minimum of the dimension of a faithful representation of $\mathbf{G}$ and e_F is the ramification degree of $F/\mathbb{Q}_p$.

3. Twisted spaces

In this section, we review basics of twisted spaces and establish a version of the theory of good product expansion by Adler–Spice ([AS08]) in twisted spaces.

3.1. **Twisted spaces.** Recall that the theory of twisted endoscopy ([KS99]) starts with fixing a triple $(\mathbf{G}, \theta, \mathbf{a})$. Here,

- $\mathbf{G}$ is a connected reductive group over F,
- θ is an F-rational quasi-semisimple automorphism of $\mathbf{G}$ (i.e., θ preserves a Borel pair), and
- $\mathbf{a} \in H^1(W_F, \mathbf{Z}_{\hat{\mathbf{G}}})$, where $\hat{\mathbf{G}}$ is the Langlands dual group of $\mathbf{G}$ over $\mathbb{C}$.

In this paper, we focus on the case where $(\mathbf{G}, \theta, \mathbf{a})$ satisfies the following conditions:

- $\mathbf{G}$ is quasi-split (hence, we may and do fix an F-splitting $\mathbf{spl}_\mathbf{G} = (\mathbf{B}, \mathbf{T}, \{X_\alpha\}_\alpha)$ of $\mathbf{G}$);

- θ preserves $\mathbf{spl_G}$ and is involutive, i.e., the order of θ is 2;
- $\mathbf{a}$ is trivial.

Example 3.1. We particularly have the following example in mind. Let $\mathbf{G}$ be the general linear group GL_n over F. Let θ be the F-rational automorphism of $\mathbf{G}$ defined by

$$\theta(g) := J_n {}^t g^{-1} J_n^{-1},$$

where J_n is an anti-diagonal matrix of size n whose $(i, n+1-i)$-th entry is given by $(-1)^{i-1}$ and ${}^t g$ denotes the transpose of g. Then θ is involutive and preserves the standard splitting of GL_n. This is the case considered in Arthur's theory of the endoscopic classification of representations of quasi-split classical groups ([Art13]).

Following Labesse ([Lab04]) and Waldspurger ([Wal08]), we work with the formalism of *twisted spaces* as follows. We put

$$\tilde{\mathbf{G}} := \mathbf{G}\theta.$$

This is a twisted space in the sense of Labesse, that is, an algebraic variety over F which is a bi-$\mathbf{G}$-torsor. As an algebraic variety, it is isomorphic to $\mathbf{G}$ by the map written by $g \mapsto g\theta$. The right and left actions of $\mathbf{G}$ on $\tilde{\mathbf{G}}$ is given by

$$g_1 \cdot (g\theta) \cdot g_2 = (g_1 g \theta(g_2))\theta.$$

Thus the conjugate action of $\mathbf{G}$ on $\tilde{\mathbf{G}}$ is given by

$$[g_1](g\theta) := g_1 \cdot (g\theta) \cdot g_1^{-1} = (g_1 g \theta(g_1)^{-1})\theta.$$

Note that the θ-twisted conjugacy in $\mathbf{G}$ (as in [KS99]) is amount to the $\mathbf{G}$-conjugacy in $\tilde{\mathbf{G}}$. The conjugate action of $\tilde{\mathbf{G}}$ on $\mathbf{G}$ is also defined by, for $\delta = g\theta \in \tilde{\mathbf{G}}$,

$$[\delta] := [g] \circ \theta \colon \mathbf{G} \to \mathbf{G}.$$

3.2. **Twisted maximal torus.** We next investigate the notion of a twisted maximal torus.

Definition 3.2 ([MW18, Section 4.1])**.** Let $(\tilde{\mathbf{S}}, \mathbf{S})$ be a pair of

- an F-rational maximal torus $\mathbf{S}$ of $\mathbf{G}$ and
- an F-rational $\mathbf{S}$-twisted subspace $\tilde{\mathbf{S}}$ of $\tilde{\mathbf{G}}$ (i.e., subvariety of $\tilde{\mathbf{G}}$ which is a bi-$\mathbf{S}$-torsor under the bi-$\mathbf{S}$-action on $\tilde{\mathbf{S}} \subset \tilde{\mathbf{G}}$).

We say that $(\tilde{\mathbf{S}}, \mathbf{S})$ is an *F-rational twisted maximal torus* of $\tilde{\mathbf{G}}$ if the following two conditions are satisfied:

1. There exists a Borel subgroup $\mathbf{B_S}$ of $\mathbf{G}$ (not necessarily defined over F) containing $\mathbf{S}$ and satisfying $\tilde{\mathbf{S}} = \mathbf{N}_{\tilde{\mathbf{G}}}(\mathbf{S}) \cap \mathbf{N}_{\tilde{\mathbf{G}}}(\mathbf{B_S})$.
2. The set $\tilde{S} = \tilde{\mathbf{S}}(F)$ of F-valued points of $\tilde{\mathbf{S}}$ is not empty.

By the condition (1) of Definition 3.2, every $\eta \in \tilde{\mathbf{S}}$ acts on $\mathbf{S}$ by the conjugation $[\eta]$. Since $\tilde{\mathbf{S}}$ is an $\mathbf{S}$-twisted space and $\mathbf{S}$ is commutative, this action is independent of the choice of η. We let $\theta_{\mathbf{S}}$ denote this automorphism of $\mathbf{S}$. Note that we can take η to be F-rational by the condition (2) of Definition 3.2, hence $\theta_{\mathbf{S}}$ is F-rational. Moreover, since θ is involutive, so is $\theta_{\mathbf{S}}$.

When $(\mathbf{S}, \tilde{\mathbf{S}})$ is an F-rational twisted maximal torus of $\tilde{\mathbf{G}}$, we often simply say that "$\tilde{\mathbf{S}}$ is an F-rational twisted maximal torus of $\tilde{\mathbf{G}}$". For an F-rational twisted maximal torus $\tilde{\mathbf{S}}$ of $\tilde{\mathbf{G}}$, we put $\mathbf{S}^\natural := \mathbf{S}^{\theta_{\mathbf{S}},\circ}$. Note that, for any $\eta \in \tilde{\mathbf{S}}$, we have $\mathbf{S}^\natural = \mathbf{S}_\eta \subset \mathbf{G}_\eta$. The relationship between $\mathbf{S}$ and $\mathbf{S}^\natural$ is described as follows:

Proposition 3.3. *For any F-rational twisted maximal torus $\tilde{\mathbf{S}}$ of $\tilde{\mathbf{G}}$, we have*

1. $\mathbf{Z}_{\mathbf{G}}(\mathbf{S}^{\natural}) = \mathbf{S}$,
2. $\mathbf{Z}_{\mathbf{G}}(\tilde{\mathbf{S}})^{\circ} = \mathbf{S}^{\natural}$, *and*
3. *for any $\eta \in \tilde{\mathbf{S}}$, $\mathbf{S}^{\natural}$ is a maximal torus of $\mathbf{G}_{\eta}$ (when $\eta \in \tilde{S}$, both $\mathbf{S}$ and $\mathbf{G}_{\eta}$ are F-rational).*

Proof. Let $\eta \in \tilde{\mathbf{S}}$. Let $\mathbf{B}_{\mathbf{S}}$ be a Borel subgroup of $\mathbf{G}$ containing $\mathbf{S}$ and satisfying $\tilde{\mathbf{S}} = \mathbf{N}_{\tilde{\mathbf{G}}}(\mathbf{S}) \cap \mathbf{N}_{\tilde{\mathbf{G}}}(\mathbf{B}_{\mathbf{S}})$. Then $[\eta]$ defines an automorphism of $\mathbf{G}$ preserving the Borel pair $(\mathbf{B}_{\mathbf{S}}, \mathbf{S})$. We apply Steinberg's result ([Ste68]), which is summarized in [KS99, Theorem 1.1.A], to the automorphism $[\eta]$. By [KS99, Theorem 1.1.A (2)], $\mathbf{S} \cap \mathbf{G}_{\eta}$ is a maximal torus of $\mathbf{G}_{\eta}$. Since $\mathbf{S}_{\eta} \subset \mathbf{S} \cap \mathbf{G}_{\eta} \subset \mathbf{S}^{\eta}$, the connectedness of $\mathbf{S} \cap \mathbf{G}_{\eta}$ implies that $\mathbf{S}_{\eta} = \mathbf{S} \cap \mathbf{G}_{\eta}$. Thus we get the assertion (3) (the F-rationality of $\mathbf{S}$ and $\mathbf{G}_{\eta}$ when $\eta \in \tilde{S}$ is clear). Moreover, by [KS99, Theorem 1.1.A (4)], we get the assertion (1).

Let us check the assertion (2). As the inclusion $\mathbf{Z}_{\mathbf{G}}(\tilde{\mathbf{S}})^{\circ} \supset \mathbf{S}^{\natural}$ is obvious, we show the converse inclusion. Let $g \in \mathbf{Z}_{\mathbf{G}}(\tilde{\mathbf{S}})^{\circ}$. Then we have $gs\eta g^{-1} = s\eta$ for any $s \in \mathbf{S}$ since $\tilde{\mathbf{S}} = \mathbf{S}\eta$. Note that, as $\eta \in \tilde{\mathbf{S}}$, we have $\mathbf{Z}_{\mathbf{G}}(\tilde{\mathbf{S}})^{\circ} \subset \mathbf{Z}_{\mathbf{G}}(\tilde{\eta})^{\circ} = \mathbf{G}_{\eta}$. Thus $gs\eta g^{-1} = s\eta$ (for any $s \in \mathbf{S}$) if and only if $gsg^{-1} = s$ (for any $s \in \mathbf{S}$), which implies that $g \in \mathbf{S}$. Hence we get $g \in \mathbf{S} \cap \mathbf{G}_{\eta} = \mathbf{S}_{\eta} = \mathbf{S}^{\natural}$. □

Recall that an element $\delta \in \tilde{\mathbf{G}}$ is said to be

- *semisimple* if $[\delta]$ is quasi-semisimple,
- *regular semisimple* if δ is semisimple and $\mathbf{G}_{\delta}$ is a torus, and
- *strongly regular semisimple* if δ is semisimple and $\mathbf{G}^{\delta}$ is abelian

(see [KS99, Sections 3.2 and 3,3]).

Let $\mathbf{A}_{\tilde{\mathbf{G}}}$ denote the maximal split subtorus of $\mathbf{Z}_{\mathbf{G}}^{\theta}$.

Definition 3.4. (1) Let $\tilde{\mathbf{S}}$ be an F-rational twisted maximal torus of $\tilde{\mathbf{G}}$. We say that $\tilde{\mathbf{S}}$ is *elliptic* if $\mathbf{S}^{\natural}$ is anisotropic modulo $\mathbf{A}_{\tilde{\mathbf{G}}}$.

(2) For any semisimple element $\delta \in \tilde{G}$, we say that δ is *elliptic* if there exists an F-rational elliptic twisted maximal torus $\tilde{\mathbf{S}}$ of $\tilde{\mathbf{G}}$ such that $\delta \in \tilde{S}$.

Remark 3.5. If $(\tilde{\mathbf{S}}, \mathbf{S})$ is an F-rational twisted maximal torus of $\tilde{\mathbf{G}}$ whose $\mathbf{S}$ is elliptic, then $(\tilde{\mathbf{S}}, \mathbf{S})$ is elliptic. Indeed, as we have an injection

$$\mathbf{S}^{\natural}/\mathbf{A}_{\tilde{\mathbf{G}}} \hookrightarrow \mathbf{S}^{\natural}/(\mathbf{Z}_{\mathbf{G}} \cap \mathbf{S}^{\natural}) \hookrightarrow \mathbf{S}/\mathbf{Z}_{\mathbf{G}},$$

the ellipticity of $\mathbf{S}$ in $\mathbf{G}$ (which is equivalent to the anisotropy of $\mathbf{S}$ modulo $\mathbf{Z}_{\mathbf{G}}$) implies that the anisotropy of $\mathbf{S}^{\natural}$ modulo $\mathbf{A}_{\tilde{\mathbf{G}}}$.

Lemma 3.6. *Let $\mathbf{S}$ be an F-rational maximal torus of $\mathbf{G}$. If there exists a semisimple element $\eta \in \tilde{G}$ and a Borel subgroup $\mathbf{B}_{\mathbf{S}}$ containing $\mathbf{S}$ such that $(\mathbf{B}_{\mathbf{S}}, \mathbf{S})$ is preserved by $[\eta]$, then $(\tilde{\mathbf{S}}, \mathbf{S}) := (\mathbf{S}\eta, \mathbf{S})$ is an F-rational twisted maximal torus of $(\tilde{\mathbf{G}}, \mathbf{G})$.*

Proof. Since $\tilde{\mathbf{G}} = \mathbf{G}\eta$ and $[\eta]$ preserves $(\mathbf{B}_{\mathbf{S}}, \mathbf{S})$, we have

$$\mathbf{N}_{\tilde{\mathbf{G}}}(\mathbf{S}) \cap \mathbf{N}_{\tilde{\mathbf{G}}}(\mathbf{B}_{\mathbf{S}}) = \big(\mathbf{N}_{\mathbf{G}}(\mathbf{S}) \cap \mathbf{N}_{\mathbf{G}}(\mathbf{B}_{\mathbf{S}})\big)\eta = \mathbf{S}\eta = \tilde{\mathbf{S}}.$$

Moreover, obviously $\tilde{S} = \tilde{\mathbf{S}}(F)$ is not empty as it contains η. □

3.3. **Steinberg's result on the structure of descended groups.** Let $\tilde{\mathbf{S}}$ be an F-rational twisted maximal torus of $\tilde{\mathbf{G}}$ and $\mathbf{B}_{\mathbf{S}}$ a Borel subgroup which contains $\mathbf{S}$ and is preserved by the action of $\tilde{\mathbf{S}}$. By fixing an element $g_{\mathbf{S}} \in \mathbf{G}$ satisfying $[g_{\mathbf{S}}](\mathbf{B}_{\mathbf{S}}, \mathbf{S}) = (\mathbf{B}, \mathbf{T})$, we get an isomorphism $[g_{\mathbf{S}}] \colon (\tilde{\mathbf{S}}, \mathbf{S}) \xrightarrow{\sim} (\tilde{\mathbf{T}}, \mathbf{T})$. Note that the isomorphism $[g_{\mathbf{S}}] \colon \mathbf{S} \xrightarrow{\sim} \mathbf{T}$ is independent of the choice of $g_{\mathbf{S}} \in \mathbf{G}$ and that the automorphism $\theta_{\mathbf{S}}$ of $\mathbf{S}$ is transported to θ on $\mathbf{T}$ via $[g_{\mathbf{S}}]$, i.e., $\theta \circ [g_{\mathbf{S}}] = [g_{\mathbf{S}}] \circ \theta_{\mathbf{S}}$.

For any $\eta \in \tilde{S}$, its connected centralizer $\mathbf{G}_\eta$ is a connected reductive group with a maximal torus $\mathbf{S}^\natural$ (Proposition 3.3). In this subsection, we review some facts about the structure of the root system $\Phi(\mathbf{G}_\eta, .\mathbf{S}^\natural)$ following [Wal08, Section 3.3] (we will review more details in Section 12.1).

We write $\mathbf{T}^\natural := \mathbf{T}^{\theta,\circ}$. We put

- $Y^*(\mathbf{T}) := X^*(\mathbf{T})/(X^*(\mathbf{T}) \cap (1-\theta)X^*(\mathbf{T})_{\mathbb{Q}})$ and
- $Y_*(\mathbf{T}) := X_*(\mathbf{T})/(X_*(\mathbf{T}) \cap (1-\theta)X_*(\mathbf{T})_{\mathbb{Q}})$.

We write $p^* \colon X^*(\mathbf{T}) \to Y^*(\mathbf{T})$ and $p_* \colon X_*(\mathbf{T}) \to Y_*(\mathbf{T})$ for the natural surjections. Then we have the following:

(1) $Y^*(\mathbf{T}) \cong X^*(\mathbf{T}^\natural)$ is the $\mathbb{Z}$-dual to $X_*(\mathbf{T})^\theta \cong X_*(\mathbf{T}^\natural)$;
(2) $Y_*(\mathbf{T})$ is the $\mathbb{Z}$-dual to $X^*(\mathbf{T})^\theta$.

We put $\Theta := \langle\theta\rangle$. Note that the action of Θ on $(\mathbf{G}, \mathbf{T})$ induces an action on $\Phi(\mathbf{G}, \mathbf{T})$. For any $\alpha \in \Phi(\mathbf{G}, \mathbf{T})$, we let l_α be the cardinality of the Θ-orbit of α in $\Phi(\mathbf{G}, \mathbf{T})$ and define an element $N(\alpha) \in \Phi(\mathbf{G}, \mathbf{T})$ by

$$N(\alpha) := \sum_{i=0}^{l_\alpha - 1} \theta^i(\alpha).$$

We also define $l_{\alpha^\vee}$ and $N(\alpha^\vee)$ for any $\alpha^\vee \in \Phi^\vee(\mathbf{G}, \mathbf{T})$ in the same manner. For $\alpha \in \Phi(\mathbf{G}, \mathbf{T})$, we shortly write $\alpha_{\mathrm{res}} := p^*(\alpha)$. We define a set $\Phi_{\mathrm{res}}(\mathbf{G}, \mathbf{T})$ by

$$\Phi_{\mathrm{res}}(\mathbf{G}, \mathbf{T}) = \{p^*(\alpha) \mid \alpha \in \Phi(\mathbf{G}, \mathbf{T})\} \subset Y^*(\mathbf{T}) \cong X^*(\mathbf{T}^\natural).$$

Then $\Phi_{\mathrm{res}}(\mathbf{G}, \mathbf{T})$ forms a (possibly non-reduced) root system. We call elements of $\Phi_{\mathrm{res}}(\mathbf{G}, \mathbf{T})$ *restricted roots*. Following [KS99, Section 1.3], we say that $\alpha \in \Phi(\mathbf{G}, \mathbf{T})$ (or its associated α_{res}) is of

- *type* 1 if $2\alpha_{\mathrm{res}}, \frac{1}{2}\alpha_{\mathrm{res}} \notin \Phi_{\mathrm{res}}(\mathbf{G}, \mathbf{T})$,
- *type* 2 if $2\alpha_{\mathrm{res}} \in \Phi_{\mathrm{res}}(\mathbf{G}, \mathbf{T})$,
- *type* 3 if $\frac{1}{2}\alpha_{\mathrm{res}} \in \Phi_{\mathrm{res}}(\mathbf{G}, \mathbf{T})$.

We put

$$\varrho_\alpha := \begin{cases} 1 & \text{if } \alpha \text{ is of type 1 or 3,} \\ 2 & \text{if } \alpha \text{ is of type 2,} \end{cases} \qquad \varsigma_\alpha := \begin{cases} 1 & \text{if } \alpha \text{ is of type 1 or 2,} \\ -1 & \text{if } \alpha \text{ is of type 3.} \end{cases}$$

We also define a set $\Phi^\vee_{\mathrm{res}}(\mathbf{G}, \mathbf{T})$ by

$$\Phi^\vee_{\mathrm{res}}(\mathbf{G}, \mathbf{T}) = \{\varrho_\alpha \cdot N(\alpha^\vee) \mid \alpha \in \Phi^\vee(\mathbf{G}, \mathbf{T})\} \subset X_*(\mathbf{T})^\theta \cong X_*(\mathbf{T}^\natural).$$

Then we have bijections

$$\Phi(\mathbf{G}, \mathbf{T})/\Theta \xrightarrow{1:1} \Phi_{\mathrm{res}}(\mathbf{G}, \mathbf{T}) \colon \alpha \mapsto \alpha_{\mathrm{res}}\ (:= p^*(\alpha)),$$
$$\Phi^\vee(\mathbf{G}, \mathbf{T})/\Theta \xrightarrow{1:1} \Phi^\vee_{\mathrm{res}}(\mathbf{G}, \mathbf{T}) \colon \alpha \mapsto \varrho_\alpha \cdot N(\alpha).$$

(We note that $\Phi_{\mathrm{res}}(\mathbf{G}, \mathbf{T})$ and $\Phi^\vee_{\mathrm{res}}(\mathbf{G}, \mathbf{T})$ are denoted by Σ^{res} and $\check{\Sigma}^{\mathrm{res}}$ in [Wal08, Section 3.3], respectively.)

Remark 3.7 ([KS99, (1.3.3)]). There exists a restricted root of type 2 or 3 only when $\Phi(\mathbf{G}, \mathbf{T})$ contains an irreducible component of Dynkin type A_{2n} which is preserved and acted by θ nontrivially.

Now let η be an element of $\tilde{S}$ and let $\nu \in \mathbf{T}$ be the element such that

$$[g_{\mathbf{S}}](\eta) = \nu\theta \in \tilde{\mathbf{T}} = \mathbf{T}\theta.$$

Then $[g_{\mathbf{S}}]\colon \mathbf{G} \to \mathbf{G}$ induces an isomorphism between $(\mathbf{G}_\eta, \mathbf{S}^\natural)$ and $(\mathbf{G}_{\nu\theta}, \mathbf{T}^\natural)$. In particular, the sets $\Phi(\mathbf{G}_\eta, \mathbf{S}^\natural)$ and $\Phi^\vee(\mathbf{G}_\eta, \mathbf{S}^\natural)$ can be identified with $\Phi(\mathbf{G}_{\nu\theta}, \mathbf{T}^\natural)$ and $\Phi^\vee(\mathbf{G}_{\nu\theta}, \mathbf{T}^\natural)$, respectively (note that here we ignore the Galois actions). The latter sets are described in terms of the restricted roots and coroots as follows:

$$\Phi(\mathbf{G}_{\nu\theta}, \mathbf{T}^\natural) = \{p^*(\alpha) \mid \alpha \in \Phi(\mathbf{G}, \mathbf{T}); N(\alpha)(\nu) = \varsigma_\alpha\} \subset \Phi_{\mathrm{res}}(\mathbf{G}, \mathbf{T}),$$
$$\Phi^\vee(\mathbf{G}_{\nu\theta}, \mathbf{T}^\natural) = \{\varrho_\alpha \cdot N(\alpha^\vee) \mid \alpha^\vee \in \Phi^\vee(\mathbf{G}, \mathbf{T}); N(\alpha)(\nu) = \varsigma_\alpha\} \subset \Phi^\vee_{\mathrm{res}}(\mathbf{G}, \mathbf{T}).$$

Note that these sets are thought of as subsets of $X^*(\mathbf{T}^\natural)$ and $X_*(\mathbf{T}^\natural)$.

3.4. **Good product expansion in twisted spaces.** We discuss a twisted version of the theory of good product expansion due to Adler and Spice ([AS08]).

We first recall the definition of a good product expansion of elements of p-adic groups in the untwisted case. We temporarily let $\mathbf{G}$ be any tamely ramified connected reductive group over F. Let $\bar{\mathbf{G}}$ be the quotient $\mathbf{G}/\mathbf{Z}_{\mathbf{G}}^\circ$ of $\mathbf{G}$ by the identity component of the center $\mathbf{Z}_{\mathbf{G}}$ of $\mathbf{G}$.

Definition 3.8 ([AS08, Definitions 4.11 and 6.1])**.**
(1) We say that an element $\gamma \in G$ is *good of depth zero* if γ is semisimple and its image $\bar{\gamma}$ in $\bar{G}$ is *absolutely semisimple*, i.e., every character value of $\bar{\gamma}$ (see [AS08, Definition A.4]) is of finite prime-to-p order.
(2) For $d \in \mathbb{R}_{>0}$, an element $\gamma \in G$ is said to be *good of depth d* if there exists an F-rational tame-modulo-center torus $\mathbf{S}$ in $\mathbf{G}$ such that
   - $\gamma \in S_d \smallsetminus S_{d+}$, and
   - for every $\alpha \in \Phi(\mathbf{G}, \mathbf{S})$, $\alpha(\gamma) = 1$ or $\mathrm{val}_F(\alpha(\gamma) - 1) = d$.

Definition 3.9 ([AS08, Definition 6.4])**.** For $r \in \widetilde{\mathbb{R}}$, a sequence $\underline{\gamma} = (\gamma_i)_{0 \leq i < r}$ of elements of G indexed by real numbers $0 \leq i < r$ is called a *good sequence* if

- γ_i is 1 or a good of depth i for each $0 \leq i < r$, and
- there exists an F-rational tame torus $\mathbf{S}$ of $\mathbf{G}$ such that $\gamma_i \in S$ for every $0 \leq i < r$.

To a good sequence $\underline{\gamma} = (\gamma_i)_{0 \leq i < r}$, we associate subgroups of $\mathbf{G}$ to $\underline{\gamma}$ as follows:

$$C_{\mathbf{G}}^{(r)}(\underline{\gamma}) := \Big(\bigcap_{0 \leq i < r} \mathbf{Z}_{\mathbf{G}}(\gamma_i)\Big)^\circ, \quad Z_{\mathbf{G}}^{(r)}(\underline{\gamma}) := \mathbf{Z}_{C_{\mathbf{G}}^{(r)}(\underline{\gamma})}.$$

Definition 3.10 ([AS08, Definition 6.8])**.** For $\gamma \in G$, a good sequence $\underline{\gamma} = (\gamma_i)_{0 \leq i < r}$ ($r \in \widetilde{\mathbb{R}}_{>0}$) is called an *$r$-approximation to* γ if there exists a point $\mathbf{x} \in \mathcal{B}(C_{\mathbf{G}}^{(r)}(\underline{\gamma}), F)$ satisfying $\gamma \in (\prod_{0 \leq i < r} \gamma_i) G_{\mathbf{x},r}$. When we have $\gamma \in C_{\mathbf{G}}^{(r)}(\underline{\gamma})$, we say that $\underline{\gamma}$ is a *normal r-approximation to* γ.

When we have a normal r-approximation $\underline{\gamma}$ to γ, we put

$$\gamma_{<r} := \prod_{0 \leq i < r} \gamma_i, \quad \gamma_{\geq r} := \gamma \cdot \gamma_{<r}^{-1}$$

and simply say that "$\gamma = \gamma_{<r} \cdot \gamma_{\geq r}$ is a normal r-approximation." Note that $\gamma_{\geq r}$ commutes with $\gamma_{<r}$ when $\gamma = \gamma_{<r} \cdot \gamma_{\geq r}$ is a normal r-approximation.

Now we move on to the setting of twisted spaces. Let $(\mathbf{G}, \theta)$ be as in Section 3.1. In particular, we have a twisted space $\tilde{\mathbf{G}} = \mathbf{G}\theta$. From now on, we assume that

$$p \textbf{ is odd.}$$

We put $\mathbf{G}^\dagger := \mathbf{G} \rtimes \langle\theta\rangle$. Note that this is a disconnected reductive group whose identity component is $\mathbf{G}$ and non-identity component is given by $\tilde{\mathbf{G}}$. Recall that $\mathbf{A}_{\tilde{\mathbf{G}}}$ is the maximal split subtorus of $\mathbf{Z}_{\mathbf{G}}^{\theta}$. To extend the theory of Adler–Spice to $\mathbf{G}^\dagger$, we utilize Spice's topological Jordan decomposition.

Definition 3.11 ([Spi08, Definition 1.6])**.** For $\gamma \in G^\dagger$, we say that a pair (γ_0, γ_+) of elements of $G^\dagger$ is a *topological p-Jordan decomposition modulo* $A_{\tilde{\mathbf{G}}}$ of γ if

- $\gamma = \gamma_0\gamma_+ = \gamma_+\gamma_0$,
- γ_0 is absolutely p-semisimple modulo $A_{\tilde{\mathbf{G}}}$, i.e., the image $\bar{\gamma}_0$ of γ_0 in $G^\dagger/A_{\tilde{\mathbf{G}}}$ is of finite prime-to-p order, and
- γ_+ is topologically p-unipotent modulo $A_{\tilde{\mathbf{G}}}$, i.e., the image $\bar{\gamma}_+$ of γ_+ in $G^\dagger/A_{\tilde{\mathbf{G}}}$ satisfies $\lim_{n\to\infty} \bar{\gamma}_+^{p^n} = 1$.

In this paper, we refer to a topological p-Jordan decomposition modulo $A_{\tilde{\mathbf{G}}}$ simply as a *topological Jordan decomposition*. Similarly, when an element γ is absolutely p-semisimple modulo $A_{\tilde{\mathbf{G}}}$ (resp. topologically p-unipotent modulo $A_{\tilde{\mathbf{G}}}$), we often simply say that γ is topologically semisimple (resp. topologically unipotent) as long as there is no risk of confusion.

Remark 3.12. Note that, for a given element $\gamma \in G^\dagger$, its topological Jordan decomposition is unique modulo $A_{\tilde{\mathbf{G}}}$ if it exists. More precisely, if both (γ_0, γ_+) and (γ_0', γ_+') are topological Jordan decompositions of γ, then we have $\bar{\gamma}_0 = \bar{\gamma}_0'$ and $\bar{\gamma}_+ = \bar{\gamma}_+'$ in $G^\dagger/A_{\tilde{\mathbf{G}}}$.

Proposition 3.13. *Let* $\mathbf{S}$ *be a torus equipped with an involutive automorphism* $\theta_{\mathbf{S}}$*. Then the order of* $\pi_0(\mathbf{S}^{\theta_{\mathbf{S}}}) = \mathbf{S}^{\theta_{\mathbf{S}}}/\mathbf{S}^{\theta_{\mathbf{S}},\circ}$ *is a power of* 2*.*

Proof. Recall that the abelian category of groups of multiplicative types (i.e., algebraic groups isomorphic to a product of $\mathbb{G}_{\mathrm{m}}$'s or μ_n's for $n \in \mathbb{Z}_{>1}$) is equivalent to the opposite of the abelian category of finitely generated $\mathbb{Z}$-modules by taking the character groups (e.g., see [Poo17, Theorem 5.5.10]). Thus, the short exact sequence

$$1 \to \mathbf{S}^{\theta_{\mathbf{S}},\circ} \to \mathbf{S}^{\theta_{\mathbf{S}}} \to \pi_0(\mathbf{S}^{\theta_{\mathbf{S}}}) \to 1$$

induces a short exact sequence

$$1 \to X^*(\pi_0(\mathbf{S}^{\theta_{\mathbf{S}}})) \to X^*(\mathbf{S}^{\theta_{\mathbf{S}}}) \to X^*(\mathbf{S}^{\theta_{\mathbf{S}},\circ}) \to 1.$$

Since $\pi_0(\mathbf{S}^{\theta_{\mathbf{S}}})$ is finite and abelian, we have $|\pi_0(\mathbf{S}^{\theta_{\mathbf{S}}})| = |X^*(\pi_0(\mathbf{S}^{\theta_{\mathbf{S}}}))|$. Thus it suffices to show that $|X^*(\pi_0(\mathbf{S}^{\theta_{\mathbf{S}}}))|$ is a power of 2.

Note that $X^*(\pi_0(\mathbf{S}^{\theta_{\mathbf{S}}}))$ is the torsion part of $X^*(\mathbf{S}^{\theta_{\mathbf{S}}})$. From the left exact sequence

$$1 \to \mathbf{S}^{\theta_{\mathbf{S}}} \to \mathbf{S} \xrightarrow{1-\theta_{\mathbf{S}}} \mathbf{S},$$

where the map $1-\theta_{\mathbf{S}}$ is given by $s \mapsto s \cdot \theta_{\mathbf{S}}(s)^{-1}$, we get a right exact sequence

$$X^*(\mathbf{S}) \xrightarrow{1-\theta_{\mathbf{S}}^*} X^*(\mathbf{S}) \to X^*(\mathbf{S}^{\theta_{\mathbf{S}}}) \to 0.$$

Hence, by fixing an identification $X^*(\mathbf{S}) \cong \mathbb{Z}^{\oplus n}$, it is enough to show that the cokernel of the homomorphism $1-\theta_{\mathbf{S}}^*\colon \mathbb{Z}^{\oplus n} \to \mathbb{Z}^{\oplus n}$ has no ℓ-torsion for any prime number $\ell \neq 2$. Equivalently, it suffices to show that $\mathrm{Cok}(1-\theta_{\mathbf{S}}^*) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$ has no torsion for any $\ell \neq 2$. Since tensoring $\mathbb{Z}_\ell$ over $\mathbb{Z}$ preserves the right-exactness, $\mathrm{Cok}(1-[\eta]^*) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$ is isomorphic to the cokernel of $1-\theta_{\mathbf{S}}^*\colon \mathbb{Z}_\ell^{\oplus n} \to \mathbb{Z}_\ell^{\oplus n}$.

We put $V := \mathbb{Z}_\ell^{\oplus n}$, on which $\theta_{\mathbf{S}}^*$ acts. Since $\theta_{\mathbf{S}}^*$ is involutive, we have a decomposition $V = V^+ \oplus V^-$ such that $\theta_{\mathbf{S}}$ acts on V^+ (resp. V^-) via identity (resp. negation). Indeed, the projection from V to $V^\pm$ is given by $v \mapsto \frac{1}{2}(v \pm \theta_{\mathbf{S}}^*(v))$. (Note that here we use $\ell \neq 2$.) From this, we immediately see that the cokernel of $1-\theta_{\mathbf{S}}^*$ is free again by noting that $\ell \neq 2$. □

We say that an elliptic semisimple element δ of $\tilde{G}$ is *tame* if there exists an F-rational elliptic twisted maximal torus $(\tilde{\mathbf{S}}, \mathbf{S})$ such that $\mathbf{S}$ is tame and $\delta \in \tilde{S}$. We note that if $\mathbf{G}$ is tamely ramified and p does not divide the order of the absolute Weyl group of $\mathbf{G}$, any F-rational maximal torus of $\mathbf{G}$ is tame by [Fin21, Theorem 3.3]. In particular, any elliptic semisimple element δ of $\tilde{G}$ is tame.

Proposition 3.14. *Let δ be a tame elliptic semisimple element of $\tilde{G}$. Then there exists a pair (δ_0, δ_+) of elements of $G^\dagger$ such that*

(1) $\delta = \delta_0\delta_+ = \delta_+\delta_0$,
(2) $\delta_0 \in \tilde{G} \subset G^\dagger$ *is absolutely p-semisimple modulo* $A_{\tilde{\mathbf{G}}}$,
(3) $\delta_+ \in G_{\delta_0,0+} \subset G^\dagger$ *is topologically p-unipotent, and*
(4) δ_0 *and* δ_+ *belong to the closure* $\overline{\langle\delta\rangle A_{\tilde{\mathbf{G}}}}$ *of* $\langle\delta\rangle A_{\tilde{\mathbf{G}}}$ *in* $G^\dagger$.

In particular, (δ_0, δ_+) is a topological Jordan decomposition of δ.

Proof. Since δ is elliptic semisimple, there exists an F-rational elliptic twisted maximal torus $\tilde{\mathbf{S}}$ of $\tilde{\mathbf{G}}$ such that $\delta \in \tilde{S}$ by definition (Definition 3.4). Since the associated automorphism $\theta_{\mathbf{S}}$ of $\mathbf{S}$ can be thought of as the conjugation by δ on $\mathbf{S}$, the element δ^2 of S is fixed by $\theta_{\mathbf{S}}$. In other words, δ^2 belongs to $S^{\theta_{\mathbf{S}}}$. Thus, by putting $k := \pi_0(\mathbf{S}^{\theta_{\mathbf{S}}})$, we see that $\delta^{2k} \in S^\natural$ (recall that $\mathbf{S}^\natural := \mathbf{S}^{\theta_{\mathbf{S}},\circ}$).

Since $\tilde{\mathbf{S}}$ is elliptic, $\mathbf{S}^\natural$ is anisotropic modulo $\mathbf{A}_{\tilde{\mathbf{G}}}$, hence $S^\natural$ is compact modulo $A_{\tilde{\mathbf{G}}}$. In particular, δ^{2k} has a topological Jordan decomposition in $S^\natural/A_{\tilde{\mathbf{G}}}$ according to [Spi08, Proposition 1.8]. More precisely, if we write $\bar{\delta}'$ for the image of δ^{2k} in $S^\natural/A_{\tilde{\mathbf{G}}}$, then we have two elements $\bar{\delta}'_0$ and $\bar{\delta}'_+$ of $S^\natural/A_{\tilde{\mathbf{G}}}$ such that

- $\bar{\delta}' = \bar{\delta}'_0\bar{\delta}'_+ = \bar{\delta}'_+\bar{\delta}'_0$,
- $\bar{\delta}'_0$ is of finite prime-to-p order,
- $\bar{\delta}'_+$ is topologically p-unipotent.

We put $\bar{\mathbf{S}}^\natural := \mathbf{S}^\natural/\mathbf{A}_{\tilde{\mathbf{G}}}$. We note that the image of $\bar{\delta}'_+$ under the natural injection

$$S^\natural/A_{\tilde{\mathbf{G}}} \hookrightarrow \bar{S}^\natural = (\mathbf{S}^\natural/\mathbf{A}_{\tilde{\mathbf{G}}})(F)$$

belongs to the pro-unipotent radical $\bar{S}^\natural_{0+}$ of the unique parahoric subgroup $\bar{S}^\natural_0$ of $\bar{S}^\natural$. Indeed, by [Spi08, Lemma 2.21], the topological p-unipotency of $\bar{\delta}'_+$ implies the topological F-unipotency of $\bar{\delta}'_+$ in the sense of [Spi08, Definition 2.15], that is, $\bar{\delta}'_+$ belongs to $\bar{\mathbf{S}}^\natural(E)_{0+}$ for the splitting field E of $\bar{\mathbf{S}}^\natural$. Since the torus $\bar{\mathbf{S}}^\natural$ is tame, we have $\bar{\mathbf{S}}^\natural(E)_{0+} \cap \bar{S}^\natural = \bar{S}^\natural_{0+}$ (see [Yu01, Proposition 2.2]).

By applying [Kal19b, Lemma 3.1.4 (2)] to the short exact sequence

$$1 \to \mathbf{A}_{\tilde{\mathbf{G}}} \to \mathbf{S}^\natural \to \bar{\mathbf{S}}^\natural \to 1,$$

we see that the map $S^{\natural}_{0+} \to \bar{S}^{\natural}_{0+}$ is surjective. Thus we can take an element $\delta'_{+} \in S^{\natural}_{0+}$ mapping to $\bar{\delta}'_{+} \in \bar{S}^{\natural}_{0+}$.

Now note that $2k$ is a power of 2 by Proposition 3.13, in particular, $2k$ is prime to p. Thus we can take $\delta_{+} \in S^{\natural}_{0+}$ satisfying $\delta_{+}^{2k} = \delta'_{+}$ as follows. Let $a \in \mathbb{Z}_{>0}$ be a positive integer such that $p^a \equiv 1 \pmod{2k}$. Then the topological p-unipotency of δ'_{+} implies that the sequence

$$\left\{ {\delta'_{+}}^{\frac{p^{an}-1}{2k}} \right\}_{n=1,2,\ldots}$$

is Cauchy. If we let $\delta_{+}^{-1} \in S^{\natural}_{0+}$ be the limit of this sequence, then we have $\delta_{+}^{2k} = \delta'_{+}$.

We put $\delta_0 := \delta \cdot \delta_{+}^{-1}$. Then obviously δ_0 belongs to $\tilde{S} \subset \tilde{G}$. Since δ_{+} belongs to $S^{\natural}$, δ_{+} commutes with δ_0. Moreover, by the construction, δ_0^{2k} belongs to $S^{\natural}$ and its image in $S^{\natural}/A_{\tilde{\mathbf{G}}}$ is of finite prime-to-p order. Hence, again by noting that $2k$ is prime to p, we conclude that δ_0 is absolutely p-semisimple modulo $A_{\tilde{\mathbf{G}}}$. Finally, in order to check that $\delta_0, \delta_{+} \in \overline{\langle\delta\rangle A_{\tilde{\mathbf{G}}}}$, it suffices to show only $\delta_{+} \in \overline{\langle\delta\rangle A_{\tilde{\mathbf{G}}}}$. Since the similar statement holds for $\bar{\delta}'_{+}$, namely, $\bar{\delta}'_{+} \in \overline{\langle\bar{\delta}'\rangle} \subset S^{\natural}/A_{\tilde{\mathbf{G}}}$, we have $\delta'_{+} \in \overline{\langle\delta^{2k}\rangle A_{\tilde{\mathbf{G}}}}$. By the construction of δ_{+}, this implies that $\delta_{+} \in \overline{\langle\delta\rangle A_{\tilde{\mathbf{G}}}}$. □

In the rest of this paper, for a tame elliptic semisimple element δ of $\tilde{G}$, we call a decomposition as in Proposition 3.14 a *topological Jordan decomposition of* δ.

Definition 3.15. Let $\delta \in \tilde{G}$ be a tame elliptic semisimple element. A *normal r-approximation to δ* ($r \in \widetilde{\mathbb{R}}_{>0}$) is a pair $(\delta = \delta_0\delta_{+}, \underline{\delta}_{+})$ of

- a topological decomposition $\delta = \delta_0\delta_{+}$ of δ and
- a normal r-approximation $\underline{\delta}_{+} = (\delta_i)_{0<i<r}$ to δ_{+} in G_{δ_0}.

For a normal r-approximation $(\delta = \delta_0\delta_{+}, \underline{\delta}_{+})$ to a tame elliptic semisimple element $\delta \in \tilde{G}$, we put

$$\delta^{+}_{<r} := \prod_{0<i<r} \delta_i, \quad \delta_{<r} := \prod_{0\leq i<r} \delta_i, \quad \delta_{\geq r} := \delta_{<r}^{-1}\delta,$$

$$C^{(r)}_{\mathbf{G}}(\delta) := C^{(r)}_{\mathbf{G}_{\delta_0}}(\delta_{+}).$$

When $(\delta = \delta_0\delta_{+}, \underline{\delta}_{+})$ is a normal r-approximation to δ, we often simply say that "$\delta = \delta_0\delta^{+}_{<r}\delta_{\geq r}$ is a normal r-approximation to δ".

Lemma 3.16. *Let δ and (δ_0, δ_{+}) be as in Proposition 3.14. If δ is elliptic regular semisimple in $\tilde{\mathbf{G}}$, then so is δ_{+} in $\mathbf{G}_{\delta_0}$.*

Proof. Let $\tilde{\mathbf{S}}$ be an F-rational elliptic twisted maximal torus of $\tilde{\mathbf{G}}$ containing δ. Then, by Proposition 3.3 (3), $\mathbf{S}^{\natural}$ is a maximal torus of $\mathbf{G}_{\delta}$. Since $\mathbf{G}_{\delta}$ is a torus by the regular semisimplicity of δ, this implies that $\mathbf{G}_{\delta} = \mathbf{S}^{\natural}$. As we have $(\mathbf{G}_{\delta_0})_{\delta_{+}} \subset \mathbf{G}_{\delta} = \mathbf{S}^{\natural}$, we see that $(\mathbf{G}_{\delta_0})_{\delta_{+}} = \mathbf{S}^{\natural}$ and δ_{+} is regular semisimple in $\mathbf{G}_{\delta_0}$.

Let us check that $\mathbf{S}^{\natural}$ is elliptic in $\mathbf{G}_{\delta_0}$. Again by Proposition 3.3 (3), $\mathbf{S}^{\natural}$ is a maximal torus of $\mathbf{G}_{\delta_0}$. Since $\tilde{\mathbf{S}}$ is elliptic, $\mathbf{S}^{\natural}$ is anisotropic modulo $\mathbf{A}_{\tilde{\mathbf{G}}}$. As $\mathbf{A}_{\tilde{\mathbf{G}}}$ is contained in the center of $\mathbf{G}_{\delta_0}$, the maximal torus $\mathbf{S}^{\natural}$ of $\mathbf{G}_{\delta_0}$ is elliptic. □

Proposition 3.17. *Suppose that $\mathbf{G}$ is tamely ramified and p does not divide the order of the absolute Weyl group $\Omega_{\mathbf{G}}$ of $\mathbf{G}$. Then any elliptic semisimple element $\delta \in \tilde{G}$ has a normal r-approximation.*

Proof. Since the existence of a topological Jordan decomposition of δ is guaranteed by Proposition 3.14, we only have to show that δ_+ has a normal r-approximation in G_{δ_0}. By [AS08, Lemma 8.1], any bounded-modulo-$Z_{\mathbf{G}_{\delta_0}}$ element of G_{δ_0} belonging to an F-rational tame maximal torus of $\mathbf{G}_{\delta_0}$ has a normal r-approximation as long as the assumption "($\mathbf{Gd}^{G_{\delta_0}}$)" is satisfied (see [AS08, Definition 6.3]). Since δ_+ is bounded-modulo-$Z_{\mathbf{G}_{\delta_0}}$, it is enough to show that δ_+ is contained in an F-rational tame maximal torus of $\mathbf{G}_{\delta_0}$ and that ($\mathbf{Gd}^{G_{\delta_0}}$) is satisfied.

As remarked above, the assumption on p implies that δ is tame, hence we can find an F-rational elliptic twisted maximal torus $(\tilde{\mathbf{S}}, \mathbf{S})$ such that $\delta \in \tilde{S}$. This implies that $\mathbf{S}^\natural$ is an F-rational tame maximal torus of $\mathbf{G}_{\delta_0}$. By construction, δ_+ belongs to $S^\natural$. As p does not divide the order of the absolute Weyl group $\Omega_{\mathbf{G}_{\delta_0}}$ of $\mathbf{G}_{\delta_0}$ (note that this is a subgroup of $\Omega_{\mathbf{G}}$; see [KS99, Section 1.1]), the assumption ($\mathbf{Gd}^{G_{\delta_0}}$) is satisfied by [Fin21, Theorem 3.6]. □

Lemma 3.18. *Let $\delta \in \tilde{G}$ be a tame elliptic semisimple element having a normal r-approximation $\delta = \delta_0\delta^+_{<r}\delta_{\geq r}$. Then δ_0 belongs to $\overline{\langle\delta_{<r}\rangle A_{\tilde{\mathbf{G}}}}$.*

Proof. We let $\bar{\delta}_0$ denote the image of δ_0 in $G^\dagger/A_{\tilde{\mathbf{G}}}$. Let p' be the order of $\bar{\delta}_0$, which is prime to p. If we take $k \in \mathbb{Z}_{>0}$ such that $p^k \equiv 1 \pmod{p'}$, then we have $\bar{\delta}_0^{p^k} = \bar{\delta}_0$. Hence, for any $n \in \mathbb{Z}_{>0}$, we have $\bar{\delta}_0^{p^{nk}} = \bar{\delta}_0^{p^{(n-1)k}} = \cdots = \bar{\delta}_0$. Since $\delta^+_{<r}$ is topologically p-unipotent and commutes with δ_0, we have

$$(\bar{\delta}_{<r})^{p^{nk}} = (\bar{\delta}_0)^{p^{nk}} \cdot (\bar{\delta}^+_{<r})^{p^{nk}} = \bar{\delta}_0 \cdot (\bar{\delta}^+_{<r})^{p^{nk}} \xrightarrow[n\to\infty]{} \bar{\delta}_0.$$

Thus $\bar{\delta}_0$ belongs to $\overline{\langle\bar{\delta}_{<r}\rangle} \subset G^\dagger/A_{\tilde{\mathbf{G}}}$. It can be easily checked that this implies that δ_0 belongs to $\overline{\langle\delta_{<r}\rangle A_{\tilde{\mathbf{G}}}} \subset \tilde{G}$. □

Lemma 3.19. *Let $\delta \in \tilde{G}$ be a tame elliptic semisimple element having a normal r-approximation $\delta = \delta_0\delta^+_{<r}\delta_{\geq r}$. Then we have*

$$(\mathbf{G}_{\delta_0})_{\delta^+_{<r}} = \mathbf{G}_{\delta_{<r}}$$

Proof. The statement can be proved by a similar argument to the untwisted case (cf. [AS08, Corollary 6.14]) as we explain in the following.

We may take an F-rational tame twisted maximal torus $\tilde{\mathbf{S}}$ containing δ, δ_0, and $\delta_{<r}$. Indeed, as $\delta_+ = \delta^+_{<r}\delta_{\geq r}$ is a normal r-approximation in $\mathbf{G}_{\delta_0}$, δ_+ belongs to $(\mathbf{G}_{\delta_0})_{\delta^+_{<r}}$, hence we can find an F-rational tame maximal torus $\mathbf{S}'$ of $(\mathbf{G}_{\delta_0})_{\delta^+_{<r}}$ containing δ_+ (note that δ_+ is semisimple). Then $\mathbf{S}'$ contains $\delta^+_{<r}$ and δ_+. By Steinberg's result (see [KS99, Theorem 1.1.A]), $\mathbf{S} := \mathbf{Z}_{\mathbf{G}}(\mathbf{S}')$ gives an F-rational tame maximal torus of $\mathbf{G}$. Moreover, $\mathbf{S}$ is $[\delta_0]$-stable and there exists an $[\delta_0]$-stable Borel subgroup containing $\mathbf{S}$. Hence, by Lemma 3.6, $\tilde{\mathbf{S}} := \mathbf{S}\delta_0$ is an F-rational tame twisted maximal torus of $\tilde{\mathbf{G}}$. (Note that then $\mathbf{S}' = \mathbf{S}^\natural$.) By construction, we have $\delta, \delta_0, \delta_{<r} \in \tilde{S}$.

Since $\delta_{<r} = \delta_0\delta^+_{<r}$, the inclusion $(\mathbf{G}_{\delta_0})_{\delta^+_{<r}} \subset \mathbf{G}_{\delta_{<r}}$ is obvious. Let us show that this inclusion is in fact the equality. As $\mathbf{S}^\natural$ is a maximal torus both in $(\mathbf{G}_{\delta_0})_{\delta^+_{<r}}$ and $\mathbf{G}_{\delta_{<r}}$, it suffices to check that $\Phi((\mathbf{G}_{\delta_0})_{\delta^+_{<r}}, \mathbf{S}^\natural)$ equals $\Phi(\mathbf{G}_{\delta_{<r}}, \mathbf{S}^\natural)$. By taking $g_{\mathbf{S}} \in \mathbf{G}$ as in Section 3.3, we put

$$\nu_{<r}\theta := {}^{g_{\mathbf{S}}}\delta_{<r} \in \mathbf{T}\theta, \quad \nu_0\theta := {}^{g_{\mathbf{S}}}\delta_0 \in \mathbf{T}\theta, \quad \nu^+_{<r} := {}^{g_{\mathbf{S}}}\delta^+_{<r} \in \mathbf{T}^\natural.$$

Then $\Phi((\mathbf{G}_{\delta_0})_{\delta_{<r}^+}, \mathbf{S}^\natural)$ and $\Phi(\mathbf{G}_{\delta_{<r}}, \mathbf{S}^\natural)$ are identified with $\Phi((\mathbf{G}_{\nu_0\theta})_{\nu_{<r}^+}, \mathbf{T}^\natural)$ and $\Phi(\mathbf{G}_{\nu_{<r}\theta}, \mathbf{T}^\natural)$, respectively. By the description explained in Section 3.3, we have

$$\Phi(\mathbf{G}_{\nu_0\theta}, \mathbf{T}^\natural) = \{\alpha_{\mathrm{res}} \mid \alpha \in \Phi(\mathbf{G}, \mathbf{T}); N(\alpha)(\nu_0) = \varsigma_\alpha\},$$

hence

$$\Phi((\mathbf{G}_{\nu_0\theta})_{\nu_{<r}^+}, \mathbf{T}^\natural) = \{\alpha_{\mathrm{res}} \mid \alpha \in \Phi(\mathbf{G}, \mathbf{T}); N(\alpha)(\nu_0) = \varsigma_\alpha \text{ and } \alpha_{\mathrm{res}}(\nu_{<r}^+) = 1\}.$$

On the other hand, we have

$$\Phi(\mathbf{G}_{\nu_{<r}\theta}, \mathbf{T}^\natural) = \{\alpha_{\mathrm{res}} \mid \alpha \in \Phi(\mathbf{G}, \mathbf{T}); N(\alpha)(\nu_{<r}) = \varsigma_\alpha\}.$$

Thus our task is to check that, for any $\alpha \in \Phi(\mathbf{G}, \mathbf{T})$ satisfying $N(\alpha)(\nu_{<r}) = \varsigma_\alpha$, we have $N(\alpha)(\nu_0) = \varsigma_\alpha$ and $\alpha_{\mathrm{res}}(\nu_{<r}^+) = 1$. Let $\alpha \in \Phi(\mathbf{G}, \mathbf{T})$ be such a root. By the definition of l_α, we have $\sum_{i=0}^{1} \theta^i(\alpha) = \frac{2}{l_\alpha} \sum_{i=0}^{l_\alpha - 1} \theta^i(\alpha)$. Thus we get

$$\begin{aligned} N(\alpha)(\nu_{<r})^{\frac{2}{l_\alpha}} &= \Bigl(\sum_{i=0}^{l_\alpha-1} \theta^i(\alpha)\Bigr)(\nu_{<r})^{\frac{2}{l_\alpha}} \\ &= \Bigl(\sum_{i=0}^{1} \theta^i(\alpha)\Bigr)(\nu_{<r}) = \alpha\Bigl(\prod_{i=0}^{1} \theta^i(\nu_{<r})\Bigr) = \alpha\bigl((\nu_{<r}\theta)^2\bigr). \end{aligned}$$

Hence, noting that $(\nu_{<r}\theta)^2 = (\nu_0\theta)^2 \cdot \nu_{<r}^{+;2}$, we have

$$\varsigma_\alpha^{\frac{2}{l_\alpha}} = \alpha\bigl((\nu_{<r}\theta)^2\bigr) = \alpha\bigl((\nu_0\theta)^2\bigr) \cdot \alpha(\nu_{<r}^+)^2.$$

Since δ_0 is of finite prime-to-p order modulo $A_{\tilde{\mathbf{G}}}$ and $\delta_{<r}^+$ is topologically p-unipotent, we see that $\alpha((\nu_0\theta)^2) \in \overline{F}^\times$ is of finite prime-to-p order (note that $A_{\tilde{\mathbf{G}}}$ is killed by any root) and $\alpha(\nu_{<r}^+)^2 \in \overline{F}^\times$ is topologically p-unipotent. Thus, by noting that $\varsigma_\alpha \in \{\pm 1\}$ and $p \neq 2$, we must have $\alpha(\nu_{<r}^+)^2 = 1$, which furthermore implies that $\alpha(\nu_{<r}^+) = 1$. Then we get

$$\varsigma_\alpha = N(\alpha)(\nu_{<r}) = N(\alpha)(\nu_0) \cdot N(\alpha)(\nu_{<r}^+) = N(\alpha)(\nu_0).$$

□

4. Regular supercuspidal representations

4.1. **Regular supercuspidal representations.** In the following, we assume that

- $\mathbf{G}$ is tamely ramified over F, and
- $p \neq 2$ and p does not divide the order of the absolute Weyl group $\Omega_{\mathbf{G}}$ of $\mathbf{G}$.

In [Yu01], Yu introduced the notion of a *cuspidal* $\mathbf{G}$-*datum* and attached an irreducible supercuspidal representation of G to each cuspidal $\mathbf{G}$-datum. Recall that a cuspidal $\mathbf{G}$-datum is a quintuple $\Sigma = (\vec{\mathbf{G}}, \vec{\vartheta}, \vec{r}, \mathbf{x}, \rho_0)$ consisting of the following objects (here we follows the convention of [HM08, Section 3.1]):

- $\vec{\mathbf{G}}$ is a sequence $\mathbf{G}^0 \subsetneq \mathbf{G}^1 \subsetneq \cdots \subsetneq \mathbf{G}^d = \mathbf{G}$ of tame Levi subgroups such that $\mathbf{Z}_{\mathbf{G}^0}/\mathbf{Z}_{\mathbf{G}}$ is anisotropic,
- $\mathbf{x}$ is a vertex of the reduced Bruhat–Tits building $\mathcal{B}^{\mathrm{red}}(\mathbf{G}^0, F)$ of $\mathbf{G}^0$,
- $\vec{r}$ is a sequence $0 \leq r_0 < \cdots < r_{d-1} \leq r_d$ such that $0 < r_0$ when $d > 0$,
- $\vec{\vartheta}$ is a sequence $(\vartheta_0, \ldots, \vartheta_d)$ of characters ϑ_i of G^i satisfying
  - for $0 \leq i < d$, ϑ_i is $\mathbf{G}^{i+1}$-generic of depth r_i at $\mathbf{x}$, and

– for $i = d$,

$$\begin{cases} \operatorname{depth}_{\mathbf{x}}(\vartheta_d) = r_d & \text{if } r_{d-1} < r_d, \\ \vartheta_d = \mathbb{1} & \text{if } r_{d-1} = r_d, \end{cases}$$

- ρ_0 is an irreducible representation of $G^0_{\mathbf{x}}$ whose restriction to $G^0_{\mathbf{x},0}$ contains the inflation of a cuspidal representation of the quotient $G^0_{\mathbf{x},0:0+}$.

We call the representations obtained from cuspidal $\mathbf{G}$-data by Yu's construction *tame supercuspidal representations*.

The "fibers" of Yu's construction was investigated by Hakim–Murnaghan; in [HM08], they introduced an equivalence relation called $\mathbf{G}$-*equivalence* and proved that two cuspidal $\mathbf{G}$-data give rise to the same (isomorphic) supercuspidal representations if and only if two data are $\mathbf{G}$-equivalent. Thus Yu's construction gives the following bijective map:

$$\{\text{cusp. }\mathbf{G}\text{-data}\}/\mathbf{G}\text{-eq.} \xrightarrow[\text{Yu's construction}]{1:1} \{\text{tame s.c. rep'ns of }G\}/\sim$$

In [Kal19b], Kaletha introduced the notion of *(extra) regularity* for cuspidal $\mathbf{G}$-data (see [Kal19b, Section 3]). Tame supercuspidal representations arising from (extra) regular cuspidal $\mathbf{G}$-data are called *(extra) regular supercuspidal representations*. Kaletha discovered that (extra) regular cuspidal $\mathbf{G}$-data can be parametrized by much simpler data called *tame elliptic (extra) regular pairs*. Let us recall the definition of a tame elliptic (extra) regular pair:

Definition 4.1 ([Kal19b, Definition 3.7.5])**.** A *tame elliptic regular* (resp. *extra regular*) *pair* is a pair $(\mathbf{S}, \vartheta)$ consisting of

- a tame elliptic F-rational maximal torus $\mathbf{S}$ of $\mathbf{G}$ and
- a character $\vartheta\colon S \to \mathbb{C}^\times$

satisfying the following conditions:

(1) By choosing a finite tamely ramified extension E of F splitting $\mathbf{S}$, we put

$$\Phi_{0+} := \{\alpha \in \Phi(\mathbf{G}, \mathbf{S}) \mid (\vartheta \circ \mathrm{Nr}_{E/F} \circ \alpha^\vee)|_{E^\times_{0+}} \equiv \mathbb{1}\}.$$

Then the action of I_F on Φ_{0+} preserves a set of positive roots.

(2) We put $\mathbf{G}^0$ to be the tame Levi subgroup of $\mathbf{G}$ with maximal torus $\mathbf{S}$ and root system Φ_{0+}. Then $\vartheta|_{S_0}$ has trivial stabilizer for the action of $N_{\mathbf{G}^0}(\mathbf{S})/S$ (resp. $\Omega_{\mathbf{G}^0}(\mathbf{S})(F)$).

Kaletha's re-parametrizing result [Kal19b, Proposition 3.7.8] asserts that $\mathbf{G}$-equivalence classes of (extra) regular cuspidal $\mathbf{G}$-data bijectively correspond to G-conjugacy classes of tame elliptic (extra) regular pairs. We write $\pi_{(\mathbf{S},\vartheta)}$ for the (extra) regular supercuspidal representation which corresponds to a tame elliptic (extra) regular pair $(\mathbf{S}, \vartheta)$.

$$\begin{array}{ccc}
\{\text{cusp. }\mathbf{G}\text{-data}\}/\mathbf{G}\text{-eq.} & \xrightarrow{1:1} & \{\text{tame s.c. rep'ns of }G\}/\sim \\
\cup & & \cup \\
\{\text{(ex.) reg. cusp. }\mathbf{G}\text{-data}\}/\mathbf{G}\text{-eq.} & \xrightarrow{1:1} & \{\text{(ex.) reg. s.c. rep'ns of }G\}/\sim \\
{\scriptstyle 1:1}\updownarrow & \nearrow_{(\mathbf{S},\vartheta)\mapsto\pi_{(\mathbf{S},\vartheta)}} & \\
\{\text{tame ell. (ex.) reg. pairs}\}/G\text{-conj.} & &
\end{array}$$

4.2. **Toral supercuspidal representations.** We next give more detailed explanation of Yu's construction in the case of *toral supercuspidal representations*, which will be mainly treated in this paper.

Definition 4.2. We say that a cuspidal $\mathbf{G}$-datum $\Sigma = (\vec{\mathbf{G}}, \vec{\vartheta}, \vec{r}, \mathbf{x}, \rho_0)$ is *toral* if it satisfies the following:

- $d = 1$ and $\mathbf{G}^0$ is an F-rational tame elliptic maximal torus $\mathbf{S}$ of $\mathbf{G}$ (thus $\vec{\mathbf{G}} = (\mathbf{S} \subset \mathbf{G})$),
- $0 < r_0 = r_1 \,(=: r)$,
- $\vec{\vartheta} = (\vartheta_0, \vartheta_1)$, where
  - ϑ_0 is a $\mathbf{G}$-generic character of S of depth r (put $\vartheta := \vartheta_0$), and
  - $\vartheta_1 = \mathbb{1}$,
- ρ_0 is the trivial representation $\mathbb{1}$.

We call a tame supercuspidal representation associated to a toral cuspidal $\mathbf{G}$-datum *toral supercuspidal representation.*

Remark 4.3. We caution that, in some literature, the terminology "toral" only means that $\mathbf{G}^0$ is a torus. For example, in [FS21], they distinguish these two versions of torality by calling the one of Definition 4.2 the "0-torality". We decided to use "toral" rather than "0-toral" in this paper following [DS18] and [Kal19b].

Under the bijection of [Kal19b, Proposition 3.7.8] mentioned above, a tame elliptic regular pair corresponding to a toral cuspidal $\mathbf{G}$-datum $((\mathbf{S} \subset \mathbf{G}), (r = r), (\vartheta, \mathbb{1}), \mathbf{x}, \mathbb{1})$ is simply given by $(\mathbf{S}, \vartheta)$. Let us call a tame elliptic regular pair obtained in this way a *tame elliptic toral pair*. We note that the torality implies the extra regularity.

In the following, we fix a tame elliptic toral pair $(\mathbf{S}, \vartheta)$. Let $\mathbf{x} \in \mathcal{B}^{\mathrm{red}}(\mathbf{G}, F)$ be the point associated to $\mathbf{S}$ and $r \in \mathbb{R}_{>0}$ be the depth of ϑ. We put $s := r/2$ and define the subgroups K, J, and J_+ of G by

$$K := SG_{\mathbf{x},s}, \quad J := (S,G)_{\mathbf{x},(r,s)}, \quad J_+ := (S,G)_{\mathbf{x},(r,s+)},$$

where $(S,G)_{\mathbf{x},(r,s)}$ and $(S,G)_{\mathbf{x},(r,s+)}$ are the groups defined according to the manner of Yu (see [Yu01, Sections 1 and 2]). Note that we have $K = SJ$.

Since the depth of ϑ is r, we can extend ϑ to a character $\hat{\vartheta}$ of J_+ satisfying $\hat{\vartheta}|_{J_+} \equiv \mathbb{1}$. Then, by the definition of the $\mathbf{G}$-genericity, there exists an element X^* of $\mathfrak{s}^*_{-r}$ which is $\mathbf{G}$-generic of depth r in the sense of [Yu01, Section 8] and satisfies

$$\hat{\vartheta}(\exp(Y)) = \psi_F(\langle Y, X^* \rangle)$$

for any $Y \in \mathfrak{g}_{\mathbf{x},s+:r+}$ (or, equivalently, for any $Y \in \mathfrak{s}_{s+:r+}$). Here, as explained in [Yu01, Section 8], we may regard $\mathfrak{s}^*$ as a subspace of $\mathfrak{g}^*$ by considering the coadjoint action of $\mathbf{S}$ on $\mathfrak{g}^*$. We recall that the definition of $\mathbf{G}$-genericity consists of two conditions **GE1** and **GE2**. The condition **GE1** requires that $\mathrm{val}_F(\langle H_\alpha, X^* \rangle) = -r$ for any $\alpha \in \Phi(\mathbf{G}, \mathbf{S})$, where $H_\alpha := d\alpha^\vee(1)$. We do not review the condition **GE2** because **GE1** implies **GE2** by [Yu01, Lemma 8.1] when p is not a torsion prime for the dual based root datum of $\mathbf{G}$. (Recall that we have assumed the $p \nmid |\Omega_{\mathbf{G}}|$, which is equivalent to $p \nmid |\Omega_{\hat{\mathbf{G}}}|$. In fact, this implies that p is not a torsion prime for the dual based root datum of $\mathbf{G}$; see [Fin21, Lemma 3.2].)

The point of the construction is that, by putting $N := \mathrm{Ker}\,\hat{\vartheta} \subset J_+$, the quotient J/N has the structure of a finite Heisenberg group:

- The center of J/N is given by J_+/N, which is isomorphic to $\mu_p \cong \mathbb{F}_p$ via ϑ (here we fix an isomorphism $\mu_p \cong \mathbb{F}_p$).
- The quotient J/J_+ has a symplectic space with respect to the pairing

$$(J/J_+) \times (J/J_+) \to \mu_p \cong \mathbb{F}_p \colon (g, g') \mapsto \hat{\vartheta}([g, g'])$$

(see [Yu01, Section 11]; we will review the structure of the symplectic space J/J_+ in more detail in Section 6.1).

Therefore, as an application of the Stone–von Neumann theorem, there exists a unique irreducible representation of J/N whose central character on J_+/N is given by $\hat{\vartheta}$. Furthermore, as the conjugate action of S on J preserves J_+ and N and induces a symplectic action on J/J_+, we can extend the (inflation of) the representation of J to the semi-direct group $S \ltimes J$, for which we write $\omega_{(\mathbf{S},\vartheta)}$ (so-called the Heisenberg–Weil representation). Then the tensor representation $\omega_{(\mathbf{S},\vartheta)} \otimes (\vartheta \ltimes \mathbb{1})$ of $S \ltimes J$ descends to $SJ = K$ (factors through the canonical map $S \ltimes J \twoheadrightarrow K$). We let $\rho_{(\mathbf{S},\vartheta)}$ be the descended representation of K. The toral supercuspidal representation $\pi_{(\mathbf{S},\vartheta)}$ is given by

$$\pi_{(\mathbf{S},\vartheta)} := \text{c-Ind}_K^G \rho_{(\mathbf{S},\vartheta)}.$$

We also recall the definitions of a few more groups and representations which will be needed later (for describing the Adler–DeBacker–Spice character formula in Sections 5 and 6):

$$K_\sigma := SG_{\mathbf{x},0+}, \quad \sigma_{(\mathbf{S},\vartheta)} := \text{Ind}_K^{K_\sigma} \rho_{(\mathbf{S},\vartheta)},$$

$$\tau_{(\mathbf{S},\vartheta)} := \text{Ind}_K^{G_\mathbf{x}} \rho_{(\mathbf{S},\vartheta)} \, (\cong \text{Ind}_{K_\sigma}^{G_\mathbf{x}} \sigma_{(\mathbf{S},\vartheta)}).$$

5. Twisted Adler–DeBacker–Spice formula: preliminary form

In this and the next sections, we discuss a twisted version of the character formula of Adler–DeBacker–Spice for toral supercuspidal representations ([AS09, DS18]). Our arguments heavily depend on the work [AS08, AS09, DS18]. We note that several technical assumptions on p are required so that the theory of Adler–DeBacker–Spice works, but it is enough to assume only the oddness and the non-badness of p (for the root system of $\mathbf{G}$ in the sense of [SS70, I.4.1], see also [AS08, Section A]) whenever $\mathbf{G}$ is tamely ramified by [Kal19b, Section 4.1]. Recall that we have assumed that p is odd and does not divide the order of the absolute Weyl group $\Omega_\mathbf{G}$ of $\mathbf{G}$; in fact, this implies the non-badness of p ([Fin21, Lemma 3.2]).

5.1. Twisted character of a θ-stable representation.

Let us first recall the basics of twisted characters of irreducible admissible representations. See [LH17, Section 2.6] for the details of the content of this subsection.

Let $\eta \in \tilde{\mathbf{G}}$. Then $[\eta]$ is an F-rational automorphism of $\mathbf{G}$. Recall that, for an irreducible admissible representation π of G realized on a $\mathbb{C}$-vector space V, its η-twist π^η is defined by the action

$$\pi^\eta(g) := \pi \circ [\eta](g) = \pi(\eta g \eta^{-1})$$

on the same representation space V. We say that π is *η-stable* if π^η is isomorphic to π as a representation of G.

Remark 5.1. Note that, if we write $\eta = \eta^\circ\theta$ with an element $\eta^\circ \in G$, then we have $[\eta] = [\eta^\circ] \circ \theta$. Hence, as $[\eta^\circ]$ does not change the isomorphism class of any representation, π is η-stable if and only if π is θ-stable. More explicitly, $\pi(\eta^\circ)$ gives an intertwiner between π^θ and π^η, i.e., $\pi^\eta(g) \circ \pi(\eta^\circ) = \pi(\eta^\circ) \circ \pi^\theta(g)$ for any $g \in G$.

Suppose that π is an η-stable irreducible admissible representation of G. We fix an intertwiner

$$I_\pi^\eta \colon \pi \xrightarrow{\sim} \pi^\eta$$

(note that such an I_π^η is unique up to $\mathbb{C}^\times$-multiple, as π is irreducible) and put

$$\tilde{\pi}(g\eta) := \pi(g) \circ I_\pi^\eta$$

for any $g\eta \in \tilde{G}$ with $g \in G$. Then we get a representation $\tilde{\pi}$ of $\tilde{G}$ on the representation space V of π, i.e., $\tilde{\pi} \colon \tilde{G} \to \mathrm{Aut}_{\mathbb{C}}(V)$ is a map satisfying the following relation for any $g_1, g_2 \in G$ and $\delta \in \tilde{G}$:

$$\tilde{\pi}(g_1 \cdot \delta \cdot g_2) = \pi(g_1) \circ \tilde{\pi}(\delta) \circ \pi(g_2).$$

For any $f \in C_c^\infty(\tilde{G})$, an operator $\tilde{\pi}(f)$ on V is defined by

$$\tilde{\pi}(f) := \int_{\delta \in \tilde{G}} f(\delta)\tilde{\pi}(\delta)\, d\delta,$$

where $d\delta$ is a measure on $\tilde{G}$ obtained by transferring a Haar measure on dg on G by the bijection $G \to \tilde{G} \colon g \mapsto g\eta$. Then, as in the untwisted case, the operator $\tilde{\pi}(f)$ is of finite rank and hence we can define its trace. In this setting, the *(η-)twisted character* $\Theta_{\tilde{\pi}}$ of π is defined to be the unique locally constant function on $\tilde{G}_{\mathrm{rs}}$ such that

$$\mathrm{tr}\, \tilde{\pi}(f) = \int_{\tilde{G}_{\mathrm{rs}}} \Theta_{\tilde{\pi}}(\delta) f(\delta)\, d\delta$$

for every $f \in C_c^\infty(\tilde{G})$ satisfying $\mathrm{supp}(f) \subseteq \tilde{G}_{\mathrm{rs}}$, where $\tilde{G}_{\mathrm{rs}}$ denotes the set of regular semisimple elements of $\tilde{G}$.

Remark 5.2.
1. We emphasize that the twisted representation $\tilde{\pi}$ and the twisted character $\Theta_{\tilde{\pi}}$ depend on the choice of an intertwiner I_π^η between π and π^η although this dependence is not reflected to the symbol $\tilde{\pi}$. For any $c \in \mathbb{C}^\times$, $cI_\pi^\eta := (c \cdot \mathrm{id}_V) \circ I_\pi^\eta$ is again an intertwiner between π and π^η. If we define a twisted representation by using cI_π^η (let us write $c\tilde{\pi}$ for it), then its twisted character is simply given by $\Theta_{c\tilde{\pi}} = c \cdot \Theta_{\tilde{\pi}}$.
2. As mentioned in Remark 5.1, for any $\eta = \eta^\circ\theta \in \tilde{G}$, π is η-stable if and only if it is θ-stable. When I_π^θ is an intertwiner between π and π^θ, then $I_\pi^\eta := \pi(\eta^\circ) \circ I_\pi^\theta$ gives an intertwiner between π and π^η. If we write $\tilde{\pi}[I_\pi^\eta]$ and $\tilde{\pi}[I_\pi^\theta]$ for the twisted representations of $\tilde{G}$ obtained from π by using these two intertwiners I_π^η and I_π^θ, then we can easily check that $\tilde{\pi}[I_\pi^\eta] = \tilde{\pi}[I_\pi^\theta]$. In particular, we have $\Theta_{\tilde{\pi}[I_\pi^\eta]} = \Theta_{\tilde{\pi}[I_\pi^\theta]}$.

5.2. **Twist and intertwiner.** Let $\eta \in \tilde{G}$. Only in this subsection, for any subgroup H of G, we let H^η denote its conjugate $[\eta]^{-1}(H) = \eta^{-1}H\eta$. We caution that this usage of notation is temporary; in other places of this paper, the upper η denotes the stabilizer of η.

For any tame elliptic toral pair $(\mathbf{S}, \vartheta)$ of $\mathbf{G}$, its η-twist

$$(\mathbf{S}, \vartheta)^\eta := ([\eta]^{-1}(\mathbf{S}), \vartheta \circ [\eta])$$

is again a tame elliptic toral pair of $\mathbf{G}$. Thus we have the toral regular supercuspidal representation $\pi_{(\mathbf{S},\vartheta)^\eta}$ associated $(\mathbf{S},\vartheta)^\eta$. On the other hand, we also have the η-twist $\pi^\eta_{(\mathbf{S},\vartheta)}$ of the toral regular supercuspidal representation $\pi_{(\mathbf{S},\vartheta)}$ associated to $(\mathbf{S},\vartheta)$. In fact, these representations are isomorphic. Let us investigate how we can construct an intertwiner $\pi_{(\mathbf{S},\vartheta)^\eta} \cong \pi^\eta_{(\mathbf{S},\vartheta)}$.

Recall from Section 4.1 that $\pi_{(\mathbf{S},\vartheta)}$ is defined to be the compact induction $\text{c-Ind}_K^G \rho_{(\mathbf{S},\vartheta)}$ of a representation $\rho_{(\mathbf{S},\vartheta)}$ of an open compact-mod-center subgroup $K = SJ$ of G. Hence the η-twisted representation $\pi^\eta_{(\mathbf{S},\vartheta)}$ is isomorphic to the compact induction of $\rho^\eta_{(\mathbf{S},\vartheta)}$ from K^η to G by the following explicit intertwiner:

$$\text{(1)} \qquad \text{c-Ind}_{K^\eta}^G \rho^\eta_{(\mathbf{S},\vartheta)} \xrightarrow{\sim} (\text{c-Ind}_K^G \rho_{(\mathbf{S},\vartheta)})^\eta = \pi^\eta_{(\mathbf{S},\vartheta)} \colon f \mapsto f \circ [\eta]^{-1}.$$

On the other hand, we can easily see that the open compact-mod-center subgroup associated to η-twisted pair $(\mathbf{S},\vartheta)^\eta$ is given by $K^\eta = S^\eta J^\eta$. Thus $\pi_{(\mathbf{S},\vartheta)^\eta}$ is given by the compact induction $\text{c-Ind}_{K^\eta}^G \rho_{(\mathbf{S},\vartheta)^\eta}$ of a representation $\rho_{(\mathbf{S},\vartheta)^\eta}$ of K^η.

Let us consider the relationship between the representations $\rho^\eta_{(\mathbf{S},\vartheta)}$ and $\rho_{(\mathbf{S},\vartheta)^\eta}$ of K^η. The representation $\rho_{(\mathbf{S},\vartheta)}$ of K is defined to be the push-out of the representation $\omega_{(\mathbf{S},\vartheta)} \otimes (\vartheta \ltimes \mathbb{1})$ of $S \ltimes J$ along the natural multiplication map $S \ltimes J \twoheadrightarrow SJ$. Hence $\rho^\eta_{(\mathbf{S},\vartheta)}$ is the push-out of $\omega^\eta_{(\mathbf{S},\vartheta)} \otimes (\vartheta^\eta \ltimes \mathbb{1})$ along $S^\eta \ltimes J^\eta \twoheadrightarrow S^\eta J^\eta$. On the other hand, $\rho_{(\mathbf{S},\vartheta)^\eta}$ is defined to be the push-out of $\omega_{(\mathbf{S},\vartheta)^\eta} \otimes (\vartheta^\eta \ltimes \mathbb{1})$ along $S^\eta \ltimes J^\eta \twoheadrightarrow S^\eta J^\eta$. We note that both of $\omega^\eta_{(\mathbf{S},\vartheta)}$ and $\omega_{(\mathbf{S},\vartheta)^\eta}$ are Heisenberg–Weil representations with central character $\hat{\vartheta}^\eta$. Hence, by the Stone–von Neumann theorem, $\omega^\eta_{(\mathbf{S},\vartheta)}$ and $\omega_{(\mathbf{S},\vartheta)^\eta}$ are isomorphic. Let us fix an intertwiner

$$I^\eta_{\omega_{(\mathbf{S},\vartheta)}} \colon \omega_{(\mathbf{S},\vartheta)^\eta} \xrightarrow{\sim} \omega^\eta_{(\mathbf{S},\vartheta)},$$

which naturally induces an intertwiner

$$I^\eta_{\rho_{(\mathbf{S},\vartheta)}} \colon \rho_{(\mathbf{S},\vartheta)^\eta} \xrightarrow{\sim} \rho^\eta_{(\mathbf{S},\vartheta)}.$$

Then we get an intertwiner between $\text{Ind}_{K^\eta}^G \rho_{(\mathbf{S},\vartheta)^\eta}$ and $\text{Ind}_{K^\eta}^G \rho^\eta_{(\mathbf{S},\vartheta)}$ given by

$$\text{(2)} \qquad \text{Ind}_{K^\eta}^G \rho_{(\mathbf{S},\vartheta)^\eta} \xrightarrow{\sim} \text{Ind}_{K^\eta}^G \rho^\eta_{(\mathbf{S},\vartheta)} \colon f \mapsto I^\eta_{\rho_{(\mathbf{S},\vartheta)}} \circ f.$$

Therefore, combining (1) with (2), we obtain an intertwiner $I^\eta_{\pi_{(\mathbf{S},\vartheta)}}$ between $\pi_{(\mathbf{S},\vartheta)^\eta}$ and $\pi^\eta_{(\mathbf{S},\vartheta)}$ given by $f \mapsto I^\eta_{\rho_{(\mathbf{S},\vartheta)}} \circ f \circ [\eta]^{-1}$:

$$\pi_{(\mathbf{S},\vartheta)^\eta} = \text{c-Ind}_{K^\eta}^G \rho_{(\mathbf{S},\vartheta)^\eta} \xrightarrow{(2)} \text{c-Ind}_{K^\eta}^G \rho^\eta_{(\mathbf{S},\vartheta)} \xrightarrow{(1)} (\text{c-Ind}_K^G \rho_{(\mathbf{S},\vartheta)})^\eta = \pi^\eta_{(\mathbf{S},\vartheta)}.$$

From now on (until the end of Section 6), suppose that we have the following:

- an F-rational tame elliptic twisted maximal torus $(\tilde{\mathbf{S}},\mathbf{S})$ of $(\tilde{\mathbf{G}},\mathbf{G})$, and
- a tame elliptic toral pair $(\mathbf{S},\vartheta)$ of depth $r \in \mathbb{R}_{>0}$ which is η-invariant, i.e., $(\mathbf{S},\vartheta) = (\mathbf{S},\vartheta)^\eta$, for $\eta \in \tilde{S}$. (Note that $(\mathbf{S},\vartheta)$ is η-invariant for some $\eta \in \tilde{S}$ if and only if so is for any $\eta \in \tilde{S}$.)

Let us fix a base point $\underline{\eta} \in \tilde{S}$ which is topologically semisimple in the following. Note that we can always find such an element since $\tilde{S}$ is nonempty by the definition of a twisted maximal torus (Definition 3.2); apply Proposition 3.14 to any element of $\tilde{S}$ and take the topologically semisimple part.

Then, by using the intertwiner $I^{\underline{\eta}}_{\pi_{(\mathbf{S},\vartheta)}}$ we just constructed, we obtain a representation $\tilde{\pi}_{(\mathbf{S},\vartheta)}$ of $\tilde{G}$ and its twisted character $\Theta_{\tilde{\pi}_{(\mathbf{S},\vartheta)}}$ (see Section 5.1).

We note that, since $[\underline{\eta}]$ preserves $\mathbf{S}$, the point $\mathbf{x} \in \mathcal{B}_{\mathrm{red}}(\mathbf{G}, F)$ associated to $\mathbf{S}$ is stabilized by the action on $\mathcal{B}_{\mathrm{red}}(\mathbf{G}, F)$ induced from $[\underline{\eta}]$. Accordingly, every group used in the construction of $\rho_{(\mathbf{S},\vartheta)}$ (such as $G_{\mathbf{x},s}$, K, J, J_+, and so on) is stabilized by $[\underline{\eta}]$. Then we also have a twisted representation $\tilde{\rho}_{(\mathbf{S},\vartheta)}$ of $\tilde{K} := K\underline{\eta}$ and its twisted character $\Theta_{\tilde{\rho}_{(\mathbf{S},\vartheta)}}$, which is a function on $\tilde{K}$ defined by

$$\Theta_{\tilde{\rho}_{(\mathbf{S},\vartheta)}}(k\underline{\eta}) := \mathrm{tr}\bigl(\rho_{(\mathbf{S},\vartheta)}(k) \circ I^{\eta}_{\rho_{(\mathbf{S},\vartheta)}}\bigr).$$

Also note that $I^{\eta}_{\rho_{(\mathbf{S},\vartheta)}}$ naturally induces an intertwiner between $\sigma_{(\mathbf{S},\vartheta)}$ and its $\underline{\eta}$-twist $\sigma^{\underline{\eta}}_{(\mathbf{S},\vartheta)}$ (recall that $\sigma_{(\mathbf{S},\vartheta)}$ is a representation of K_σ defined by $\mathrm{Ind}_K^{K_\sigma} \rho_{(\mathbf{S},\vartheta)}$). Thus we get a twisted representation $\tilde{\sigma}_{(\mathbf{S},\vartheta)}$ of $\tilde{K}_\sigma := K_\sigma\underline{\eta}$ and its twisted character $\Theta_{\tilde{\sigma}_{(\mathbf{S},\vartheta)}}$, which is a function on $\tilde{K}_\sigma$ defined by

$$\Theta_{\tilde{\sigma}_{(\mathbf{S},\vartheta)}}(k\underline{\eta}) := \mathrm{tr}\bigl(\sigma_{(\mathbf{S},\vartheta)}(k) \circ I^{\eta}_{\sigma_{(\mathbf{S},\vartheta)}}\bigr).$$

We emphasize that the construction of the intertwiner $I^{\eta}_{\pi_{(\mathbf{S},\vartheta)}}$ explained above involves the unspecified choice of an intertwiner $I^{\eta}_{\omega_{(\mathbf{S},\vartheta)}} : \omega_{(\mathbf{S},\vartheta)^{\underline{\eta}}} \cong \omega^{\underline{\eta}}_{(\mathbf{S},\vartheta)}$ of Heisenberg–Weil representations. In Section 6.2, we will choose $I^{\eta}_{\omega_{(\mathbf{S},\vartheta)}}$ in an explicit way.

We finally recall that, by the torality of ϑ, there exists a $\mathbf{G}$-generic element $X^* \in \mathfrak{s}^*_{-r}$ of depth r which lifts a unique element of $\mathfrak{s}^*_{-r:-r+}$ satisfying $\vartheta(\exp(Y)) = \psi_F(\langle Y, X^*\rangle)$ for any $Y \in \mathfrak{s}_{s+:r+}$. We note that

$$\vartheta \circ [\eta](\exp(Y)) = \vartheta(\exp([\eta](Y))) = \psi_F(\langle [\eta](Y), X^*\rangle) = \psi_F(\langle Y, [\eta](X^*)\rangle),$$

where we again write $[\eta]$ for the action on $\mathfrak{s}$ induced by $[\eta]$ and used that $\exp\colon \mathfrak{s}_{s+:r+} \cong S_{s+:r+}$ is $[\eta]$-equivariant (the action of $[\eta]$ on X^* is, by definition, given by the identity $\langle [\eta](Y), X^*\rangle = \langle Y, [\eta](X^*)\rangle$). Thus, as we have $\vartheta \circ [\eta] = \vartheta$ by the assumption, we see that $[\eta](X^*)$ equals X^* in $\mathfrak{s}^*_{-r:-r+}$.

Lemma 5.3. *We may take $X^* \in \mathfrak{s}^*_{-r}$ to be $[\eta]$-invariant.*

Proof. Since $p \neq 2$, we have $\frac{1}{2}(X^* + [\eta](X^*)) \in \mathfrak{s}^*_{-r}$. Note that this element is $[\eta]$-invariant. Moreover, as the image of X^* in $\mathfrak{s}^*_{-r:-r+}$ is $[\eta]$-invariant, the image of $\frac{1}{2}(X^* + [\eta](X^*))$ in $\mathfrak{s}^*_{-r:-r+}$ is equal to the image of X^*. Thus, by replacing X^* with $\frac{1}{2}(X^* + [\eta](X^*))$, we get a desired element. □

In the following, by this lemma, we assume that an element $X^* \in \mathfrak{s}^*_{-r}$ representing the character $\vartheta|_{S_r}$ is invariant under $[\eta]|_{\mathbf{S}} = \theta_{\mathbf{S}}$.

We finish this subsection by showing one more lemma.

Lemma 5.4. *For any $\eta \in \tilde{S}$, the restriction $(\mathbf{S}^\natural, \vartheta^\natural := \vartheta|_{S^\natural})$ gives a tame elliptic toral pair of $\mathbf{G}_\eta$ (of depth r).*

Proof. Note that, as we have $\mathbf{S}^\natural \subset \mathbf{S}$, we have $\mathfrak{s}^\natural \subset \mathfrak{s}$, hence $\mathfrak{s}^* \twoheadrightarrow \mathfrak{s}^{\natural *}$. We can take an element of $\mathfrak{s}^{\natural *}_{-r}$ representing the character $\vartheta^\natural$ to be the image of X^* taken above via the natural map $\mathfrak{s}^* \twoheadrightarrow \mathfrak{s}^{\natural *}$. Our task is to show that X^* is an $\mathbf{G}_\eta$-generic element of depth r. We note that our assumption that $p \nmid |\Omega_{\mathbf{G}}|$ implies that $p \nmid |\Omega_{\mathbf{G}_\eta}|$ (recall that $\Omega_{\mathbf{G}_\eta}$ is regarded as a subgroup of $\Omega_{\mathbf{G}}$). Thus it is enough to only check that **GE1** is satisfied, which requires that $\mathrm{val}_F(\langle H_{\alpha_{\mathrm{res}}}, X^*\rangle) = -r$ for any $\alpha_{\mathrm{res}} \in \Phi(\mathbf{G}_\eta, \mathbf{S}^\natural)$, where $H_{\alpha_{\mathrm{res}}} = d\alpha^\vee_{\mathrm{res}}(1)$ (see Section 4.2).

By the description of $\Phi(\mathbf{G}_\eta, \mathbf{S}^\natural)$ and $\Phi^\vee(\mathbf{G}_\eta, \mathbf{S}^\natural)$ as in Section 3.3, we have $H_{\alpha_{\mathrm{res}}} = \varrho_\alpha \cdot \sum_{i=0}^{l_\alpha - 1} H_{\theta^i_{\mathbf{S}}(\alpha)}$. As X^* is $\theta_{\mathbf{S}}$-invariant, we have $\langle H_{\theta^i_{\mathbf{S}}(\alpha)}, X^*\rangle = \langle H_\alpha, \theta^i_{\mathbf{S}}(X^*)\rangle =$

$\langle H_\alpha, X^*\rangle$. Hence $\langle H_{\alpha_{\mathrm{res}}}, X^*\rangle = \varrho_\alpha \cdot l_\alpha \cdot \langle H_\alpha, X^*\rangle$. Since X^* is $\mathbf{G}$-generic of depth r and $p \nmid \varrho_\alpha \cdot l_\alpha$, we get $\mathrm{val}_F(\langle H_{\alpha_{\mathrm{res}}}, X^*\rangle) = -r$. □

5.3. **Separation lemma.** In this subsection, we prove some technical lemma and propositions which will be needed later.

The following follows from [KP23, Theorem 12.7.1] by using the tamely ramified descent for the Bruhat–Tits buildings ([KP23, Section 12.9]).

Proposition 5.5. *Let $\delta_0 \in \tilde{S}$ be absolutely p-semisimple modulo $A_{\tilde{\mathbf{G}}}$. There exist an identification between the building $\mathcal{B}(\mathbf{G}_{\delta_0}, F)$ and the fixed points of $\mathcal{B}(\mathbf{G}, F)$ under the action induced by $[\delta_0]$ such that $\mathcal{A}(\mathbf{S}^\natural, F)$ is mapped to $\mathcal{A}(\mathbf{S}, F)^{\delta_0}$:*

$$\begin{array}{ccccc}
\mathcal{B}(\mathbf{G}_{\delta_0}, F) & \xrightarrow{\cong} & \mathcal{B}(\mathbf{G}, F)^{\delta_0} & \subset & \mathcal{B}(\mathbf{G}, F) \\
\cup & & \cup & & \cup \\
\mathcal{A}(\mathbf{S}^\natural, F) & \xrightarrow{\cong} & \mathcal{A}(\mathbf{S}, F)^{\delta_0} & \subset & \mathcal{A}(\mathbf{S}, F)
\end{array}$$

Proposition 5.6. *Let $\delta_0 \in \tilde{S}$ be absolutely p-semisimple modulo $A_{\tilde{\mathbf{G}}}$. Suppose that the point $\mathbf{x}$ associated to $\mathbf{S}$ belongs to $\mathcal{A}(\mathbf{S}^\natural, F)$ under the identification as in Proposition 5.5. Then we have the following for any $r, s \in \widetilde{\mathbb{R}}_{>0}$ satisfying $r < s$:*

1. $S^\natural_r = (S_r)^{\delta_0}$ $(= (S_r)^{\theta_{\mathbf{S}}})$ *and* $S^\natural_{0+:r} = (S_{0+:r})^{\delta_0}$,
2. $G_{\delta_0,\mathbf{x},r} = (G_{\mathbf{x},r})^{\delta_0}$ *and* $(S^\natural, G_{\delta_0})_{\mathbf{x},(r,s(+))} = (S, G)^{\delta_0}_{\mathbf{x},(r,s(+))}$,
3. $S^\natural_{0+}G_{\delta_0,\mathbf{x},r} = (S_{0+}G_{\mathbf{x},r})^{\delta_0}$,
4. $(S^\natural, G_{\delta_0})_{\mathbf{x},(r,s):(r,s+)} = (S, G)^{\delta_0}_{\mathbf{x},(r,s):(r,s+)}$.

Proof. Let us first show (1). Recall that δ_0 acts on $\mathbf{S}$ via $\theta_{\mathbf{S}}$ and we put $\mathbf{S}^\natural := \mathbf{S}^{\theta_{\mathbf{S}},\circ}$. The r-th filtration of S is defined by

$$S_r := \{t \in S^0 \mid \mathrm{val}_F(\chi(t) - 1) \geq r \text{ for any } \chi \in X^*(\mathbf{S})\},$$

where S^0 denotes the Iwahori subgroup of S (here note that $\mathbf{S}$ is tamely ramified; see [KP23, Definitions 2.5.13 and B.5.1]). Similarly, the r-th filtration of $S^\natural$ is defined by

$$S^\natural_r := \{t \in S^{\natural,0} \mid \mathrm{val}_F(\chi(t) - 1) \geq r \text{ for any } \chi \in X^*(\mathbf{S}^\natural)\}.$$

Thus, noting that $S^{\natural,0}$ is contained in S^0, we have $S^\natural_r \subset (S_r)^{\theta_{\mathbf{S}}}$. To show the converse inclusion, we take any element $t \in (S_r)^{\theta_{\mathbf{S}}}$. By Proposition 3.13, there exists a power of 2 (say $k \in \mathbb{Z}_{>0}$) satisfying $t^k \in S^\natural_r$. Then, as discussed in the proof of Proposition 3.14, we can remove k and get $t \in S^\natural_r$ since $p \neq 2$ and t is topologically p-unipotent. We consider the latter part of (1). By the former part which we just showed, we have $S^\natural_{0+:r} = (S_{0+})^{\delta_0}/(S_r)^{\delta_0}$. Note that we have $(S_{0+})^{\delta_0}/(S_r)^{\delta_0} \hookrightarrow (S_{0+:r})^{\delta_0}$. Let $\bar{s}$ be an element of $(S_{0+:r})^{\delta_0}$ represented by $s \in S_{0+}$. Then $s\theta_{\mathbf{S}}(s)$ is an element of $(S_{0+})^{\delta_0}$. Again by noting that $s\theta_{\mathbf{S}}(s)$ is topologically p-unipotent and $p \neq 2$, we can find an element $t \in (S_{0+})^{\delta_0}$ satisfying $t^2 = s\theta_{\mathbf{S}}(s)$. Then we have $\bar{t}^2 = \bar{s}^2$, hence $\bar{t} = \bar{s}$ since the order of $(S_{0+:r})^{\delta_0}$ is prime to 2. Hence we obtained the surjectivity of the map $(S_{0+})^{\delta_0}/(S_r)^{\delta_0} \hookrightarrow (S_{0+:r})^{\delta_0}$.

The assertion (2) follows from [KP23, Proposition 12.8.5] (together with the tamely ramified descent of Bruhat–Tits theory). Note that the assumption of [KP23, Proposition 12.8.5] is satisfied by (1).

Let us show (3). The inclusion $S^\natural_{0+}G_{\delta_0,\mathbf{x},r} \subset (S_{0+}G_{\mathbf{x},r})^{\delta_0}$ is obvious. To check the converse inclusion, let us take an element g of $(S_{0+}G_{\mathbf{x},r})^{\delta_0}$. Since we have

$S_{0+} \cap G_{\mathbf{x},r} = S_r$ (see [AS08, Proposition 4.6]), we have a bijection

$$S_{0+:r} = S_{0+}/S_r \xrightarrow{1:1} S_{0+}G_{\mathbf{x},r}/G_{\mathbf{x},r}.$$

This implies that the coset $gG_{\mathbf{x},r}$ is represented by an element s of S_{0+}. As g is $[\delta_0]$-invariant and the above bijection is $[\delta_0]$-equivariant, the coset sS_r is also $[\delta_0]$-invariant. Since we have $(S_{0+:r})^{\delta_0} = S^{\natural}_{0+:r}$ by (1), we know that s can be taken to be an element of $S^{\natural}_{0+}$. Now let us write $g = sg'$ with $s \in S^{\natural}_{0+}$ and $g' \in G_{\mathbf{x},r}$. Since g and s are δ_0-invariant, so is g'. By (2), this implies that $g' \in G_{\delta_0,\mathbf{x},r}$.

The assertion (4) follows from the same argument as in the proof of assertion (1) by using (2). □

Lemma 5.7. *Let δ be an elliptic regular semisimple element of $\tilde{G}$ with a topological Jordan decomposition $\delta = \delta_0\delta_+$. If δ belongs to $\tilde{S}G_{\mathbf{x},r}$, then δ_0 belongs to ${}^{G_{\mathbf{x},r}}\tilde{S} := \{{}^{g}s \mid g \in G_{\mathbf{x},r}, s \in \tilde{S}\}$.*

Proof. For any element $g \in G^{\dagger}$, we write $\bar{g}$ for its image in $G^{\dagger}/A_{\tilde{\mathbf{G}}}$. Similarly, we write $\overline{G_{\mathbf{x},r}}$ and $\overline{S}$ for the images of $G_{\mathbf{x},r}$ and S in $G^{\dagger}/A_{\tilde{\mathbf{G}}}$, respectively.

Since $\overline{\delta_0}$ belongs to the closure of $\langle\overline{\delta}\rangle$ in $G^{\dagger}/A_{\tilde{\mathbf{G}}}$ (see Proposition 3.14), the assumption $\delta \in \tilde{S}G_{\mathbf{x},r}$ implies that $\overline{\delta_0} \in \tilde{S}G_{\mathbf{x},r}/A_{\tilde{\mathbf{G}}}$. Let us take elements $s_0 \in \tilde{S}$ and $g_+ \in G_{\mathbf{x},r}$ satisfying $\delta_0 = g_+s_0$. If we let p' be the order of $\overline{\delta_0}$, which is prime to p, then we have

$$1 = \overline{\delta_0}^{p'} = \prod_{i=0}^{p'-1}[s_0]^i(\overline{g_+}) \cdot \overline{s_0}^{p'}.$$

Since $\prod_{i=0}^{p'-1}[s_0]^i(\overline{g_+}) \in \overline{G_{\mathbf{x},r}}$, this implies that $s_0^{p'}$ lies in $A_{\tilde{\mathbf{G}}}(S \cap G_{\mathbf{x},r}) = A_{\tilde{\mathbf{G}}}S_r$ (see [AS08, Proposition 4.6] for the equality). Furthermore, by noting that $s_0^{p'}$ is fixed by $[s_0]$ and $[s_0]$ acts on S as $\theta_{\mathbf{S}}$ and on $A_{\tilde{\mathbf{G}}}$ trivially, we have $s_0^{p'} \in A_{\tilde{\mathbf{G}}}(S_r)^{\theta_{\mathbf{S}}}$. Thus, by Proposition 5.6 (1), we get $s_0^{p'} \in A_{\tilde{\mathbf{G}}}S^{\natural}_r$. As p' is prime to p, we can find an element $s_r \in S^{\natural}_r$ such that $s_0^{p'} \in A_{\tilde{\mathbf{G}}} \cdot s_r^{p'}$ (see the proof of Proposition 3.14, the same argument as in the construction of δ_+ works). Then, by replacing $s_0 \in \tilde{S}$ with $s_0s_r^{-1} \in \tilde{S}$ and $g_+ \in G_{\mathbf{x},r}$ with g_+s_r, respectively, we may assume that

$$\prod_{i=0}^{p'-1}[s_0]^i(\overline{g_+}) = 1 \quad \text{and} \quad \overline{s_0}^{p'} = 1.$$

In other words, we have an action of a finite cyclic group $\mathbb{Z}/p'\mathbb{Z}$ on $\overline{G_{\mathbf{x},r}}$ given by $\bar{i} \cdot g = [s_0]^i(g)$ and a 1-cocycle $\mathbb{Z}/p'\mathbb{Z} \to \overline{G_{\mathbf{x},r}}$ given by $\bar{1} \mapsto \overline{g_+}$. Since p' is prime to p and $\overline{G_{\mathbf{x},r}}$ is a pro-p group, the first group cohomology $H^1(\mathbb{Z}/p'\mathbb{Z}, \overline{G_{\mathbf{x},r}})$ is trivial. (This follows from a standard argument by using that the action of $\mathbb{Z}/p'\mathbb{Z}$ is filtration-preserving; see the proof of [KP23, Theorem 13.8.5] for the details). Hence the cohomology class of the 1-cocycle $[\bar{1} \mapsto \overline{g_+}]$ is trivial. Namely, there exists an element $k \in G_{\mathbf{x},r}$ such that $kg_+[s_0](k)^{-1} \in A_{\tilde{\mathbf{G}}}$. This means that

$${}^{k}\delta_0 = kg_+s_0k^{-1} = kg_+[s_0](k)^{-1} \cdot s_0 \in A_{\tilde{\mathbf{G}}} \cdot \tilde{S} = \tilde{S}.$$

□

Proposition 5.8. *Suppose that the point $\mathbf{x}$ associated to $\mathbf{S}$ belongs to $\mathcal{A}(\mathbf{S}^\natural, F)$ under the identification as in Proposition 5.5. Let δ be an elliptic regular semisimple element of $\tilde{G}$ with a normal r-approximation $\delta = \delta_0\delta_{<r}^{+}\delta_{\geq r}$. If δ belongs to ${}^{G_{\mathbf{x},0+}}(\tilde{S}G_{\mathbf{x},r})$, then there exists $k \in G_{\mathbf{x},0+}$ such that*

$$\delta_0 \in \tilde{S}', \quad \delta_{<r}^{+} \in S'^{\natural}, \quad \delta_{\geq r} \in G_{\delta_{<r},\mathbf{x},r},$$

where $(\tilde{\mathbf{S}}', \mathbf{S}') := {}^{k}(\tilde{\mathbf{S}}, \mathbf{S})$. Here, the point $\mathbf{x} \in \mathcal{B}(\mathbf{G}_{\delta_0}, F)$ is regarded as a point of $\mathcal{B}(\mathbf{G}_{\delta_{<r}}, F)$ by an embedding $\mathcal{B}(\mathbf{G}_{\delta_{<r}}, F) \hookrightarrow \mathcal{B}(\mathbf{G}_{\delta_0}, F)$.

Proof. By replacing δ with its $G_{\mathbf{x},0+}$-conjugate, we may assume that δ itself belongs to $\tilde{S}G_{\mathbf{x},r}$. By Lemma 5.7, we have $\delta_0 \in {}^{k}\tilde{S}$ for some element $k \in G_{\mathbf{x},r}$. We put $(\tilde{\mathbf{S}}', \mathbf{S}') := {}^{k}(\tilde{\mathbf{S}}, \mathbf{S})$. Let us show that $\delta_{<r}^{+} \in S'^{\natural}$ and $\delta_{\geq r} \in G_{\delta_0,\mathbf{x},r}$.

Note that we have $\tilde{S}'G_{\mathbf{x},r} = {}^{k}\tilde{S}G_{\mathbf{x},r} = \tilde{S}G_{\mathbf{x},r}$. Indeed, for any $s \in \tilde{S}$, we have ${}^{k}s = s \cdot s^{-1}ks \cdot k^{-1}$. As the s-conjugation on G preserves $G_{\mathbf{x},r}$, $s^{-1}ks$ lies in $G_{\mathbf{x},r}$, which implies that ${}^{k}s \in \tilde{S}G_{\mathbf{x},r}$. Thus we get ${}^{k}\tilde{S}G_{\mathbf{x},r} \subset \tilde{S}G_{\mathbf{x},r}$. By the same argument for k^{-1}, we also get $\tilde{S}G_{\mathbf{x},r} \subset {}^{k}\tilde{S}G_{\mathbf{x},r}$, hence ${}^{k}\tilde{S}G_{\mathbf{x},r} = \tilde{S}G_{\mathbf{x},r}$.

Since we have $\delta_0 \in \tilde{S}'$ and $\delta = \delta_0\delta_+ \in \tilde{S}G_{\mathbf{x},r} = \tilde{S}'G_{\mathbf{x},r}$, we know that $\delta_+ \in S'G_{\mathbf{x},r}$. On the other hand, by the construction of a topological Jordan decomposition 3.14, δ_+ belongs to $G_{\delta_0,0+} \subset G_{0+}$. Since we have $S'G_{\mathbf{x},r} \cap G_{0+} = S'_{0+}G_{\mathbf{x},r}$ (see [AS08, Proposition 4.6]), we have $\delta_+ \in S'_{0+}G_{\mathbf{x},r}$. Furthermore, as δ_+ commutes with δ_0, we get $\delta_+ \in S'^{\natural}_{0+}G_{\delta_0,\mathbf{x},r}$ by Proposition 5.6 (3).

Now the situation is reduced to the untwisted setting. By applying [AS08, Corollary 9.16] to $\delta_+ \in S'^{\natural}_{0+}G_{\delta_0,\mathbf{x},r}$, we get $\delta_{<r}^{+} \in {}^{G_{\delta_0,\mathbf{x},0+}}S'^{\natural}$. Thus, by taking $k' \in G_{\delta_0,\mathbf{x},0+}$ such that $\delta_{<r}^{+} \in {}^{k'}S'^{\natural}$ and replacing k with $k'k$, we have $\delta_{<r}^{+} \in S'^{\natural}$.

On the other hand, [AS08, Lemma 9.13] implies that the point $\mathbf{x}$ belongs to the set "$\mathcal{B}_r(\delta_+)$" (which is considered in the group $\mathbf{G}_{\delta_0}$; see [AS08, Definition 9.5] for the definition). By the description of the set $\mathcal{B}_r(\delta_+)$ in [AS08, Lemma 9.6], we have

$$\mathcal{B}_r(\delta_+) = \{\mathbf{y} \in \mathcal{B}(C^{(r)}_{\mathbf{G}_{\delta_0}}(\delta_+), F) \mid \delta_{\geq r} \in G_{\delta_0,\mathbf{y},r}\}.$$

Hence $\mathbf{x}$ belongs to the building of $C^{(r)}_{\mathbf{G}_{\delta_0}}(\delta_+) = (\mathbf{G}_{\delta_0})_{\delta_{<r}^{+}}$ (see [AS08, Corollary 6.14]), which furthermore equals $\mathbf{G}_{\delta_{<r}}$ by Lemma 3.19, and we have $\delta_{\geq r} \in G_{\delta_0,\mathbf{x},r}$. By the definition of a normal approximation, $\delta_{\geq r}$ belongs to $(\mathbf{G}_{\delta_0})_{\delta_{<r}^{+}} = \mathbf{G}_{\delta_{<r}}$. Thus $\delta_{\geq r}$ lies in $\mathbf{G}_{\delta_0,\mathbf{x},r} \cap \mathbf{G}_{\delta_{<r}}$, which equals $\mathbf{G}_{\delta_{<r},\mathbf{x},r}$ by [AS08, Proposition 4.6]. □

5.4. **Twisted character formula of 1st form.** Our aim in this and subsequent sections is to establish a formula of the twisted character of $\Theta_{\tilde{\pi}_{(\mathbf{S},\vartheta)}}$ as in the untwisted case by Adler–DeBacker–Spice ([AS09, DS18]).

Since the pair $(\mathbf{S}, \vartheta)$ is always fixed in the following, we simply write ω, ρ, σ, τ, π for the representations $\omega_{(\mathbf{S},\vartheta)}$, $\rho_{(\mathbf{S},\vartheta)}$, $\sigma_{(\mathbf{S},\vartheta)}$, $\tau_{(\mathbf{S},\vartheta)}$, and $\pi_{(\mathbf{S},\vartheta)}$ (see Section 4.2), respectively. Similarly, we simply write $\tilde{\rho}$, $\tilde{\sigma}$, and $\tilde{\pi}$ for the twisted representations as introduced in Section 5.2. We use the identification of Bruhat–Tits buildings and apartments as in Proposition 5.5 in the following. We may suppose that the point $\mathbf{x}$ associated to $\mathbf{S}$ comes from $\mathcal{A}(\mathbf{S}^\natural, F)$.

In the following, we fix an elliptic regular semisimple element $\delta \in \tilde{G}$ and a normal r-approximation $\delta = \delta_0\delta_{<r}^{+}\delta_{\geq r}$ to δ, which exists by Proposition 3.17. We simply write $\eta := \delta_{<r}$.

We start by showing the following lemma, which is a twisted version of [AS09, Lemma 6.1]:

Lemma 5.9. *The set $S\backslash\{g \in G \mid {}^{g}\eta \in \tilde{S}\}/G_{\eta}$ is finite.*

Proof. First we note that $\mathbf{Z}_{\mathbf{G}}(\mathbf{S}^{\natural}) = \mathbf{S}$ (Proposition 3.3 (1)). Thus, if an element $n \in \mathbf{G}$ belongs to $\mathbf{N}_{\mathbf{G}}(\mathbf{S}^{\natural})$, i.e., satisfies $n\mathbf{S}^{\natural}n^{-1} \subset \mathbf{S}^{\natural}$, then we get $n\mathbf{S}n^{-1} \supset \mathbf{S}$ by taking the centralizer groups in $\mathbf{G}$. Hence $n^{-1} \in \mathbf{N}_{\mathbf{G}}(\mathbf{S})$, which implies $n \in \mathbf{N}_{\mathbf{G}}(\mathbf{S})$. Thus we get $\mathbf{N}_{\mathbf{G}}(\mathbf{S}^{\natural}) \subset \mathbf{N}_{\mathbf{G}}(\mathbf{S})$. Since $\mathbf{S}$ is contained in $\mathbf{N}_{\mathbf{G}}(\mathbf{S}^{\natural})$ and of finite index in $\mathbf{N}_{\mathbf{G}}(\mathbf{S})$, S is of finite index in $N_{\mathbf{G}}(\mathbf{S}^{\natural})$. Thus it is enough to show that $N_{\mathbf{G}}(\mathbf{S}^{\natural})\backslash\{g \in G \mid {}^{g}\eta \in \tilde{S}\}/G_{\eta}$ is finite.

For any element $g \in G$ satisfying ${}^{g}\eta \in \tilde{S}$, by Proposition 3.3 (2), we have

$$ {}^{g}\mathbf{G}_{\eta} = \mathbf{G}_{{}^{g}\eta} = \mathbf{Z}_{\mathbf{G}}({}^{g}\eta)^{\circ} \supset \mathbf{Z}_{\mathbf{G}}(\tilde{\mathbf{S}})^{\circ} = \mathbf{S}^{\natural}. $$

In other words, we have ${}^{g^{-1}}\mathbf{S}^{\natural} \subset \mathbf{G}_{\eta}$. Since $\mathbf{S}^{\natural}$ is an F-rational maximal torus of $\mathbf{G}_{\eta}$ (see Proposition 3.3 (3)), so is ${}^{g^{-1}}\mathbf{S}^{\natural}$. Therefore we get an injection

$$ N_{G}(\mathbf{S}^{\natural})\backslash\{g \in G \mid {}^{g}\eta \in \tilde{S}\} \hookrightarrow \{F\text{-rational maximal tori of } \mathbf{G}_{\eta}\} \colon g \mapsto {}^{g^{-1}}\mathbf{S}^{\natural}. $$

By taking the quotients with respect to the action of G_{η}, we furthermore get

$$ N_{G}(\mathbf{S}^{\natural})\backslash\{g \in G \mid {}^{g}\eta \in \tilde{S}\}/G_{\eta} \hookrightarrow \{F\text{-rational maximal tori of } \mathbf{G}_{\eta}\}/{\sim_{G_{\eta}}}, $$

where the symbol $\sim_{G_{\eta}}$ denotes the equivalence class given by G_{η}-conjugation. As the right-hand side is finite, $N_{G}(\mathbf{S}^{\natural})\backslash\{g \in G \mid {}^{g}\eta \in \tilde{S}\}/G_{\eta}$ is also finite. □

Recall that $\tilde{\sigma}$ is a representation of $\tilde{K}_{\sigma} = \tilde{S}G_{\mathbf{x},0+}$ and that $\Theta_{\tilde{\sigma}}$ is its twisted character with respect to the intertwiner as chosen in Section 5.2. Let $\dot{\Theta}_{\tilde{\sigma}}$ be the zero extension of $\Theta_{\tilde{\sigma}}$ from $\tilde{K}_{\sigma} = \tilde{S}G_{\mathbf{x},0+}$ to $\tilde{G}$.

The following lemma is a twisted version of [AS09, Proposition 4.3]. In fact, the same proof as in [AS09, Proposition 4.3] works as we present in the following.

Lemma 5.10. *For any $g \in G$, if $\dot{\Theta}_{\tilde{\sigma}}({}^{g}\delta) \neq 0$, then we have ${}^{g}\delta \in {}^{G_{\mathbf{x},0+}}(\tilde{S}G_{\mathbf{x},r})$.*

Proof. We put $\delta' := {}^{g}\delta$. We obtain a normal r-approximation $\delta' = \delta'_{0}\delta'^{+}_{<r}\delta'_{\geq r}$ to δ' by taking the g-conjugation of $\delta = \delta_{0}\delta^{+}_{<r}\delta_{\geq r}$. Suppose that $\dot{\Theta}_{\tilde{\sigma}}({}^{g}\delta) \neq 0$, in particular, δ' belongs to $\tilde{K}_{\sigma} = \tilde{S}G_{\mathbf{x},0+}$. Then, by Proposition 5.8 (take r in Proposition 5.8 to be $0+$), we know that $\delta'_{0} \in {}^{G_{\mathbf{x},0+}}\tilde{S}$ and $\delta'_{+} \in G_{\delta'_{0},\mathbf{x},0+}$.

Let $t \in \mathbb{R}_{>0}$ be the largest number such that $\delta'_{+} \in G_{\delta'_{0},\mathbf{x},t} \smallsetminus G_{\delta'_{0},\mathbf{x},t+}$. Then it suffices to show that $t \geq r$. Let us suppose that $t < r$ for a contradiction.

We take $k \in G_{\mathbf{x},0+}$ satisfying $\delta'_{0} \in {}^{k}\tilde{S}$ and put $(\tilde{\mathbf{S}}', \mathbf{S}') := ({}^{k}\tilde{\mathbf{S}}, {}^{k}\mathbf{S})$. By [AS09, Lemma 9.13] (we take $(\mathbf{G}', \mathbf{G})$ to be $(\mathbf{S}'^{\natural}, \mathbf{G}_{\delta'_{0}})$), we know that $\mathbf{x} \in \mathcal{B}_{t}(\delta'_{+})$. In other words, $\mathbf{x}$ belongs to the building of $C^{(t)}_{\mathbf{G}_{\delta'_{0}}}(\delta'_{+}) = (\mathbf{G}_{\delta'_{0}})_{\delta'^{+}_{<t}} = \mathbf{G}_{\delta'_{<t}}$ and we have $\delta'_{\geq t} \in G_{\delta'_{<t},\mathbf{x},t}$ (cf. the proof of Proposition 5.8). For any $h \in C^{(t)}_{G_{\delta'_{0}}}(\delta'_{+})_{\mathbf{x},r-t} = G_{\delta'_{<t},\mathbf{x},r-t}$, we have $[\delta'^{-1}, h] = [\delta'^{-1}_{\geq t}, h] \in G_{\delta'_{0},\mathbf{x},r}$. Thus, by noting that σ is $\hat{\vartheta}$-isotypic on $G_{\mathbf{x},r}$ ([AS08, Lemma 2.5]) and that $\Theta_{\tilde{\sigma}}$ is invariant under K_{σ}-conjugation, we get

$$ \Theta_{\tilde{\sigma}}(\delta') = \Theta_{\tilde{\sigma}}({}^{h}\delta') = \Theta_{\tilde{\sigma}}(\delta' \cdot [\delta'^{-1}, h]) = \Theta_{\tilde{\sigma}}(\delta') \cdot \hat{\vartheta}([\delta'^{-1}_{\geq t}, h]) $$

for any $h \in G_{\delta'_{<r},\mathbf{x},r-t}$. Since $\hat{\vartheta}([\delta'^{-1}_{\geq t}, -])$ is nontrivial on $G_{\delta'_{<r},\mathbf{x},r-t}$ as proved in the final paragraph of the proof of [AS09, Proposition 4.3], we conclude that $\Theta_{\tilde{\sigma}}(\delta')$ equals zero. This is a contradiction. □

We next establish a twisted version of [AS08, Lemma 6.3].

Lemma 5.11. *Let $\mathcal{K}_\eta$ be an open compact subgroup of G_η. Then the function*

$$G/Z_{\mathbf{G}} \to \mathbb{C}\colon g \mapsto \int_{\mathcal{K}_\eta} \dot{\Theta}_{\tilde{\sigma}}\big({}^{gk}\delta\big)\, dk$$

is compactly supported.

Proof. We let $\mathcal{F}\colon G \to \mathbb{C}$ be the function given by

$$\mathcal{F}(g) := \int_{\mathcal{K}_\eta} \dot{\Theta}_{\tilde{\sigma}}\big({}^{gk}\delta\big)\, dk.$$

Our task is to show that $\mathcal{F}$ is compactly supported modulo $Z_{\mathbf{G}}$.

We first note that the support of $\mathcal{F}$ is contained the following set:

$$\big\{g \in G \mid {}^{g}\eta \in {}^{G_{\mathbf{x},0+}}\tilde{S}\big\}.$$

Indeed, if ${}^{gk}\delta$ belongs to the support of $\dot{\Theta}_{\tilde{\sigma}}$, then ${}^{gk}\delta$ have to lie in ${}^{G_{\mathbf{x},0+}}(\tilde{S}G_{\mathbf{x},r})$ by Lemma 5.10. On the other hand, as $\mathcal{K}_\eta$ is a subset of G_η, every $k \in \mathcal{K}_\eta$ commutes with $\eta = \delta_{<r}$. Hence we have ${}^{gk}\delta = {}^{g}\eta \cdot {}^{gk}\delta_{\geq r}$. Thus, by Proposition 5.8, ${}^{g}\eta$ necessarily belongs to ${}^{G_{\mathbf{x},0+}}\tilde{S}$.

We consider the following double quotient:

$$K_\sigma \backslash \big\{g \in G \mid {}^{g}\eta \in {}^{G_{\mathbf{x},0+}}\tilde{S}\big\}/G_\eta.$$

Since $K_\sigma = SG_{\mathbf{x},0+}$, we have a natural surjection

$$S\backslash\big\{g \in G \mid {}^{g}\eta \in \tilde{S}\big\}/G_\eta \twoheadrightarrow K_\sigma\backslash\big\{g \in G \mid {}^{g}\eta \in {}^{G_{\mathbf{x},0+}}\tilde{S}\big\}/G_\eta.$$

As the former set is finite by Lemma 5.9, so is the latter set. Therefore, in order to show that $\mathcal{F}$ is compactly supported modulo $Z_{\mathbf{G}}$, it is enough to show that $\mathcal{F}$ is compactly supported modulo $Z_{\mathbf{G}}$ on each double coset $K_\sigma g G_\eta$. From now on, we fix an element $g \in G$ satisfying ${}^{g}\eta \in {}^{G_{\mathbf{x},0+}}\tilde{S}$. By replacing g with some other representative in the double coset $K_\sigma g G_\eta$ if necessary, we may suppose that g satisfies ${}^{g}\eta \in \tilde{S}$.

We define a function $\mathcal{F}_g\colon G_\eta \to \mathbb{C}$ by

$$\mathcal{F}_g(h) := \int_{\mathcal{K}_\eta} \dot{\Theta}_{\tilde{\sigma}}\big({}^{ghk}\delta\big)\, dk.$$

Note that the function $\dot{\Theta}_{\tilde{\sigma}}$ is invariant under K_σ-conjugation. Thus the function $\mathcal{F}$ is left-K_σ-invariant, and the restriction of $\mathcal{F}$ to the double coset $K_\sigma g G_\eta$ is given by $\mathcal{F}|_{K_\sigma g G_\eta}(lgh) = \mathcal{F}_g(h)$. As K_σ is compact modulo $Z_{\mathbf{G}}$, it is enough to show that $\mathcal{F}_g$ is compactly supported modulo $G_\eta \cap Z_{\mathbf{G}}$. Since $\mathbf{A}_{\tilde{\mathbf{G}}}$ is defined to be the maximal split subtorus of $\mathbf{Z}_{\mathbf{G}}^{\theta}$, we have $\mathbf{A}_{\tilde{\mathbf{G}}} \subset \mathbf{G}_\eta \cap \mathbf{Z}_{\mathbf{G}} \subset \mathbf{Z}_{\mathbf{G}}^{\theta}$. Hence it suffices to show that $\mathcal{F}_g$ is compactly supported modulo $A_{\tilde{\mathbf{G}}}$.

We compute $\dot{\Theta}_{\tilde{\sigma}}({}^{ghk}\delta)$ in the integrand of $\mathcal{F}_g$. Since g is chosen to satisfy ${}^{g}\eta \in \tilde{S}$, by also noting that $h, k \in G_\eta$, we have ${}^{ghk}\eta = {}^{g}\eta \in \tilde{S}$. On the other hand, if $\dot{\Theta}_{\tilde{\sigma}}({}^{ghk}\delta)$ is not zero, then we have ${}^{ghk}\delta_{\geq r} \in G_{{}^{g}\eta,\mathbf{x},r}$ by Lemma 5.10 and Proposition 5.8. Therefore, by noting that the restriction of σ on $G_{\mathbf{x},r}$ is $\hat{\vartheta}$-isotypic ([AS09, Lemma 2.5]), we get

$$\dot{\Theta}_{\tilde{\sigma}}({}^{ghk}\delta) = \dot{\Theta}_{\tilde{\sigma}}({}^{g}\eta)\mathbb{1}_{G_{{}^{g}\eta,\mathbf{x},r}}({}^{ghk}\delta_{\geq r})\hat{\vartheta}({}^{ghk}\delta_{\geq r}).$$

Since the term $\dot{\Theta}_{\tilde{\sigma}}({}^{g}\eta)$ does not depend on h or k, it suffices to show that the function
$$\tilde{\mathcal{F}}_g\colon G_\eta \to \mathbb{C};\quad h \mapsto \int_{\mathcal{K}_\eta} \mathbb{1}_{G_{{}^{g}\eta,\mathbf{x},r}}({}^{ghk}\delta_{\geq r})\hat{\vartheta}({}^{ghk}\delta_{\geq r})\,dk$$
is compactly supported modulo $A_{\tilde{\mathbf{G}}}$.

In the following, we put $\eta' := {}^{g}\eta$. Now recall that our toral cuspidal $\mathbf{G}$-datum is given by $((\mathbf{S}\subset\mathbf{G}),\mathbf{x},(r,r),(\vartheta,\mathbb{1}),\mathbb{1})$. We consider a toral cuspidal $\mathbf{G}_{\eta'}$-datum $((\mathbf{S}^\natural\subset\mathbf{G}_{\eta'}),\mathbf{x},(r,r),(\vartheta^\natural,\mathbb{1}),\mathbb{1})$, where we put $\vartheta^\natural := \vartheta|_{S^\natural}$. (Note that the torality is guaranteed by Lemma 5.4.) We express various objects appearing in Yu's construction for this cuspidal $\mathbf{G}_{\eta'}$-datum by adding a subscript η' to the notation used in Section 4.1. Then, again by using Lemma 5.10, Proposition 5.8, and [AS09, Lemma 2.5], we have
$$\dot{\Theta}_{\sigma_{\eta'}}({}^{ghk}\delta_{\geq r}) = \mathbb{1}_{G_{\eta',\mathbf{x},r}}({}^{ghk}\delta_{\geq r})\hat{\vartheta}_{\eta'}({}^{ghk}\delta_{\geq r}).$$
Namely, we get
$$\tilde{\mathcal{F}}_g(h) = \int_{\mathcal{K}_\eta} \dot{\Theta}_{\sigma_{\eta'}}({}^{ghk}\delta_{\geq r})\,dk.$$
Since the representation $\text{c-Ind}_{K_{\sigma_{\eta'}}}^{G_{\eta'}}\sigma_{\eta'}$ is supercuspidal by Yu's theory, this function is compactly supported modulo $Z_{\mathbf{G}_{\eta'}}$, by Harish-Chandra's well-known result ([HC70, Lemma 23]). Therefore, now our assertion is reduced to the compactness of the quotient $Z_{\mathbf{G}_{\eta'}}/A_{\tilde{\mathbf{G}}}$.

Since we have $\mathbf{G}_{\eta'}\supset\mathbf{S}^\natural$, we have $\mathbf{Z}_{\mathbf{G}_{\eta'}}\subset\mathbf{S}^\natural$. As $\tilde{\mathbf{S}}$ is an F-rational elliptic twisted maximal torus of $\tilde{\mathbf{G}}$, $\mathbf{S}^\natural$ is anisotropic modulo $\mathbf{A}_{\tilde{\mathbf{G}}}$ (see Definition 3.4), hence $S_{\eta'}$ is compact modulo $A_{\tilde{\mathbf{G}}}$. Thus $Z_{\mathbf{G}_{\eta'}}$ is compact modulo $A_{\tilde{\mathbf{G}}}$. □

Before we state the "first form" of a twisted version of Adler–DeBacker–Spice character formula, we introduce some notation. Recall that, for any connected reductive group $\mathbf{J}$ and a regular semisimple element $X_J^*\in\mathfrak{j}^*$, the Fourier transform of the orbital integral $\hat{\mu}^{\mathbf{J}}_{X_J^*}$ is defined as follows (see [Kal19b, Section 4.2] for the details). We consider a distribution $O_{X_J^*}(-)$ on $\mathfrak{j}^*$ given by
$$O_{X_J^*}(f) := \int_{J/Z_{\mathbf{J}}(X^*)^\circ} f^*(hX_J^*h^{-1})\,dh$$
for $f^*\in C_c^\infty(\mathfrak{j}^*)$, where we fix a Haar measure dh on J. For any element $f\in C_c^\infty(\mathfrak{j})$, we let $\hat{f}$ denote its Fourier transform with respect to the fixed additive character ψ_F, that is, $\hat{f}$ is an element of $C_c^\infty(\mathfrak{j}^*)$ given by
$$\hat{f}(Y^*) := \int_{\mathfrak{j}} f(Y)\cdot\psi_F(\langle Y,Y^*\rangle)\,dY,$$
where dY is a Haar measure on $\mathfrak{j}$. Then the distribution $f\mapsto O_{X_J^*}(\hat{f})$ on $\mathfrak{j}$ is represented by a function $\hat{\mu}^{\mathbf{J}}_{X_J^*}$ on $\mathfrak{j}$, i.e., we have
$$O_{X_J^*}(\hat{f}) = \int_{\mathfrak{j}} \hat{\mu}^{\mathbf{J}}_{X_J^*}(Y)\cdot f(Y)\,dY$$
for any $f\in C_c^\infty(\mathfrak{j})$. We emphasize that the function $\hat{\mu}^{\mathbf{J}}_{X_J^*}$ does not depend on the choice of dY, but depends on the choice of dh.

Recall that, as discussed in the proof of Lemma 5.11, we have a tame elliptic toral pair $(\mathbf{S}^\natural,\vartheta^\natural)$ of $\mathbf{G}_{\eta'}$ (here, $\eta' := {}^{g}\eta$ for an element $g\in G$ satisfying ${}^{g}\eta\in\tilde{S}$)

represented by $X^* \in \mathfrak{s}^{\natural *}_{-r}$, which is the image of the fixed element $X^* \in (\mathfrak{s}^*_{-r})^{\theta_{\mathbf{S}}}$ representing the character $\vartheta|_{S_r}$ (see Section 5.2). Since $X^* \in \mathfrak{s}^{\natural *}_{-r}$, which is regarded as an element of $\mathfrak{g}^*_{\eta'}$, is $\mathbf{G}_{\eta'}$-generic of depth r, hence regular semisimple in $\mathbf{G}_{\eta'}$. Hence we have $\mathbf{Z}_{\mathbf{G}_{\eta'}}(X^*)^\circ = \mathbf{S}^\natural$. By noting that $G_{\eta'}/S^\natural$ is the quotient of $G_{\eta'}/(G_{\eta'} \cap Z_{\mathbf{G}})$ by $S^\natural/(G_{\eta'} \cap Z_{\mathbf{G}})$, we choose a measure on $G_{\eta'}/S^\natural$ which is the quotient of the following two measures:

- the Haar measure dh on $G_{\eta'}/(G_{\eta'} \cap Z_{\mathbf{G}})$ satisfying $dh\big((G_{\eta'} \cap K_\sigma)/(G_{\eta'} \cap Z_{\mathbf{G}})\big) = 1$;
- the Haar measure on $S^\natural/(G_{\eta'} \cap Z_{\mathbf{G}})$ whose total volume is 1 (note that $G_{\eta'} \cap Z_{\mathbf{G}}$ is co-compact in $S^\natural$, which follows from that $S^\natural$ is compact modulo $Z_{\mathbf{G}_{\eta'}}$; cf. the final step of the proof of Lemma 5.11).

The following is a twisted version of [AS09, Theorem 6.4]:

Theorem 5.12. *We have*

$$\Theta_{\tilde{\pi}}(\delta) = \sum_{\substack{g \in S\backslash G/G_\eta \\ {}^g\eta \in \tilde{S}}} \Theta_{\tilde{\sigma}}({}^g\eta) \cdot \hat{\mu}^{\mathbf{G}_{{}^g\eta}}_{X^*}\big(\log({}^g\delta_{\geq r})\big). \tag{6}$$

Here, note that the condition ${}^g\eta \in \tilde{S}$ implies that $\mathbf{S}^\natural \subset \mathbf{G}_{{}^g\eta}$, hence the function $\hat{\mu}^{\mathbf{G}_{{}^g\eta}}_{X^*}(-)$ makes sense as explained above. In the definition of $\hat{\mu}^{\mathbf{G}_{{}^g\eta}}_{X^*}$, we use the Haar measure on $G_{{}^g\eta}/S^\natural$ explained above.

Proof. The starting point of the proof is the twisted version of Harish-Chandra's integration formula (see [LH17, Partie I, Théorème 6.2.1 (2)]):

$$\Theta_{\tilde{\pi}}(\delta) = \frac{\deg \pi}{\dim \sigma} \int_{G/Z_{\mathbf{G}}} \int_{\mathcal{K}} \dot{\Theta}_{\tilde{\sigma}}({}^{\dot{g}k}\delta)\, dk\, d\dot{g},$$

where $\mathcal{K}$ is an open compact subgroup of G, dk is the Haar measure on $\mathcal{K}$ satisfying $dk(\mathcal{K}) = 1$, and $d\dot{g}$ is a Haar measure on $G/Z_{\mathbf{G}}$ and $\deg \pi$ denotes the formal degree of π with respect to the measure $d\dot{g}$.

We take an open compact subgroup $\mathcal{K}_\eta$ of G_η to be $\mathcal{K}_\eta = \mathcal{K} \cap G_\eta$. We let dc be the Haar measure of $\mathcal{K}_\eta$ satisfying $dc(\mathcal{K}_\eta) = 1$. Then we can replace the integral over $\mathcal{K}$ in Harish-Chandra's integration formula with an integral over $\mathcal{K}_\eta$ by the following standard argument. First, since $\mathcal{K}_\eta \subset \mathcal{K}$ and $dc(\mathcal{K}_\eta) = 1$, we have

$$\int_{G/Z_{\mathbf{G}}} \int_{\mathcal{K}} \dot{\Theta}_{\tilde{\sigma}}({}^{\dot{g}k}\delta)\, dk\, d\dot{g} = \int_{G/Z_{\mathbf{G}}} \int_{\mathcal{K}_\eta} \int_{\mathcal{K}} \dot{\Theta}_{\tilde{\sigma}}({}^{\dot{g}kc}\delta)\, dk\, dc\, d\dot{g}.$$

By applying Fubini's theorem to the inner double integral (note that both of $\mathcal{K}_\eta$ and $\mathcal{K}$ are compact), we get

$$\int_{G/Z_{\mathbf{G}}} \int_{\mathcal{K}_\eta} \int_{\mathcal{K}} \dot{\Theta}_{\tilde{\sigma}}({}^{\dot{g}kc}\delta)\, dk\, dc\, d\dot{g} = \int_{G/Z_{\mathbf{G}}} \int_{\mathcal{K}} \int_{\mathcal{K}_\eta} \dot{\Theta}_{\tilde{\sigma}}({}^{\dot{g}kc}\delta)\, dc\, dk\, d\dot{g}.$$

Then, since the inner integral over $\mathcal{K}_\eta$ is compactly supported as a function on $\dot{g} \in G/Z_{\mathbf{G}}$ (Lemma 5.11), we can apply Fubini's theorem to the outer double integral:

$$\int_{G/Z_{\mathbf{G}}} \int_{\mathcal{K}} \int_{\mathcal{K}_\eta} \dot{\Theta}_{\tilde{\sigma}}({}^{\dot{g}kc}\delta)\, dc\, dk\, d\dot{g} = \int_{\mathcal{K}} \int_{G/Z_{\mathbf{G}}} \int_{\mathcal{K}_\eta} \dot{\Theta}_{\tilde{\sigma}}({}^{\dot{g}kc}\delta)\, dc\, d\dot{g}\, dk.$$

Finally, by using that $d\dot{g}$ is right G-invariant and that $dk(\mathcal{K})=1$, we get

$$\int_{\mathcal{K}}\int_{G/Z_{\mathbf{G}}}\int_{\mathcal{K}_\eta}\dot{\Theta}_{\tilde{\sigma}}({}^{\dot{g}kc}\delta)\,dc\,d\dot{g}\,dk=\int_{G/Z_{\mathbf{G}}}\int_{\mathcal{K}_\eta}\dot{\Theta}_{\tilde{\sigma}}({}^{\dot{g}c}\delta)\,dc\,d\dot{g}.$$

Now we consider the following partition of $G/Z_{\mathbf{G}}$ into double cosets:

$$\int_{G/Z_{\mathbf{G}}}\int_{\mathcal{K}_\eta}\dot{\Theta}_{\tilde{\sigma}}({}^{\dot{g}c}\delta)\,dc\,d\dot{g}=\sum_{g\in K_\sigma\backslash G/G_\eta}\int_{K_\sigma gG_\eta/Z_{\mathbf{G}}}\int_{\mathcal{K}_\eta}\dot{\Theta}_{\tilde{\sigma}}({}^{\dot{g}c}\delta)\,dc\,d\dot{g}.$$

Note that, by Proposition 5.8 and Lemma 5.10, if the contribution of the summand with respect to $g\in K_\sigma\backslash G/G_\eta$ is nonzero, then there exists an element g' in the double coset satisfying ${}^{g'}\eta\in{}^{G_{\mathbf{x},0+}}\tilde{S}$. By Lemma 5.13, which will be proved later, the natural surjective map

$$S\backslash\bigl\{g\in G\mid {}^{g}\eta\in\tilde{S}\bigr\}/G_\eta\twoheadrightarrow K_\sigma\backslash\bigl\{g\in G\mid {}^{g}\eta\in{}^{G_{\mathbf{x},0+}}\tilde{S}\bigr\}/G_\eta$$

is in fact bijective. Hence we see that the above sum of double integrals equals

$$\sum_{\substack{g\in S\backslash G/G_\eta\\ {}^{g}\eta\in\tilde{S}}}\int_{K_\sigma gG_\eta/Z_{\mathbf{G}}}\int_{\mathcal{K}_\eta}\dot{\Theta}_{\tilde{\sigma}}({}^{\dot{g}c}\delta)\,dc\,d\dot{g}. \tag{7}$$

Let us compute each summand by fixing $g\in S\backslash G/G_\eta$ satisfying ${}^{g}\eta\in\tilde{S}$. We put

$$y:=\dot{g}g^{-1}\quad\text{and}\quad c':={}^{g}c=gcg^{-1}.$$

Then, letting dy and dc' be the Haar measures on $K_\sigma{}^{g}G_\eta/Z_{\mathbf{G}}$ and ${}^{g}\mathcal{K}_\eta$ naturally induced from $d\dot{g}$ and dc, respectively, we get

$$\int_{K_\sigma gG_\eta/Z_{\mathbf{G}}}\int_{\mathcal{K}_\eta}\dot{\Theta}_{\tilde{\sigma}}({}^{\dot{g}c}\delta)\,dc\,d\dot{g}=\int_{K_\sigma{}^{g}G_\eta/Z_{\mathbf{G}}}\int_{{}^{g}\mathcal{K}_\eta}\dot{\Theta}_{\tilde{\sigma}}({}^{yc'g}\delta)\,dc'\,dy.$$

By putting $\delta':={}^{g}\delta$, $\eta':={}^{g}\eta$, and $\delta'_{\geq r}:={}^{g}\delta_{\geq r}$, we get

$$\int_{K_\sigma{}^{g}G_\eta/Z_{\mathbf{G}}}\int_{{}^{g}\mathcal{K}_\eta}\dot{\Theta}_{\tilde{\sigma}}({}^{yc'g}\delta)\,dc'\,dy=\int_{K_\sigma G_{\eta'}/Z_{\mathbf{G}}}\int_{\mathcal{K}_{\eta'}}\dot{\Theta}_{\tilde{\sigma}}({}^{yc'}\delta')\,dc'\,dy.$$

We let dh be the Haar measure on $G_{\eta'}/(G_{\eta'}\cap Z_{\mathbf{G}})\cong G_{\eta'}Z_{\mathbf{G}}/Z_{\mathbf{G}}$ normalized so that $dh\bigl((G_{\eta'}\cap K_\sigma)/(G_{\eta'}\cap Z_{\mathbf{G}})\bigr)=1$. Let $d\dot{y}$ be the quotient measure on $K_\sigma G_{\eta'}/G_{\eta'}Z_{\mathbf{G}}$ of dy by dh. Then we have

$$\begin{aligned}&\int_{K_\sigma G_{\eta'}/Z_{\mathbf{G}}}\int_{\mathcal{K}_{\eta'}}\dot{\Theta}_{\tilde{\sigma}}({}^{yc'}\delta')\,dc'\,dy\\&\qquad=\int_{K_\sigma G_{\eta'}/G_{\eta'}Z_{\mathbf{G}}}\int_{G_{\eta'}/G_{\eta'}\cap Z_{\mathbf{G}}}\int_{\mathcal{K}_{\eta'}}\dot{\Theta}_{\tilde{\sigma}}({}^{\dot{y}hc'}\delta')\,dc'\,dh\,d\dot{y}.\end{aligned}$$

Since $\dot{\Theta}_{\tilde{\sigma}}$ is left-K_σ-invariant, this triple integral equals

$$d\dot{y}(K_\sigma G_{\eta'}/G_{\eta'}Z_{\mathbf{G}})\int_{G_{\eta'}/G_{\eta'}\cap Z_{\mathbf{G}}}\int_{\mathcal{K}_{\eta'}}\dot{\Theta}_{\tilde{\sigma}}({}^{hc'}\delta')\,dc'\,dh. \tag{8}$$

Let us compute the volume $d\dot{y}(K_\sigma G_{\eta'}/G_{\eta'}Z_{\mathbf{G}})$. Since $K_\sigma G_{\eta'}/G_{\eta'}Z_{\mathbf{G}}\cong K_\sigma/(G_{\eta'}\cap K_\sigma)Z_{\mathbf{G}}$ is equal to the quotient of $K_\sigma/Z_{\mathbf{G}}$ by $(G_{\eta'}\cap K_\sigma)Z_{\mathbf{G}}/Z_{\mathbf{G}}\cong(G_{\eta'}\cap K_\sigma)/(G_{\eta'}\cap Z_{\mathbf{G}})$, the volume $d\dot{y}(K_\sigma G_{\eta'}/G_{\eta'}Z_{\mathbf{G}})$ is given by

$$dy(K_\sigma/Z_{\mathbf{G}})\cdot dh\bigl((G_{\eta'}\cap K_\sigma)/(G_{\eta'}\cap Z_{\mathbf{G}})\bigr)^{-1}.$$

By our choice of dh, we have $dh((G_{\eta'} \cap K_\sigma)/(G_{\eta'} \cap Z_{\mathbf{G}})) = 1$. On the other hand, we have

$$dy(K_\sigma/Z_{\mathbf{G}}) = d\dot{g}(K_\sigma g/Z_{\mathbf{G}}) = d\dot{g}(K_\sigma/Z_{\mathbf{G}}) = \frac{\dim\sigma}{\deg\pi}.$$

(The final equality is a well-known formula for the formal degree of a compactly induced supercuspidal representation; see, for example, [LH17, Partie I, Théorème 6.2.1 (1)]. Recall that π is the compact induction of σ from K_σ to G.) Hence we obtain $d\dot{y}(K_\sigma G_{\eta'}/G_{\eta'}Z_{\mathbf{G}}) = \dim\sigma/\deg\pi$.

Let us next compute the double integral in (8). Recall that in the proof of Lemma 5.11 we showed that

$$\dot{\Theta}_{\tilde{\sigma}}({}^{hc'}\delta') = \Theta_{\tilde{\sigma}}(\eta')\mathbb{1}_{G_{\eta',\mathbf{x},r}}({}^{hc'}\delta'_{\geq r})\hat{\vartheta}({}^{hc'}\delta'_{\geq r}).$$

Note that δ'_+ is regular semisimple in $\mathbf{G}_{\delta'_0}$ by Lemma 3.16. As $\delta'_+ = \delta'^{+}_{<r}\delta'_{\geq r}$ is a normal r-approximation in $\mathbf{G}_{\delta'_0}$, the regular semisimplicity of δ'_+ in $\mathbf{G}_{\delta'_0}$ implies that of $\delta'_{\geq r}$ in $(\mathbf{G}_{\delta'_0})_{\delta'^{+}_{<r}} = \mathbf{G}_{\eta'}$ (Lemma 3.19 and [AS08, Corollary 6.14]). Thus $\log(\delta'_{\geq r}) \in \mathfrak{g}_{\eta',\mathbf{x},r}$ is also regular semisimple. By the orbital integral formula of Adler–Spice [AS09, Lemma B.4], we get

$$\int_{G_{\eta'}/G_{\eta'}\cap Z_{\mathbf{G}}}\int_{\mathcal{K}_{\eta'}} \mathbb{1}_{G_{\eta',\mathbf{x},r}}({}^{hc'}\delta'_{\geq r})\hat{\vartheta}({}^{hc'}\delta'_{\geq r})\, dc'\, dh = \hat{\mu}^{\mathbf{G}_{\eta'}}_{X^*}\big(\log(\delta'_{\geq r})\big).$$

□

Lemma 5.13. *The natural surjective map*

$$S\backslash\{g \in G \mid {}^{g}\eta \in \tilde{S}\}/G_\eta \twoheadrightarrow K_\sigma\backslash\{g \in G \mid {}^{g}\eta \in {}^{G_{\mathbf{x},0+}}\tilde{S}\}/G_\eta$$

is bijective.

Proof. Suppose that two double cosets SgG_η and $Sg'G_\eta$ map to the same double coset $K_\sigma gG_\eta$. Then, as $K_\sigma = SG_{\mathbf{x},0+}$ and S normalizes $G_{\mathbf{x},0+}$, we may assume that g' is given by kg with some $k \in G_{\mathbf{x},0+}$. We write $\eta' := {}^{g}\eta$. As we have $SgG_\eta = SG_{\eta'}g$ and $SkgG_\eta = SkG_{\eta'}g$, it suffices to show that $SG_{\eta'} = SkG_{\eta'}$ (for $\eta' \in \tilde{S}$ and $k \in G_{\mathbf{x},0+}$ satisfying ${}^{k}\eta' \in \tilde{S}$).

Let $\eta' = \eta'_0\eta'_+$ and ${}^{k}\eta' = {}^{k}\eta'_0{}^{k}\eta'_+$ be the topological Jordan decompositions induced from $\eta = \eta_0\eta_+$. Let p' be the order of η_0 modulo $A_{\tilde{\mathbf{G}}}$, which is prime-to-p. By Lemma 3.18, the conditions $\eta', {}^{k}\eta' \in \tilde{S}$ implies that $\eta'_0, {}^{k}\eta'_0 \in \tilde{S}$. Thus there exists an element $s_+ \in S$ such that ${}^{k}\eta'_0 = s_+\eta'_0$. By noting that

$$s_+ = {}^{k}\eta'_0 \cdot {\eta'_0}^{-1} = k \cdot (\eta'_0 k {\eta'_0}^{-1})^{-1} \in G_{\mathbf{x},0+},$$

s_+ belongs to $S \cap G_{\mathbf{x},0+} = S_{0+}$. Since the order of η'_0 and ${}^{k}\eta'_0$ modulo $A_{\tilde{\mathbf{G}}}$ is given by p', we get

$$1 = {}^{k}{\eta'_0}^{p'} = \prod_{i=0}^{p'-1}[\eta'_0]^i(s_+) \cdot {\eta'_0}^{p'} = \prod_{i=0}^{p'-1}[\eta'_0]^i(s_+)$$

in $S/A_{\tilde{\mathbf{G}}}$. Thus, by the same argument as in the proof of Lemma 5.7 (i.e., using the vanishing of $H^1(\mathbb{Z}/p'\mathbb{Z}, \overline{S_{0+}})$), we can find an element $t_+ \in S_{0+}$ satisfying $t_+s_+[\eta'_0](t_+)^{-1} \in A_{\tilde{\mathbf{G}}}$, hence $t_+s_+[\eta'_0](t_+)^{-1} \in A_{\tilde{\mathbf{G}},0+}$. Then we have

$${}^{t_+k}\eta'_0 = {}^{t_+}(s_+\eta'_0) = t_+s_+[\eta'_0](t_+)^{-1} \cdot \eta'_0.$$

Hence, by replacing k with t_+k, we may assume that ${}^k\eta_0' = a\eta_0'$ for some $a \in A_{\tilde{\mathbf{G}},0+}$. In other words, the image $\overline{k}$ of k in $G_{\mathbf{x},0+}/A_{\tilde{\mathbf{G}},0+}$ is fixed by $[\eta_0']$. We note that the short exact sequence

$$1 \to A_{\tilde{\mathbf{G}},0+} \to G_{\mathbf{x},0+} \to G_{\mathbf{x},0+}/A_{\tilde{\mathbf{G}},0+} \to 1$$

induces an exact sequence

$$1 \to A_{\tilde{\mathbf{G}},0+}^{\eta_0'} \to G_{\mathbf{x},0+}^{\eta_0'} \to (G_{\mathbf{x},0+}/A_{\tilde{\mathbf{G}},0+})^{\eta_0'} \to H^1(\langle[\eta_0']\rangle, A_{\tilde{\mathbf{G}},0+})$$

and that $H^1(\langle[\eta_0']\rangle, A_{\tilde{\mathbf{G}},0+})$ vanishes since $\langle[\eta_0']\rangle$ is of order 2 and $A_{\tilde{\mathbf{G}},0+}$ is a pro-p group. Thus we can find an element $k' \in G_{\mathbf{x},0+}^{\eta_0'}$ whose image $\overline{k'}$ in $G_{\mathbf{x},0+}/A_{\tilde{\mathbf{G}},0+}$ equals $\overline{k}$. As we have $G_{\mathbf{x},0+}^{\eta_0'} = G_{\eta_0',\mathbf{x},0+}$ by Proposition 5.6 (2), this implies that k belongs to $G_{\eta_0',\mathbf{x},0+}A_{\tilde{\mathbf{G}},0+} = G_{\eta_0',\mathbf{x},0+}$.

Now we utilize [AS08, Lemma 9.10], which asserts that if

- $(\mathbf{G}', \mathbf{G})$ is a tame reductive F-sequence,
- $\gamma \in G$ is an element having a normal r-approximation,
- $\mathbf{x} \in \mathcal{B}(C_{\mathbf{G}}^{(r)}(\gamma), F) \cap \mathcal{B}(\mathbf{G}', F)$,
- $k \in G_{\mathbf{x},0+}$, and
- $Z_G^{(r)}(\gamma) \subset G'$ and ${}^kZ_G^{(r)}(\gamma) \subset G'$,

the element k belongs to $G'_{\mathbf{x},0+}C_G^{(r)}(\gamma)_{\mathbf{x},0+}$. If we take

- $(\mathbf{G}, \mathbf{G}') := (\mathbf{G}_{\eta_0'}, \mathbf{S}^\natural)$,
- $\gamma := \eta_+'$ (then $C_{\mathbf{G}_{\eta_0'}}^{(r)}(\gamma) = (\mathbf{G}_{\eta_0'})_{\eta_+'} = \mathbf{G}_{\eta'}$ by Lemma 3.19),
- $\mathbf{x}$ to be the point $\mathbf{x}$ belonging to $\mathcal{A}(\mathbf{S}^\natural, F)$, and
- $k \in G_{\eta_0',\mathbf{x},0+}$,

then the assumptions of [AS08, Lemma 9.10] are satisfied. Indeed, we have $Z_G^{(r)}(\gamma) = Z_{C_{\mathbf{G}}^{(r)}(\gamma)} = Z_{\mathbf{G}_{\eta'}}$. As we have $\eta' \in \tilde{S}$, we have $G_{\eta'} \supset S^\natural$, hence $Z_G^{(r)}(\gamma) \subset S^\natural$. Similarly, we have ${}^kZ_G^{(r)}(\gamma) \subset S^\natural$. Thus we conclude that k belongs to $S_{0+}^\natural G_{\eta',\mathbf{x},0+} = G_{\eta',\mathbf{x},0+}$. In particular, we get $SG_{\eta'} = SkG_{\eta'}$. □

Now, by Theorem 5.12, our task is to describe each summand of the right-hand side of (6). We next show the following proposition, which is a twisted version of [AS09, Proposition 5.3.2]:

Proposition 5.14. *For any element $\eta' \in \tilde{S}$ with a topological Jordan decomposition $\eta' = \eta_0'\eta_+'$, we have*

$$\Theta_{\tilde{\sigma}}(\eta') = \sum_{g \in S_{0+}^\natural G_{\eta_0',\mathbf{x},s} \backslash [\![\eta_+';\mathbf{x},r]\!]_{G_{\eta_0'}}^{(s)}} \Theta_{\tilde{\rho}}({}^g\eta'),$$

where $[\![\eta_+';\mathbf{x},r]\!]_{G_{\eta_0'}}^{(s)}$ is the subgroup as in [AS08, Definition 6.6] *(taken in $G_{\eta_0'}$).*

Proof. Recall that the representation σ of $SG_{\mathbf{x},0+}$ is defined by inducing the representation ρ of $SG_{\mathbf{x},s}$. Thus, by the Frobenius character formula for induced representations, we have

$$\Theta_{\tilde{\sigma}}(\eta') = \sum_{\substack{g \in SG_{\mathbf{x},s} \backslash SG_{\mathbf{x},0+} \\ {}^g\eta' \in \tilde{S}G_{\mathbf{x},s}}} \Theta_{\tilde{\rho}}({}^g\eta') = \sum_{\substack{g \in S_{0+}G_{\mathbf{x},s} \backslash G_{\mathbf{x},0+} \\ {}^g\eta' \in \tilde{S}G_{\mathbf{x},s}}} \Theta_{\tilde{\rho}}({}^g\eta').$$

Let us show that the index set on the most-right-hand side can be replaced with the set $\{g \in S^{\natural}_{0+}G_{\eta'_0,\mathbf{x},s}\backslash G_{\eta'_0,\mathbf{x},0+} \mid {}^{g}\eta'_{+} \in S^{\natural}_{0+}G_{\eta'_0,\mathbf{x},s}\}$. Suppose that a coset in $S_{0+}G_{\mathbf{x},s}\backslash G_{\mathbf{x},0+}$ contains an element g satisfying ${}^{g}\eta' \in \tilde{S}G_{\mathbf{x},s}$. Then, since ${}^{g}\eta' = {}^{g}\eta'_0 {}^{g}\eta'_{+}$ gives a topological Jordan decomposition of ${}^{g}\eta'$, Proposition 3.14 (4) implies that ${}^{g}\eta'_0 \in \tilde{S}G_{\mathbf{x},s}$. On the other hand, as η' belongs to $\tilde{S}$, we have $\eta'_0 \in \tilde{S}$ (this also follows from Proposition 3.14 (4)). Thus there exists an element $g_{+} \in SG_{\mathbf{x},s}$ satisfying ${}^{g}\eta'_0 = g_{+}\eta'_0$. As we have

$$g_{+} = {}^{g}\eta'_0\eta'^{-1}_0 = g \cdot \eta'_0 g^{-1}\eta'^{-1}_0 \in G_{\mathbf{x},0+},$$

g_{+} belongs to $SG_{\mathbf{x},s} \cap G_{\mathbf{x},0+} = S_{0+}G_{\mathbf{x},s}$. Then, by the same argument as in the proofs of Lemma 5.7 or Lemma 5.13 using the vanishing of $H^1(\mathbb{Z}/p'\mathbb{Z}, S_{0+}G_{\mathbf{x},s})$, we can find an element $k \in S_{0+}G_{\mathbf{x},s}$ such that ${}^{kg}\eta'_0 = \eta'_0$. Therefore, by replacing g with kg, we may assume that g belongs to $G_{\mathbf{x},0+} \cap G^{\eta'_0} = G_{\eta'_0,\mathbf{x},0+}$ (Proposition 5.6 (2)). In other words, we may assume that our coset in $S_{0+}G_{\mathbf{x},s}\backslash G_{\mathbf{x},0+}$ comes from a coset in $S^{\natural}_{0+}G_{\eta'_0,\mathbf{x},s}\backslash G_{\eta'_0,\mathbf{x},0+}$ (note that $S_{0+}G_{\mathbf{x},s} \cap G_{\eta'_0,\mathbf{x},0+} = S^{\natural}_{0+}G_{\eta'_0,\mathbf{x},s}$ by Proposition 5.6 (3)). Furthermore, as ${}^{g}\eta'_{+}$ belongs to $S_{0+}G_{\mathbf{x},s}$ and commutes with ${}^{g}\eta'_0 = \eta'_0$, we get ${}^{g}\eta'_{+} \in S_{0+}G_{\mathbf{x},s} \cap G^{\eta'_0} = S^{\natural}_{0+}G_{\eta'_0,\mathbf{x},s}$.

Now the same argument as in the proof of [AS09, Proposition 5.3.2] can be applied to the descended group $G_{\eta'_0}$. (One of the most important inputs in the proof of [AS09, Proposition 5.3.2] is the $\hat{\vartheta}$-isotypicity of the representation. In the current situation, ρ is $\hat{\vartheta}$-isotypic on $G_{\mathbf{x},r}$, hence also on $G_{\eta'_0,\mathbf{x},r}$.) Then we get

$$\Theta_{\tilde{\sigma}}(\eta') = \sum_{\substack{g \in S^{\natural}_{0+}G_{\eta'_0,\mathbf{x},s}\backslash G_{\eta'_0,\mathbf{x},0+} \\ {}^{g}\eta'_{+} \in S^{\natural}_{0+}G_{\eta'_0,\mathbf{x},s}}} \Theta_{\tilde{\rho}}({}^{g}\eta') = \sum_{g \in S^{\natural}_{0+}G_{\eta'_0,\mathbf{x},s}\backslash [\![\eta'_{+};\mathbf{x},r]\!]^{(s)}_{G_{\eta'_0}}} \Theta_{\tilde{\rho}}({}^{g}\eta').$$

(We caution that our notation are different from those of Adler–Spice. Especially, the representation $\tilde{\rho}$ in [DS18, Proposition 5.3.2] is nothing but our $\rho_{(\mathbf{S},\vartheta)}$. In our notation, the symbol $\sim$ basically denotes the twist of a representation.) □

Corollary 5.15. *Let $\eta \in \tilde{S}$ be an element with a topological Jordan decomposition $\eta' = \eta'_0\eta'_{+}$. Then we have*

$$\Theta_{\tilde{\sigma}}(\eta') = \Theta_{\tilde{\rho}}(\eta') \cdot |\tilde{\mathfrak{G}}_{\mathbf{G}_{\eta'_0}}(\vartheta, \eta'_{+})| \cdot \mathfrak{G}_{\mathbf{G}_{\eta'_0}}(\vartheta, \eta'_{+}),$$

where $\tilde{\mathfrak{G}}_{\mathbf{G}_{\eta'_0}}(\vartheta, \eta'_{+})$ and $\mathfrak{G}_{\mathbf{G}_{\eta'_0}}(\vartheta, \eta'_{+})$ are the quantities defined in [AS09, Definition 5.2.4] *(in the group $\mathbf{G}_{\eta'_0}$).*

Proof. By Proposition 5.14, $\Theta_{\tilde{\sigma}}(\eta')$ is given by the sum of $\Theta_{\tilde{\rho}}({}^{g}\eta')$ over the set $g \in S^{\natural}_{0+}G_{\eta'_0,\mathbf{x},s}\backslash [\![\eta'_{+};\mathbf{x},r]\!]^{(s)}_{G_{\eta'_0}}$. For any element $g \in [\![\eta'_{+};\mathbf{x},r]\!]^{(s)}_{G_{\eta'_0}}$, we have

$${}^{g}\eta' = g\eta' g^{-1} = \eta' \cdot \eta'^{-1} g\eta' g^{-1} = \eta' \cdot [\eta'^{-1}, g] = \eta' \cdot [\eta'^{-1}_{+}, g].$$

As g belongs to $[\![\eta_{+};\mathbf{x},r]\!]^{(s)}_{G_{\eta_0}}$, we know that $[\eta^{-1}_{+}, g]$ belongs to J_{+}. (Note that this is fact is necessary also in [AS09, Definition 5.2.4] and essentially proved in [AS09, Section 5]; one can verify this property by using [AS08, Lemmas 5.30 and 5.32]). By $\hat{\vartheta}$-isotypicity of ρ on J_{+}, we have

$$\sum_{g \in S^{\natural}_{0+}G_{\eta'_0,\mathbf{x},s}\backslash [\![\eta'_{+};\mathbf{x},r]\!]^{(s)}_{G_{\eta'_0}}} \Theta_{\tilde{\rho}}({}^{g}\eta') = \Theta_{\tilde{\rho}}(\eta') \sum_{g \in S^{\natural}_{0+}G_{\eta'_0,\mathbf{x},s}\backslash [\![\eta'_{+};\mathbf{x},r]\!]^{(s)}_{G_{\eta'_0}}} \hat{\vartheta}([\eta'^{-1}_{+}, g]).$$

The sum on the right-hand side is nothing but $\tilde{\mathfrak{G}}_{\mathbf{G}_{\eta'_0}}(\vartheta, \eta'_+)$ by definition. Since we have $\tilde{\mathfrak{G}}_{\mathbf{G}_{\eta'_0}}(\vartheta, \eta'_+) = |\tilde{\mathfrak{G}}_{\mathbf{G}_{\eta'_0}}(\vartheta, \eta'_+)| \cdot \mathfrak{G}_{\mathbf{G}_{\eta'_0}}(\vartheta, \eta'_+)$, we get the assertion. $\square$

In summary, by combining Theorem 5.12 with Corollary 5.15, we obtain the following:

Theorem 5.16. *We have*

$$(9)\quad \Theta_{\tilde{\pi}}(\delta) = \sum_{\substack{g \in S\backslash G/G_\eta \\ {}^g\eta \in \tilde{S}}} \Theta_{\tilde{\rho}}({}^g\eta) \cdot |\tilde{\mathfrak{G}}_{\mathbf{G}_{{}^g\eta_0}}(\vartheta, {}^g\eta_+)| \cdot \mathfrak{G}_{\mathbf{G}_{{}^g\eta_0}}(\vartheta, {}^g\eta_+) \cdot \hat{\mu}^{\mathbf{G}_{{}^g\eta}}_{X^*}\bigl(\log({}^g\delta_{\geq r})\bigr).$$

6. Twisted Adler–DeBacker–Spice formula

Let us keep the notation as in the previous section. Our aim in this section is to compute $\Theta_{\tilde{\rho}}({}^g\eta)$ in each summand of (9). Recall that the representation $\rho = \rho_{(\mathbf{S},\vartheta)}$ is defined by descending $\omega_{(\mathbf{S},\vartheta)} \otimes (\vartheta \ltimes \mathbb{1})$ from $S \ltimes J$ to $K = SJ$. Hence, noting that ${}^g\eta \in \tilde{S}$, the computation of $\Theta_{\tilde{\rho}}({}^g\eta)$ is reduced to the computation of the twisted character of the Weil representation $\omega_{(\mathbf{S},\vartheta)}$. For this, we repeat the computations in the proofs of [AS09, Proposition 3.8] and [DS18, Proposition 4.21] by taking the effect of the "twist" into consideration.

In the following (the rest of this paper), we assume that

no restricted root of type 2 or 3 appears in $\Phi_{\mathrm{res}}(\mathbf{G}, \mathbf{T})$.

Remark 6.1. We believe that this assumption is harmless for our purpose, namely, study of the θ-stable toral supercuspidal representations. As explained in Remark 3.7, restricted roots of type 2 or 3 appear only when $\mathbf{G}$ contains a factor of type A_{2n} on which θ acts nontrivially. However, it is known that GL_{2n+1} does not have θ-stable irreducible supercuspidal representation for such a θ whenever $p \neq 2$ (see, e.g., [Pra99, Proposition 4]). Hence, since we are assuming that $p \neq 2$, this assumption does not cause any additional constraint.

6.1. **Structure of the Heisenberg quotient.** We first recall the description of the group J/J_+ according to Adler–Spice ([AS09, Proof of Proposition 3.8]). By fixing a finite tamely ramified extension E of F splitting $\mathbf{S}$, we put

$$\mathbf{V} := \mathrm{Lie}(\mathbf{S}, \mathbf{G})(E)_{\mathbf{x},(r,s):(r,s+)} \quad \text{and} \quad V := \mathbf{V}^\Gamma.$$

Recall that we have

$$J/J_+ \cong (S, G)_{\mathbf{x},(r,s):(r,s+)} = (\mathbf{S}, \mathbf{G})(E)^\Gamma_{\mathbf{x},(r,s):(r,s+)}.$$

Thus the exponential map $\mathrm{Lie}(\mathbf{S}, \mathbf{G})(E)_{\mathbf{x},(r,s):(r,s+)} \xrightarrow{\sim} (\mathbf{S}, \mathbf{G})(E)_{\mathbf{x},(r,s):(r,s+)}$ induces an identification

$$V \xrightarrow{\sim} J/J_+.$$

Let us investigate the space V by using the root space decomposition of $\mathfrak{g}$ with respect to the maximal torus $\mathbf{S}$ in $\mathbf{G}$. For $\alpha \in \Phi(\mathbf{G}, \mathbf{S})$, we put $\mathbf{V}_\alpha$ to be the image of $\mathfrak{g}_\alpha(E) \cap \mathrm{Lie}(\mathbf{S}, \mathbf{G})(E)_{\mathbf{x},(r,s)}$ in $\mathrm{Lie}(\mathbf{S}, \mathbf{G})(E)_{\mathbf{x},(r,s):(r,s+)}$. Then the root space decomposition $\mathfrak{g} \cong \mathfrak{s} \oplus \bigoplus_{\alpha \in \Phi(\mathbf{G},\mathbf{S})} \mathfrak{g}_\alpha$ naturally induces a decomposition

$$\mathbf{V} = \bigoplus_{\alpha \in \Phi(\mathbf{G},\mathbf{S})} \mathbf{V}_\alpha.$$

For each $\alpha \in \Phi(\mathbf{G}, \mathbf{S})$, we put $V_\alpha := \mathbf{V}_\alpha^{\Gamma_\alpha}$ (recall that Γ_α is the stabilizer of α in Γ). Here, note that $\mathbf{V}_\alpha$ and V_α might be zero depending on $\alpha \in \Phi(\mathbf{G}, \mathbf{S})$. We define a subset $\Xi(\mathbf{G}, \mathbf{S})$ of $\Phi(\mathbf{G}, \mathbf{S})$ by

$$\Xi(\mathbf{G}, \mathbf{S}) := \{\alpha \in \Phi(\mathbf{G}, \mathbf{S}) \mid V_\alpha \neq 0\}.$$

Note that, for any $\alpha \in \Xi(\mathbf{G}, \mathbf{S})$, the space V_α is (noncanonically) isomorphic to the residue field k_α of F_α. Also note that $\Xi(\mathbf{G}, \mathbf{S})$ is preserved by the action of $\Sigma = \Gamma \times \{\pm 1\}$ on $\Phi(\mathbf{G}, \mathbf{S})$. In the following, we simply write Ξ for $\Xi(\mathbf{G}, \mathbf{S})$.

For $\Gamma\alpha \in \dot{\Phi}(\mathbf{G}, \mathbf{S})$, we put

$$\mathbf{V}_{\Gamma\alpha} := \bigoplus_{\beta \in \Gamma\alpha} \mathbf{V}_\beta \quad \text{and} \quad V_{\Gamma\alpha} := \mathbf{V}_{\Gamma\alpha}^{\Gamma}.$$

Then, for any $\Gamma\alpha \in \dot{\Phi}(\mathbf{G}, \mathbf{S})$, we have

$$V_\alpha \xrightarrow{\sim} V_{\Gamma\alpha} = \Big(\bigoplus_{\beta \in \Gamma\alpha} \mathbf{V}_\beta\Big)^{\Gamma} : X_\alpha \mapsto \sum_{\sigma \in \Gamma/\Gamma_\alpha} \sigma(X_\alpha).$$

Therefore we get

$$V \cong \bigoplus_{\Gamma\alpha \in \dot{\Xi}} V_{\Gamma\alpha} = \bigoplus_{\Sigma\alpha \in \ddot{\Xi}} V_{\Sigma\alpha}, \tag{10}$$

where we put $V_{\Sigma\alpha} := V_{\Gamma\alpha} \oplus V_{-\Gamma\alpha}$ for $\alpha \in \Xi_{\mathrm{asym}}$ and $V_{\Sigma\alpha} := V_{\Gamma\alpha}$ for $\alpha \in \Xi_{\mathrm{sym}}$.

Recall that $V \cong J/J_+$ has a structure of a symplectic $\mathbb{F}_p$-vector space given by

$$(J/J_+) \times (J/J_+) \to \mu_p \cong \mathbb{F}_p \colon (g, g') \mapsto \hat{\vartheta}([g, g'])$$

(see Section 4.2). In fact, the above decomposition (10) gives a orthogonal decomposition of V into symplectic subspaces. Each symplectic subspace $V_{\Sigma\alpha}$ is described as follows.

Asymmetric case: Suppose that $\alpha \in \Xi_{\mathrm{asym}}$. We put $V_{\pm\alpha} := V_\alpha \oplus V_{-\alpha}$. Under the identification $V \cong J/J_+$, the symplectic form on J/J_+ is transformed into the symplectic form on V given by

$$V \times V \to \mathbb{F}_p \colon (X_1, X_2) \mapsto c \cdot \mathrm{Tr}_{k/\mathbb{F}_p}(\langle X^*, [X_1, X_2]\rangle),$$

where $c \in \mathbb{F}_p^\times$ is a constant determined by the fixed identification $\mu_p \cong \mathbb{F}_p$. Here, $\langle X^*, [X_1, X_2]\rangle \in k$ denotes the pairing of $X^* \in \mathfrak{s}^*_{r:r+}$ with the $\mathfrak{s}$-part of $[X_1, X_2] \in \mathfrak{g}_{\mathbf{x},r:r+}$ (i.e., the trivial isotypic component with respect to the $\mathbf{S}$-action). Recall that the identification $V_\alpha \cong V_{\Gamma\alpha} \subset V$ is given by $X_\alpha \mapsto \sum_{\sigma \in \Gamma/\Gamma_\alpha} \sigma(X_\alpha)$. By noting that, for any $\alpha_1, \alpha_2 \in \Xi$, we have $\langle X^*, [X_{\alpha_1}, X_{\alpha_2}]\rangle \neq 0$ only if $\alpha_2 = -\alpha_1$, the resulting symplectic form $V_{\pm\alpha} \times V_{\pm\alpha} \to \mathbb{F}_p$ maps $(X_\alpha + X_{-\alpha}, Y_\alpha + Y_{-\alpha})$ to

$$\begin{aligned} &c \cdot \sum_{\sigma \in \Gamma/\Gamma_\alpha} \sum_{\sigma' \in \Gamma/\Gamma_\alpha} \mathrm{Tr}_{k/\mathbb{F}_p}(\langle X^*, [\sigma(X_\alpha + X_{-\alpha}), \sigma'(Y_\alpha + Y_{-\alpha})]\rangle) \\ &= c \cdot \sum_{\sigma \in \Gamma/\Gamma_\alpha} \mathrm{Tr}_{k/\mathbb{F}_p}(\langle X^*, \sigma([X_\alpha, Y_{-\alpha}] + [X_{-\alpha}, Y_\alpha])\rangle). \end{aligned}$$

Since X^* is F-rational, this equals $c \cdot e_\alpha \cdot \mathrm{Tr}_{k_\alpha/\mathbb{F}_p}(\langle X^*, [X_\alpha, Y_{-\alpha}] + [X_{-\alpha}, Y_\alpha]\rangle)$.

We recall that $V_\alpha \cong k_\alpha$. Hence, by fixing nonzero elements $X_\alpha \in V_\alpha$ and $X_{-\alpha} \in V_{-\alpha}$ so that $X_\alpha \in V_\alpha$ and $X_{-\alpha} \in V_{-\alpha}$ are identified with $1 \in k_\alpha$, we may think of the above symplectic form as the symplectic form on $k_\alpha \oplus k_\alpha$ which maps $(x_+ + x_-, y_+ + y_-)$ to

$$\mathrm{Tr}_{k_\alpha/\mathbb{F}_p}(C \cdot (x_+y_- - x_-y_+)),$$

where we put $C := c \cdot e_\alpha \cdot \langle X^*, [X_\alpha, X_{-\alpha}]\rangle \in k_\alpha^\times$.

Symmetric case: Suppose that $\alpha \in \Xi_{\mathrm{sym}}$. Let $\tau_\alpha \in \Gamma/\Gamma_\alpha$ be the unique element satisfying $\tau_\alpha(\alpha) = -\alpha$. By the same discussion as in the asymmetric case, we see that the symplectic form on V_α induced from that on J/J_+ is given by

$$\begin{aligned}(X_\alpha, Y_\alpha) \mapsto c \cdot \sum_{\sigma\in\Gamma/\Gamma_\alpha} \sum_{\sigma'\in\Gamma/\Gamma_\alpha} \mathrm{Tr}_{k/\mathbb{F}_p}(\langle X^*, [\sigma(X_\alpha), \sigma'(Y_\alpha)]\rangle) \\ = c \cdot \sum_{\sigma\in\Gamma/\Gamma_\alpha} \mathrm{Tr}_{k/\mathbb{F}_p}(\langle X^*, \sigma([X_\alpha, \tau_\alpha(Y_\alpha)])\rangle) \\ = c \cdot e_\alpha \cdot \mathrm{Tr}_{k_\alpha/\mathbb{F}_p}(\langle X^*, [X_\alpha, \tau_\alpha(Y_\alpha)]\rangle).\end{aligned}$$

By recalling that $V_\alpha \cong k_\alpha$ and fixing a nonzero element $X_\alpha \in V_\alpha$, we may think of the above symplectic form as the symplectic form on k_α which maps (x, y) to

$$\mathrm{Tr}_{k_\alpha/\mathbb{F}_p}(C \cdot x\tau_\alpha(y)),$$

where we put $C := c \cdot e_\alpha \cdot [X_\alpha, \tau_\alpha(X_\alpha)] \in k_\alpha^\times$. Note that $\tau_\alpha(C) = -C$.

Let us introduce one particular property of the set Ξ deduced from the above description of the symplectic form:

Lemma 6.2. *The set Ξ does not contain any symmetric ramified root.*

Proof. This fact is explained in the proof of [DS18, Proposition 4.21]. For the sake of completeness, we explain it here. Let $\alpha \in \Xi_{\mathrm{sym}}$. Then, as explained above, we have $V_\alpha \cong k_\alpha$ and the symplectic form on $V_\alpha \times V_\alpha$ is given by

$$k_\alpha \times k_\alpha \to \mathbb{F}_p \colon (x, y) \mapsto \mathrm{Tr}_{k_\alpha/\mathbb{F}_p}(C \cdot x\tau_\alpha(y))$$

with an element $C \in k_\alpha^\times$ satisfying $\tau_\alpha(C) = -C$. If α is ramified, then τ_α acts trivially on k_α, hence there cannot exist such an element C. Thus α must be unramified. □

6.2. **Intertwiner of Heisenberg–Weil representations.** Recall that we fixed a topologically semisimple element $\underline{\eta} \in \tilde{S}$ (Section 5.2). Hence any element $\eta' \in \tilde{S}$ is written as $\eta' = s \cdot \underline{\eta}$ with a unique element $s \in S$. Note that the action of $[\underline{\eta}]$ on $\mathfrak{g}$ induces an action on the set $\Phi(\mathbf{G}, \mathbf{S})$ of order 2, which does not depend on the choice of $\underline{\eta} \in \tilde{S}$. By abuse of notation, let us write $\theta_{\mathbf{S}}$ for this action and $\Theta_{\mathbf{S}}$ for the group $\langle\theta_{\mathbf{S}}\rangle$ generated by $\theta_{\mathbf{S}}$. To be more precise, for any $\alpha \in \Phi(\mathbf{G}, \mathbf{S})$, $\theta_{\mathbf{S}}(\alpha)$ is the root given by $\theta_{\mathbf{S}}(\alpha) = \alpha \circ [\underline{\eta}]^{-1}$. Whenever there is no risk of confusion, we abbreviate $\theta_{\mathbf{S}}(\alpha)$ even as $\theta(\alpha)$. We note that, for $X_\alpha \in \mathfrak{g}_\alpha$, $[\underline{\eta}](X_\alpha)$ belongs to $\mathfrak{g}_{\theta(\alpha)}$. We also note that, as $\underline{\eta}$ is F-rational, the actions of $\Theta_{\mathbf{S}}$ and $\Sigma = \Gamma \times \{\pm 1\}$ on $\Phi(\mathbf{G}, \mathbf{S})$ commute. Especially, the symmetry of $\Phi(\mathbf{G}, \mathbf{S})$ is preserved by $\Theta_{\mathbf{S}}$.

Let us investigate the action $[\underline{\eta}]$ on J/J_+ through the isomorphism $V \cong J/J_+$ and the above decomposition (10) of V. Note that $[\underline{\eta}]$ preserves the symplectic structure of V. Indeed, for any $g, g' \in J/J_+$, we have

$$\hat{\vartheta}([[\underline{\eta}](g), [\underline{\eta}](g')]) = \hat{\vartheta}([\underline{\eta}g\underline{\eta}^{-1}, \underline{\eta}g'\underline{\eta}^{-1}]) = \hat{\vartheta}([\underline{\eta}]([g, g'])) = \hat{\vartheta}([g, g']).$$

Moreover, each $V_{\Sigma\alpha}$ is mapped onto $V_{\Sigma\theta(\alpha)}$, respectively. In particular, the action of $\Theta_{\mathbf{S}}$ on $\Phi(\mathbf{G},\mathbf{S})$ preserves Ξ.

As in Section 3.3, for any $\alpha \in \Phi(\mathbf{G},\mathbf{S})$, we let l_α be the cardinality of the $\Theta_{\mathbf{S}}$-orbit of α. We furthermore introduce a number denoted by m_α as follows:

Definition 6.3. For $\alpha \in \Phi(\mathbf{G},\mathbf{S})$, let m_α be the order of $\Sigma\backslash(\Sigma \times \Theta_{\mathbf{S}})\alpha$. In other words, m_α is the smallest positive integer such that $\Sigma\theta^{m_\alpha}(\alpha) = \Sigma\alpha$ (hence $m_\alpha \mid l_\alpha$).

For $\Sigma\alpha \in \ddot{\Xi}$, let us write $(\omega_{\Sigma\alpha}, W_{\Sigma\alpha})$ for a Heisenberg–Weil representation of $\mathrm{Sp}(V_{\Sigma\alpha}) \ltimes \mathbb{H}(V_{\Sigma\alpha})$ with central character given by ϑ, which is unique up to isomorphism (see Section A.1). Since the action $[\underline{\eta}]$ on V induces an symplectic isomorphism from $V_{\Sigma\alpha}$ to $V_{\Sigma\theta(\alpha)}$, an isomorphism from $\mathrm{Sp}(V_{\Sigma\alpha}) \ltimes \mathbb{H}(V_{\Sigma\alpha})$ to $\mathrm{Sp}(V_{\Sigma\theta(\alpha)}) \ltimes \mathbb{H}(V_{\Sigma\theta(\alpha)})$ is induced (for which we write $[\underline{\eta}]_*$). Then the pull back $(\omega^{\underline{\eta}}_{\Sigma\theta(\alpha)}, W_{\Sigma\theta(\alpha)})$ of the Heisenberg–Weil representation $W_{\Sigma\theta(\alpha)}$ of $\mathrm{Sp}(V_{\Sigma\theta(\alpha)}) \ltimes \mathbb{H}(V_{\Sigma\theta(\alpha)})$ to $\mathrm{Sp}(V_{\Sigma\alpha}) \ltimes \mathbb{H}(V_{\Sigma\alpha})$ via $[\underline{\eta}]_*$ is isomorphic to $(\omega_{\Sigma\alpha}, W_{\Sigma\alpha})$. For each $\Sigma\alpha \in \ddot{\Xi}$, we fix an intertwiner

$$I^{\underline{\eta}}_{\Sigma\alpha} \colon \omega_{\Sigma\alpha} \xrightarrow{\sim} \omega^{\underline{\eta}}_{\Sigma\theta(\alpha)}.$$

Recall that the representation $\omega = \omega_{(\mathbf{S},\vartheta)}$ is a Heisenberg–Weil representation of $\mathrm{Sp}(V) \ltimes \mathbb{H}(V)$ with central character ϑ. Since we have the decomposition (10), ω can be realized by tensoring Heisenberg–Weil representations $\omega_{\Sigma\alpha}$ for $\Sigma\alpha \in \ddot{\Xi}$ (see Section A.1). Furthermore, by tensoring the fixed intertwiners $I^{\underline{\eta}}_{\Sigma\alpha}$, we get an intertwiner between ω and its $[\underline{\eta}]_*$-twist $\omega^{\underline{\eta}}$. Let us write $I^{\underline{\eta}}_{\omega}$ for the intertwiner obtained in this way:

$$I^{\underline{\eta}}_{\omega} = I^{\underline{\eta}}_{\omega_{(\mathbf{S},\vartheta)}} \colon \omega_{(\mathbf{S},\vartheta)} \xrightarrow{\sim} \omega^{\underline{\eta}}_{(\mathbf{S},\vartheta)}.$$

Then we have the following:

Proposition 6.4. *For each $\Sigma\alpha \in \ddot{\Xi}$, we put*

$$I^{\underline{\eta}}_{\Sigma\Theta(\alpha)} := I^{\underline{\eta}}_{\Sigma\theta^{m_\alpha - 1}(\alpha)} \circ \cdots \circ I^{\underline{\eta}}_{\Sigma\alpha}.$$

Then, for any $\eta' = s\underline{\eta} \in \tilde{S}$, we have

$$\mathrm{tr}\big(\omega([s]) \circ I^{\underline{\eta}}_{\omega}\big) = \prod_{\alpha \in \Theta_{\mathbf{S}}\backslash\ddot{\Xi}} \mathrm{tr}\Big(\omega_{\Sigma\alpha}([\eta']^{m_\alpha} \circ [\underline{\eta}]^{-m_\alpha}) \circ I^{\underline{\eta}}_{\Sigma\Theta(\alpha)}\Big).$$

Proof. Let us fix a set $\{\alpha_0, \ldots, \alpha_r\}$ of representatives of $\Theta_{\mathbf{S}}\backslash\ddot{\Xi}$. Then we can utilize the results of Section A.1, by taking $\iota := [\underline{\eta}]$, $l_i := m_{\alpha_i} - 1$ for each $0 \leq i \leq r$ and putting $V^i_j := V_{\Sigma\theta^j(\alpha_i)}$. Since the symplectic automorphism $[s]$ preserves each V^i_j, Proposition A.2 implies that the trace of $\omega([s]) \circ I^{\underline{\eta}}_{\omega}$ is given by

$$\prod_{\alpha \in \Theta_{\mathbf{S}}\backslash\ddot{\Xi}} \mathrm{tr}\Big(\omega_{\Sigma\alpha}\big([s] \circ [\underline{\eta}]_*([s]) \circ \cdots \circ [\underline{\eta}]_*^{m_\alpha - 1}([s])\big) \circ I^{\underline{\eta}}_{\Sigma\Theta(\alpha)}\Big).$$

By noting that $[\underline{\eta}]^i_*([s]) = [\underline{\eta}]^i \circ [s] \circ [\underline{\eta}]^{-i}$, we have

$$[s] \circ [\underline{\eta}]_*([s]) \circ \cdots \circ [\underline{\eta}]_*^{m_\alpha - 1}([s]) = ([s] \circ [\underline{\eta}])^{m_\alpha} \circ [\underline{\eta}]^{-m_\alpha} = [s\underline{\eta}]^{m_\alpha} \circ [\underline{\eta}]^{-m_\alpha}.$$

Thus we get the desired result. $\square$

We still have not specified the choice of each $I^{\eta}_{\Sigma\alpha}$ so far. This means that also I^{η}_{ω} still has an ambiguity of a scalar multiple. Now we explain our choice of $I^{\eta}_{\Sigma\alpha}$. For any $\Sigma\alpha \in \dot{\Xi}$, note that $I^{\eta}_{\Sigma\Theta(\alpha)}$ is an automorphism of $W_{\Sigma\alpha}$ such that

$$\begin{array}{ccc} W_{\Sigma\alpha} & \xrightarrow{I^{\eta}_{\Sigma\Theta(\alpha)}} & W_{\Sigma\alpha} \\ {\scriptstyle \omega_{\Sigma\alpha}(g,h)}\downarrow & & \downarrow{\scriptstyle \omega_{\Sigma\alpha}([\underline{\eta}]^{m_\alpha}_{*}(g,h))} \\ W_{\Sigma\alpha} & \xrightarrow[I^{\eta}_{\Sigma\Theta(\alpha)}]{} & W_{\Sigma\alpha} \end{array}$$

is commutative for any $(g,h) \in \mathrm{Sp}(V_{\Sigma\alpha}) \ltimes \mathbb{H}(V_{\Sigma\alpha})$. Note that $[\underline{\eta}]^{m_\alpha}$ is a symplectic automorphism of V preserving $V_{\Sigma\alpha}$. Hence $I^{\eta}_{\Sigma\Theta(\alpha)}$ must be a scalar multiple of the Heisenberg–Weil action $\omega_{\Sigma\alpha}([\underline{\eta}]^{m_\alpha})$. We choose $I^{\eta}_{\Sigma\alpha}$ for $\Sigma\alpha \in \dot{\Xi}_{\mathrm{sym}}$ so that we have

$$I^{\eta}_{\Sigma\Theta(\alpha)} = \omega_{\Sigma\alpha}([\underline{\eta}]^{m_\alpha}).$$

Corollary 6.5. *With the above choice of an intertwiner I^{η}_{ω}, for any $\eta' = s\underline{\eta} \in \tilde{S}$ with topological Jordan decomposition $\eta'_0\eta'_+$,*

$$\mathrm{tr}\bigl(\omega([s]) \circ I^{\eta}_{\omega}\bigr) = \prod_{\alpha \in \Theta_{\mathbf{S}}\backslash\ddot{\Xi}} \Theta_{\omega_{\Sigma\alpha}}([\eta'_0]^{m_\alpha}).$$

Proof. With the choice of an intertwiner I^{η}_{ω} explained as above, we get

$$\mathrm{tr}\bigl(\omega([s]) \circ I^{\eta}_{\omega}\bigr) = \prod_{\alpha \in \Theta_{\mathbf{S}}\backslash\ddot{\Xi}} \Theta_{\omega_{\Sigma\alpha}}([\eta']^{m_\alpha})$$

by Proposition 6.4. Noting that the topologically unipotent part η'_+ acts on $V_{\Gamma\alpha}$ and $V_{\Sigma\alpha}$ trivially via conjugation, we get the assertion. □

6.3. **Descent of the Heisenberg quotient.** Recall that we have fixed an elliptic regular semisimple element $\delta \in \tilde{S}$ and written η for $\delta_{<r}$ so far. However, to make the notation lighter, we temporarily (until the end of Section 6.6) let $\eta \in \tilde{S}$ denote any topologically semisimple element.

Recall that, in Section 3.3, we introduced the notion of a restricted root. Although we discussed it for $\Phi(\mathbf{G},\mathbf{T})$ in Section 3.3, the same can be done for $\Phi(\mathbf{G},\mathbf{S})$, i.e., we have the set of restricted roots $\Phi_{\mathrm{res}}(\mathbf{G},\mathbf{S})$ equipped with a natural map

$$\Phi(\mathbf{G},\mathbf{S}) \twoheadrightarrow \Phi(\mathbf{G},\mathbf{S})/\Theta_{\mathbf{S}} \xrightarrow{1:1} \Phi_{\mathrm{res}}(\mathbf{G},\mathbf{S})\colon \alpha \mapsto \alpha_{\mathrm{res}}.$$

Note that, since $\Phi_{\mathrm{res}}(\mathbf{G},\mathbf{S})$ carries a Galois action induced from that of $\Phi(\mathbf{G},\mathbf{S})$, we can also discuss the symmetry of a restricted root. For any $\alpha \in \Phi(\mathbf{G},\mathbf{S})$, we put $\Gamma_{\alpha_{\mathrm{res}}}$ to be the stabilizer in Γ of the restricted root α_{res} (or, equivalently, the $\Theta_{\mathbf{S}}$-orbit $\Theta_{\mathbf{S}}\alpha$ of α):

$$\Gamma_{\alpha_{\mathrm{res}}} := \{\sigma \in \Gamma \mid \sigma(\alpha_{\mathrm{res}}) = \alpha_{\mathrm{res}}\} = \{\sigma \in \Gamma \mid \sigma(\Theta_{\mathbf{S}}\alpha) = \Theta_{\mathbf{S}}\alpha\}.$$

Similarly, we put $\Gamma_{\pm\alpha_{\mathrm{res}}}$ to be the stabilizer in Γ of the set $\{\pm\alpha_{\mathrm{res}}\}$:

$$\Gamma_{\pm\alpha_{\mathrm{res}}} := \{\sigma \in \Gamma \mid \sigma(\{\pm\alpha_{\mathrm{res}}\}) = \{\pm\alpha_{\mathrm{res}}\}\} = \{\sigma \in \Gamma \mid \sigma(\{\pm\Theta_{\mathbf{S}}\alpha\}) = \{\pm\Theta_{\mathbf{S}}\alpha\}\}.$$

Let $F_{\alpha_{\mathrm{res}}}$ and $F_{\pm\alpha_{\mathrm{res}}}$ denote the subfields of $\overline{F}$ fixed by $\Gamma_{\alpha_{\mathrm{res}}}$ and $\Gamma_{\pm\alpha_{\mathrm{res}}}$, respectively.

$$\begin{array}{ccc} F_{\alpha_{\mathrm{res}}} & \subset & F_{\alpha} \\ \cup & & \cup \\ F_{\pm\alpha_{\mathrm{res}}} & \subset & F_{\pm\alpha} \end{array} \qquad\qquad \begin{array}{ccc} \Gamma_{\alpha_{\mathrm{res}}} & \supset & \Gamma_{\alpha} \\ \cap & & \cap \\ \Gamma_{\pm\alpha_{\mathrm{res}}} & \supset & \Gamma_{\pm\alpha} \end{array}$$

As reviewed in Section 3.3, the group $\mathbf{G}_\eta$ is a connected reductive group with a maximal torus $\mathbf{S}^\natural$. Furthermore, $\Phi(\mathbf{G}_\eta, \mathbf{S}^\natural)$ is regarded as a subset of the set $\Phi_{\mathrm{res}}(\mathbf{G}, \mathbf{S})$. By Proposition 5.5, the point $\mathbf{x}$ can be regarded as a point of $\mathcal{A}(\mathbf{S}^\natural, F) \subset \mathcal{B}(\mathbf{G}_\eta, F)$. We introduce the subgroups J_η (resp. $J_{\eta,+}$) in the same way as J (resp. J_+) by using $(\mathbf{G}_\eta, \mathbf{S}^\natural, \mathbf{x}, r, s(+))$ instead of $(\mathbf{G}, \mathbf{S}, \mathbf{x}, r, s(+))$, i.e.,

$$J_\eta := (S^\natural, G_\eta)_{\mathbf{x},(r,s)} \quad \text{and} \quad J_{\eta,+} := (S^\natural, G_\eta)_{\mathbf{x},(r,s+)}.$$

By the same discussion as in Section 6.1, if we put

$$\mathbf{V}_\eta := \mathrm{Lie}(\mathbf{S}_\eta, \mathbf{G}_\eta)(E)_{\mathbf{x},(r,s):(r,s+)} \quad \text{and} \quad V_\eta := \mathbf{V}_\eta^\Gamma$$

then we have $J_\eta/J_{\eta,+} \cong V_\eta$ and a root space decomposition similar to (10):

$$V_\eta \cong \bigoplus_{\Sigma\alpha_{\mathrm{res}} \in \ddot{\Xi}_\eta} V_{\eta,\Sigma\alpha_{\mathrm{res}}},$$

where we use the notation defined in the same way as in Section 6.1, e.g.,

$$\Xi_\eta := \Xi(\mathbf{G}_\eta, \mathbf{S}^\natural) := \{\alpha_{\mathrm{res}} \in \Phi(\mathbf{G}_\eta, \mathbf{S}^\natural) \mid V_{\eta,\alpha_{\mathrm{res}}} \neq 0\}.$$

By Proposition 5.6 (4), we have a natural identification

$$V_\eta \cong (S^\natural, G_\eta)_{\mathbf{x},(r,s):(r,s+)} \cong (S, G)^\eta_{\mathbf{x},(r,s):(r,s+)} \cong V^\eta,$$

where $(S, G)^\eta_{\mathbf{x},(r,s):(r,s+)}$ and V^η denote the set of $[\eta]$-fixed points in $(S, G)_{\mathbf{x},(r,s):(r,s+)}$ and V, respectively. Let us investigate this identification more precisely. The Lie algebra $\mathfrak{g}_\eta$ of $\mathbf{G}_\eta$ is naturally identified with the $[\eta]$-fixed points $\mathfrak{g}^\eta$ of the Lie algebra $\mathfrak{g}$ of $\mathbf{G}$. If $\alpha_{\mathrm{res}} \in \Phi(\mathbf{G}_\eta, \mathbf{S}^\natural)$ is a restricted root obtained from $\alpha \in \Phi(\mathbf{G}, \mathbf{S})$, then the root subspace $\mathfrak{g}_{\eta,\alpha_{\mathrm{res}}}$ of $\mathfrak{g}_\eta$ is identified with the $[\eta]$-fixed points in the sum $\bigoplus_{\alpha' \in \Theta\alpha} \mathfrak{g}_{\alpha'}$ of root subspaces of $\mathfrak{g}$:

$$\mathfrak{g}_{\eta,\alpha_{\mathrm{res}}} \cong \Big(\bigoplus_{\alpha' \in \Theta\alpha} \mathfrak{g}_\alpha\Big)^\eta.$$

This induces an identification

$$V_{\eta,\Sigma\alpha_{\mathrm{res}}} \cong \Big(\bigoplus_{\Sigma\alpha' \in \Sigma\backslash(\Sigma\times\Theta)\alpha} V_{\Sigma\alpha'}\Big)^\eta$$

for any $\alpha_{\mathrm{res}} \in \Xi_\eta$. Let us put $V_{\Sigma\Theta(\alpha)} := \bigoplus_{\Sigma\alpha' \in \Sigma\backslash(\Sigma\times\Theta)\alpha} V_{\Sigma\alpha'}$. In particular, by letting Ξ_{res} be the set of restricted roots associated to Ξ, the set Ξ_η can be thought of as the set of restricted roots $\alpha_{\mathrm{res}} \in \Xi_{\mathrm{res}}$ such that the $[\eta]$-action has a nonzero fixed point in $V_{\Sigma\Theta(\alpha)}$.

$$\begin{array}{ccccccc} \Phi(\mathbf{G}, \mathbf{S}) & \twoheadrightarrow & \Phi(\mathbf{G}, \mathbf{S})/\Theta_{\mathbf{S}} & \xrightarrow{1:1} & \Phi(\mathbf{G}, \mathbf{S})_{\mathrm{res}} & \supset & \Phi(\mathbf{G}_\eta, \mathbf{S}^\natural) \\ \cup & & \cup & & \cup & & \cup \\ \Xi & \twoheadrightarrow & \Xi/\Theta_{\mathbf{S}} & \xrightarrow{1:1} & \Xi_{\mathrm{res}} & \supset & \Xi_\eta \end{array}$$

6.4. **Twisted characters of Weil representations: asymmetric roots.** Let $\alpha \in \Xi_{\mathrm{asym}}$. We compute $\Theta_{\omega_{\Sigma\alpha}}([\eta]^{m_\alpha})$, which constitutes the right-hand side of Corollary 6.5. Recall that $V_{\Sigma\alpha} = V_{\Gamma\alpha} \oplus V_{-\Gamma\alpha} \cong V_\alpha \oplus V_{-\alpha}$, where V_α and $V_{-\alpha}$ are 1-dimensional k_α-vector spaces, which are identified with k_α by fixing nonzero elements $X_\alpha \in V_\alpha$ and $X_{-\alpha} \in V_{-\alpha}$ (see Section 6.1). As the order of θ is 2, there are exactly 4 possibilities:

(1) $\theta(\alpha) = \alpha$ (thus $l_\alpha = m_\alpha = 1$);
(2) $\theta(\alpha) \neq \alpha$ and $\theta(\alpha) \notin \Sigma\alpha$ (thus $l_\alpha = m_\alpha = 2$);
(3) $\theta(\alpha) \neq \alpha$ and $\theta(\alpha) \in \Gamma\alpha$ (thus $l_\alpha = 2$ and $m_\alpha = 1$);
(4) $\theta(\alpha) \neq \alpha$ and $\theta(\alpha) \in -\Gamma\alpha$ (thus $l_\alpha = 2$ and $m_\alpha = 1$).

Note that the cases (3) and (4) are exclusive to each other since $-\alpha \notin \Gamma\alpha$.

6.4.1. *The case where $\theta(\alpha) = \alpha$.* In this case, we have $F_\alpha = F_{\pm\alpha} = F_{\alpha_{\mathrm{res}}} = F_{\pm\alpha_{\mathrm{res}}}$.

$$\begin{array}{ccc} F_{\alpha_{\mathrm{res}}} & = & F_\alpha \\ \| & & \| \\ F_{\pm\alpha_{\mathrm{res}}} & = & F_{\pm\alpha} \end{array}$$

The action of $[\eta]$ on $V_{\Sigma\alpha}$ preserves $V_{\Gamma\alpha}$ and $V_{-\Gamma\alpha}$. Moreover, it is k_α-linear. We let η_α (resp. $\eta_{-\alpha}$) be the element of $k_\alpha^\times$ such that $[\eta](X_\alpha) = \eta_\alpha X_\alpha$ (resp. $[\eta](X_{-\alpha}) = \eta_\alpha X_{-\alpha}$). By noting that $[\eta]$ preserves the symplectic form described as in Section 6.1, we necessarily have $\eta_{-\alpha} = \eta_\alpha^{-1}$. Then, as an element of $\mathrm{Sp}(V_{\Sigma\alpha}) \cong \mathrm{Sp}(k_\alpha \oplus k_\alpha)$, $[\eta]$ is given by

$$x_+ + x_- \mapsto \eta_\alpha x_+ + \eta_{-\alpha} x_-.$$

Hence, by Corollary A.8, we get

$$\Theta_{\omega_{\Sigma\alpha}}([\eta]) = \mathrm{sgn}_{\mathbb{F}_p^\times}(\det(\eta_\alpha \mid k_\alpha)) \cdot |V_{\Sigma\alpha}^{\eta}|^{\frac{1}{2}} = \mathrm{sgn}_{k_\alpha^\times}(\eta_\alpha) \cdot |V_{\Sigma\alpha}^{\eta}|^{\frac{1}{2}}.$$

6.4.2. *The case where $\theta(\alpha) \neq \alpha$ and $\theta(\alpha) \notin \Sigma\alpha$.* In this case, we have $F_\alpha = F_{\pm\alpha} = F_{\alpha_{\mathrm{res}}} = F_{\pm\alpha_{\mathrm{res}}}$.

$$\begin{array}{ccc} F_{\alpha_{\mathrm{res}}} & = & F_\alpha \\ \| & & \| \\ F_{\pm\alpha_{\mathrm{res}}} & = & F_{\pm\alpha} \end{array}$$

Since $\eta^2 \in S$, the action of $[\eta]^2 = [\eta^2]$ preserves $V_{\Gamma\alpha}$ and $V_{-\Gamma\alpha}$ and is k_α-linear. Hence the same argument as in the previous case works. If we let η_α^2 be the element of $k_\alpha^\times$ such that $[\eta^2](X_\alpha) = \eta_\alpha^2 X_\alpha$, then we get

$$\Theta_{\omega_{\Sigma\alpha}}([\eta]^2) = \mathrm{sgn}_{k_\alpha^\times}(\eta_\alpha^2) \cdot |V_{\Sigma\alpha}^{\eta^2}|^{\frac{1}{2}} = \mathrm{sgn}_{k_\alpha^\times}(\eta_\alpha^2) \cdot |V_{\Sigma\Theta(\alpha)}^{\eta}|^{\frac{1}{2}}.$$

6.4.3. *The case where $\theta(\alpha) \neq \alpha$ and $\theta(\alpha) \in \Gamma\alpha$.* In this case, we have $F_\alpha = F_{\pm\alpha}$ and $[F_\alpha : F_{\alpha_{\mathrm{res}}}] = 2$. Let σ_α be the unique nontrivial element of $\mathrm{Gal}(F_\alpha/F_{\alpha_{\mathrm{res}}})$, hence we have $\sigma_\alpha(\alpha) = \theta(\alpha)$. By noting that $[F_{\alpha_{\mathrm{res}}} : F_{\pm\alpha_{\mathrm{res}}}] \leq 2$ and $[F_{\pm\alpha} : F_{\pm\alpha_{\mathrm{res}}}] \leq 2$, we see that $F_{\alpha_{\mathrm{res}}} = F_{\pm\alpha_{\mathrm{res}}}$ and $[F_{\pm\alpha} : F_{\pm\alpha_{\mathrm{res}}}] = 2$.

$$\begin{array}{ccc} F_{\alpha_{\mathrm{res}}} & \overset{\text{quad}}{\underset{\langle\sigma_\alpha\rangle}{\text{------}}} & F_\alpha \\ \| & & \| \\ F_{\pm\alpha_{\mathrm{res}}} & \overset{\text{quad}}{\underset{\langle\sigma_\alpha\rangle}{\text{------}}} & F_{\pm\alpha} \end{array}$$

The action of $[\eta]$ on $V_{\Sigma\alpha}$ preserves $V_{\Gamma\alpha}$ and $V_{-\Gamma\alpha}$. Since $[\eta](X_\alpha)$ belongs to $V_{\theta(\alpha)} = V_{\sigma_\alpha(\alpha)}$, the induced action of $[\eta]$ on V_α is σ_α-linear (note that $[\eta]$ is k_α-linear on $V_{\Gamma\alpha}$):

$$\begin{array}{ccc} V_\alpha & \xrightarrow{\cong} & V_{\Gamma\alpha} \\ & & \big\downarrow [\eta] \\ V_\alpha & \xleftarrow[\cong]{} & V_{\Gamma\alpha} \end{array} \qquad \begin{array}{ccc} X_\alpha & \longmapsto & \sum_{\sigma\in\Gamma/\Gamma_\alpha} \sigma(X_\alpha) \\ & & \big\downarrow \\ [\eta]\circ\sigma_\alpha(X_\alpha) & \longleftarrow\!\!\shortmid & \sum_{\sigma\in\Gamma/\Gamma_\alpha} [\eta]\circ\sigma(X_\alpha) \end{array}$$

More explicitly, if we let η_α (resp. $\eta_{-\alpha}$) be the element of $k_\alpha^\times$ such that $[\eta]\circ\sigma_\alpha(X_\alpha) = \eta_\alpha X_\alpha$ (resp. $[\eta]\circ\sigma_\alpha(X_{-\alpha}) = \eta_{-\alpha} X_{-\alpha}$), then $[\eta]$ is given by

$$x_+ + x_- \mapsto \eta_\alpha\sigma_\alpha(x_+) + \eta_{-\alpha}\sigma_\alpha(x_-)$$

as an element of $\mathrm{Sp}(V_{\Sigma\alpha}) \cong \mathrm{Sp}(k_\alpha \oplus k_\alpha)$. Hence, by Corollary A.8, we get

$$\begin{aligned} \Theta_{\omega_{\Sigma\alpha}}([\eta]) &= \mathrm{sgn}_{\mathbb{F}_p^\times}(\det(\eta_\alpha\circ\sigma_\alpha \mid k_\alpha)) \cdot |V_{\Sigma\alpha}^\eta|^{\frac{1}{2}} \\ &= \mathrm{sgn}_{\mathbb{F}_p^\times}(\det(\sigma_\alpha \mid k_\alpha)) \cdot \mathrm{sgn}_{k_\alpha^\times}(\eta_\alpha) \cdot |V_{\Sigma\alpha}^\eta|^{\frac{1}{2}}. \end{aligned}$$

(1) If $F_\alpha/F_{\alpha_{\mathrm{res}}}$ is unramified, we have $\det(\sigma_\alpha \mid k_\alpha) = (-1)^{[k_{\alpha_{\mathrm{res}}}:\mathbb{F}_p]}$. By noting that $\mathrm{sgn}_{\mathbb{F}_p^\times}(\det(\sigma_\alpha \mid k_\alpha)) = \mathrm{sgn}_{\mathbb{F}_p^\times}(-1)^{[k_{\alpha_{\mathrm{res}}}:\mathbb{F}_p]} = \mathrm{sgn}_{k_{\alpha_{\mathrm{res}}}^\times}(-1)$, we get

$$\Theta_{\omega_{\Sigma\alpha}}([\eta]) = \mathrm{sgn}_{k_{\alpha_{\mathrm{res}}}^\times}(-1) \cdot \mathrm{sgn}_{k_\alpha^\times}(\eta_\alpha) \cdot |V_{\Sigma\alpha}^\eta|^{\frac{1}{2}}.$$

(2) If $F_\alpha/F_{\alpha_{\mathrm{res}}}$ is ramified, it acts on k_α via the identity. Thus we get

$$\Theta_{\omega_{\Sigma\alpha}}([\eta]) = \mathrm{sgn}_{k_\alpha^\times}(\eta_\alpha) \cdot |V_{\Sigma\alpha}^\eta|^{\frac{1}{2}}.$$

6.4.4. *The case where $\theta(\alpha) \neq \alpha$ and $\theta(\alpha) \in -\Gamma\alpha$.* In this case, we have $F_\alpha = F_{\pm\alpha}$ and $F_\alpha = F_{\alpha_{\mathrm{res}}}$. However, we have $[F_{\alpha_{\mathrm{res}}} : F_{\pm\alpha_{\mathrm{res}}}] = 2$ (thus $[F_{\pm\alpha} : F_{\pm\alpha_{\mathrm{res}}}] = 2$). Let σ_α be the unique nontrivial element of $\mathrm{Gal}(F_{\alpha_{\mathrm{res}}}/F_{\pm\alpha_{\mathrm{res}}})$, hence we have $\sigma_\alpha(\alpha) = -\theta(\alpha)$.

$$\begin{array}{ccc} F_{\alpha_{\mathrm{res}}} & = & F_\alpha \\ {\scriptstyle\text{quad}}\big| {\scriptstyle\langle\sigma_\alpha\rangle} & & \| \\ F_{\pm\alpha_{\mathrm{res}}} & \overset{\text{quad}}{\underset{\langle\sigma_\alpha\rangle}{\text{------}}} & F_{\pm\alpha} \end{array}$$

The action of $[\eta]$ on $V_{\Sigma\alpha}$ swaps $V_{\Gamma\alpha}$ and $V_{-\Gamma\alpha}$. Since $[\eta](X_\alpha)$ belongs to $V_{\theta(\alpha)} = V_{-\sigma_\alpha(\alpha)}$, the isomorphism from V_α to $V_{-\alpha}$ induced by $[\eta]$ is σ_α-linear (note that $[\eta]$

is k_α-linear on $V_{\Gamma\alpha}$):

$$\begin{array}{ccc} V_\alpha & \xrightarrow{\cong} & V_{\Gamma\alpha} \\ & & \downarrow [\eta] \\ V_{-\alpha} & \xleftarrow[\cong]{} & V_{-\Gamma\alpha} \end{array} \qquad \begin{array}{ccc} X_\alpha & \longmapsto & \sum_{\sigma\in\Gamma/\Gamma_\alpha} \sigma(X_\alpha) \\ & & \downarrow \\ [\eta]\circ\sigma_\alpha(X_\alpha) & \longleftarrow & \sum_{\sigma\in\Gamma/\Gamma_\alpha} [\eta]\circ\sigma(X_\alpha) \end{array}$$

More explicitly, if we let $\eta_{-\alpha}$ (resp. η_α) be the element of $k_\alpha^\times$ such that $[\eta]\circ\sigma_\alpha(X_\alpha) = \eta_{-\alpha}X_{-\alpha}$ (resp. $[\eta]\circ\sigma_\alpha(X_{-\alpha}) = \eta_\alpha X_\alpha$), then $[\eta]$ is given by

$$x_+ + x_- \mapsto \eta_\alpha\sigma_\alpha(x_-) + \eta_{-\alpha}\sigma_\alpha(x_+)$$

as an element of $\mathrm{Sp}(V_{\Sigma\alpha}) \cong \mathrm{Sp}(k_\alpha \oplus k_\alpha)$. Since this automorphism preserves the symplectic form $(x_+ + x_-, y_+ + y_-) \mapsto \mathrm{Tr}_{k_\alpha/\mathbb{F}_p}(C\cdot(x_+y_- - x_-y_+))$ (see Section 6.1), we must have

$$\mathrm{Tr}_{k_\alpha/\mathbb{F}_p}(C\cdot(x_+y_- - x_-y_+)) = \mathrm{Tr}_{k_\alpha/\mathbb{F}_p}(C\cdot\eta_\alpha\eta_{-\alpha}\cdot\sigma_\alpha(x_-y_+ - x_+y_-))$$

for any $x_+, x_-, y_+, y_- \in k_\alpha$. In other words, we have $\eta_\alpha\eta_{-\alpha} = -\sigma_\alpha(C)\cdot C^{-1}$.

Here we note the following lemma, which can be proved by a straightforward computation:

Lemma 6.6. *Let $V = V_1 \oplus V_2$ be a finite dimensional vector space equipped with isomorphisms $A_1\colon V_1 \to V_2$ and $A_2\colon V_2 \to V_1$. If we put $A := A_1 \oplus A_2$, then we have*

$$\det(T\cdot\mathrm{id}_V - A_1\oplus A_2 \mid V) = \det(T^2\cdot\mathrm{id}_V - A_2\circ A_1 \mid V_1).$$

By this lemma, we see that the eigenvalues of $[\eta] \in \mathrm{Sp}(k_\alpha\oplus k_\alpha)$ are given by the square roots of the eigenvalues of the action of $(\eta_\alpha\circ\sigma_\alpha)\circ(\eta_{-\alpha}\circ\sigma_\alpha)$ on k_α. As

$$(\eta_\alpha\circ\sigma_\alpha)\circ(\eta_{-\alpha}\circ\sigma_\alpha) = \eta_\alpha\cdot\sigma_\alpha(\eta_{-\alpha}) = -\sigma_\alpha(\eta_{-\alpha}C)/(\eta_{-\alpha}C),$$

the multi-set of eigenvalues of $(\eta_\alpha\circ\sigma_\alpha)\circ(\eta_{-\alpha}\circ\sigma_\alpha)$ on k_α is given by

$$\{-\tau(\gamma_\alpha) \in k_\alpha \mid \tau \in \mathrm{Gal}(k_\alpha/\mathbb{F}_p)\},$$

where we put $\gamma_\alpha := \sigma_\alpha(\eta_{-\alpha}C)/(\eta_{-\alpha}C) \in k_\alpha^\times$. Hence that of $[\eta]$ is given by

$$\{\pm(-\tau(\gamma_\alpha))^{\frac{1}{2}} \in \overline{\mathbb{F}}_p \mid \tau \in \mathrm{Gal}(k_\alpha/\mathbb{F}_p)\}. \tag{11}$$

Unramified case: We first consider the case where α_{res} is unramified.

(1) If $[\eta]$ has a fixed point in $V_{\Sigma\alpha}$ (note that this is equivalent to that $\gamma_\alpha = -1$), the multi-set (11) is given by $\{\pm1,\ldots,\pm1\}$, where ±1 is contained $[k_\alpha:\mathbb{F}_p]$-times. We take an $\mathbb{F}_p$-rational maximal torus T of $\mathrm{Sp}(k_\alpha\oplus k_\alpha)$ to be $\mathbb{G}_{\mathrm{m}}^{[k_\alpha:\mathbb{F}_p]}$. Then $[\eta]$ is $\mathrm{Sp}(k_\alpha\oplus k_\alpha)$-conjugate to the element

$$t := (\underbrace{1,\ldots,1}_{[k_{\alpha_{\mathrm{res}}}:\mathbb{F}_p]},\underbrace{-1,\ldots,-1}_{[k_{\alpha_{\mathrm{res}}}:\mathbb{F}_p]}) \in (\mathbb{F}_p^\times)^{[k_\alpha:\mathbb{F}_p]} = T(\mathbb{F}_p)$$

by Lemma A.4. We utilize Proposition A.3 (and Lemma A.5). We have $l(k_\alpha\oplus k_\alpha, T; t) = 2[k_{\alpha_{\mathrm{res}}}:\mathbb{F}_p]$. Since any Ω is asymmetric and $q_\Omega = p$, we have $\chi^T(t) = (-1)^{\frac{1-p}{2}\cdot[k_{\alpha_{\mathrm{res}}}:\mathbb{F}_p]}$. By letting $q_{\alpha_{\mathrm{res}}}$ be the order of $k_{\alpha_{\mathrm{res}}}$, we have

$$(-1)^{\frac{1-p}{2}\cdot[k_{\alpha_{\mathrm{res}}}:\mathbb{F}_p]} = (-1)^{\frac{1-q_{\alpha_{\mathrm{res}}}}{2}} = -(-1)^{\frac{1+q_{\alpha_{\mathrm{res}}}}{2}} = -\operatorname{sgn}_{k_\alpha^1}(\gamma_\alpha).$$

Recalling that $\gamma_\alpha := \sigma_\alpha(\eta_{-\alpha}C)/(\eta_{-\alpha}C)$, we get

$$\Theta_{\omega_{\Sigma\alpha}}([\eta]) = -|V_{\Sigma\alpha}^{\eta}|^{\frac{1}{2}} \cdot \mathrm{sgn}_{k_\alpha^\times}(\eta_{-\alpha}C).$$

(2) If $\gamma_\alpha = 1$ (hence $[\eta]$ does not have a fixed point in $V_{\Sigma\alpha}$), the multi-set (11) is given by $\{\pm\sqrt{-1}, \ldots, \pm\sqrt{-1}\}$, where $\pm\sqrt{-1}$ is contained $[k_\alpha : \mathbb{F}_p]$-times.

(a) When $\sqrt{-1} \in \mathbb{F}_p$, or equivalently, $p - 1 \equiv 0 \pmod 4$, we take an $\mathbb{F}_p$-rational maximal torus T of $\mathrm{Sp}(k_\alpha \oplus k_\alpha)$ to be $\mathbb{G}_\mathrm{m}^{[k_\alpha:\mathbb{F}_p]}$. Then $[\eta]$ is $\mathrm{Sp}(k_\alpha \oplus k_\alpha)$-conjugate to the element

$$t := (\sqrt{-1}, \ldots, \sqrt{-1}) \in (\mathbb{F}_p^\times)^{[k_\alpha:\mathbb{F}_p]} = T(\mathbb{F}_p)$$

by Lemma A.4. We utilize Proposition A.3 (and Lemma A.5). We have $l(k_\alpha \oplus k_\alpha, T; t) = 2[k_\alpha : \mathbb{F}_p]$. Since any Ω is asymmetric and $q_\Omega = p$, we have $\chi^T(t) = \sqrt{-1}^{\frac{1-p}{2}\cdot[k_\alpha:\mathbb{F}_p]}$. By noting that

$$\sqrt{-1}^{\frac{1-p}{2}\cdot[k_\alpha:\mathbb{F}_p]} = (\sqrt{-1}^{\frac{1-p}{2}})^{2[k_{\alpha_{\mathrm{res}}}:\mathbb{F}_p]} = 1,$$

we get

$$\Theta_{\omega_{\Sigma\alpha}}([\eta]) = 1.$$

(b) When $\sqrt{-1} \notin \mathbb{F}_p$, or equivalently, $p - 1 \equiv 2 \pmod 4$, we take an $\mathbb{F}_p$-rational maximal torus T of $\mathrm{Sp}(k_\alpha \oplus k_\alpha)$ to be $\mathrm{Ker}(\mathrm{Nr}_{\mathbb{F}_{p^2}/\mathbb{F}_p} \colon \mathrm{Res}_{\mathbb{F}_{p^2}/\mathbb{F}_p}\mathbb{G}_\mathrm{m} \to \mathbb{G}_\mathrm{m})^{[k_\alpha\,:\,\mathbb{F}_p]}$. Then $[\eta]$ is $\mathrm{Sp}(k_\alpha \oplus k_\alpha)$-conjugate to the element

$$t := (\sqrt{-1}, \ldots, \sqrt{-1}) \in (\mathbb{F}_{p^2}^1)^{[k_\alpha:\mathbb{F}_p]} = T(\mathbb{F}_p)$$

by Lemma A.4. We utilize Proposition A.3 (and Lemma A.5). We have $l(k_\alpha \oplus k_\alpha, T; t) = [k_\alpha : \mathbb{F}_p] = 2[k_{\alpha_{\mathrm{res}}} : \mathbb{F}_p]$. Since any Ω is symmetric and $q_\Omega = p$, we have $\chi^T(t) = \sqrt{-1}^{\frac{1+p}{2}\cdot[k_\alpha:\mathbb{F}_p]}$. By noting that

$$\sqrt{-1}^{\frac{1+p}{2}\cdot[k_\alpha:\mathbb{F}_p]} = (\sqrt{-1}^{\frac{1+p}{2}})^{2[k_{\alpha_{\mathrm{res}}}:\mathbb{F}_p]} = 1,$$

we get

$$\Theta_{\omega_{\Sigma\alpha}}([\eta]) = 1.$$

Note that, in both cases, $\Theta_{\omega_{\Sigma\alpha}}([\eta])$ can be also thought of as $\mathrm{sgn}_{k_\alpha^\times}(\eta_{-\alpha}C)$ since $\mathrm{sgn}_{k_\alpha^\times}(\eta_{-\alpha}C) = \mathrm{sgn}_{k_\alpha^1}(\gamma_\alpha) = 1$.

(3) If $\gamma_\alpha \neq \pm 1$ (hence $[\eta]$ does not have a fixed point in $V_{\Sigma\alpha}$), γ_α does not belong to $k_{\alpha_{\mathrm{res}}}$ since $\gamma_\alpha \in k_\alpha^1$ (otherwise, γ_α must be ± 1). Thus, if we put $k_\gamma := \mathbb{F}_p[\gamma_\alpha]$, then k_γ is not contained in $k_{\alpha_{\mathrm{res}}}$, or equivalently, $[k_\alpha : k_\gamma]$ is odd. We put $k_\gamma^\circ := k_{\alpha_{\mathrm{res}}} \cap k_\gamma$. As $\gamma_\alpha \in k_\alpha^1$, we also have $\gamma_\alpha \in k_\gamma^1$. In other words, by putting $q_\gamma := |k_\gamma|$ and $q_\gamma^\circ := |k_\gamma^\circ|$, we have $\gamma_\alpha^{q_\gamma^\circ+1} = 1$. This implies that $(-\gamma_\alpha)^{\frac{q_\gamma-1}{2}} = (\gamma_\alpha^{q_\gamma^\circ+1})^{\frac{q_\gamma^\circ-1}{2}} = 1$, i.e., $-\gamma_\alpha$ belongs to $k_\gamma^{\times 2}$. If we let $\delta_\alpha \in k_\gamma^\times$ be an element satisfying $\delta_\alpha^2 = -\gamma_\alpha$, then the multi-set (11) is given by $\{\pm\tau(\delta_\alpha) \mid \tau \in \mathrm{Gal}(k_\alpha/\mathbb{F}_p)\}$. Note that, since $\mathrm{Gal}(k_\alpha/k_\gamma)$ acts on δ_α trivially, this set is the union of $[k_\alpha : k_\gamma]$-copies of $\{\pm\tau(\delta_\alpha) \mid \tau \in \mathrm{Gal}(k_\gamma/\mathbb{F}_p)\}$. We take an $\mathbb{F}_p$-rational maximal torus T of $\mathrm{Sp}(k_\alpha \oplus k_\alpha)$ to be

$$(\mathrm{Ker}(\mathrm{Nr}_{k_\gamma/k_\gamma^\circ} \colon \mathrm{Res}_{k_\gamma/\mathbb{F}_p}\mathbb{G}_\mathrm{m} \to \mathrm{Res}_{k_\gamma^\circ/\mathbb{F}_p}\mathbb{G}_\mathrm{m})^2)^{[k_\alpha\,:\,k_\gamma]}$$

(there exists such a torus by Lemma A.5). Then $[\eta]$ is $\mathrm{Sp}(k_\alpha \oplus k_\alpha)$-conjugate to the element

$$t := (\underbrace{(\delta_\alpha, -\delta_\alpha), \ldots, (\delta_\alpha, -\delta_\alpha)}_{[k_\alpha : k_\gamma]}) \in (k_\gamma^1 \times k_\gamma^1)^{[k_\alpha : k_\gamma]} = T(\mathbb{F}_p)$$

by Lemma A.4. We utilize Proposition A.3 (and Lemma A.5). We have $l(k_\alpha \oplus k_\alpha, T; t) = 2[k_{\alpha_{\mathrm{res}}} : k_\gamma]$. Since any Ω is symmetric and $q_\Omega = q_\gamma^\circ$, we have

$$\chi^T(t) = \delta_\alpha^{\frac{1+q_\gamma^\circ}{2} \cdot [k_\alpha : k_\gamma]} \cdot (-\delta_\alpha)^{\frac{1+q_\gamma^\circ}{2} \cdot [k_\alpha : k_\gamma]} = \gamma_\alpha^{\frac{1+q_{\alpha_{\mathrm{res}}}}{2}}$$

(here we used that $[k_\alpha : k_\gamma]$ is odd). By noting that $\gamma_\alpha^{\frac{1+q_{\alpha_{\mathrm{res}}}}{2}} = \mathrm{sgn}_{k_\alpha^1}(\gamma_\alpha) = \mathrm{sgn}_{k_\alpha^\times}(\eta_{-\alpha} C)$, we get

$$\Theta_{\omega_{\Sigma\alpha}}([\eta]) = \mathrm{sgn}_{k_\alpha^\times}(\eta_{-\alpha} C)$$

Ramified case: We next consider the case where α_{res} is ramified. In this case, by noting that σ_α acts on k_α trivially, the multi-set (11) is given by $\{\pm\sqrt{-1}, \ldots, \pm\sqrt{-1}\}$, where $\pm\sqrt{-1}$ is contained $[k_\alpha : \mathbb{F}_p]$-times. Hence a similar computation to the case (2) (b) works. Consequently, we get

$$\Theta_{\omega_{\Sigma\alpha}}([\eta]) = \begin{cases} \sqrt{-1}^{\frac{1-p}{2} \cdot [k_\alpha : \mathbb{F}_p]} & \text{if } \sqrt{-1} \in \mathbb{F}_p, \\ (-1)^{[k_\alpha : \mathbb{F}_p]} \cdot \sqrt{-1}^{\frac{1+p}{2} \cdot [k_\alpha : \mathbb{F}_p]} & \text{if } \sqrt{-1} \notin \mathbb{F}_p. \end{cases}$$

Let us write $\Theta_{\omega_{\Sigma\alpha}}([\eta]) = \sqrt{-1}^{M_\alpha}$ in short.

6.5. **Twisted characters of Weil representations: symmetric roots.** We next compute $\Theta_{\omega_{\Sigma\alpha}}([\eta]^{m_\alpha})$ in the case where $\alpha \in \Xi_{\mathrm{sym}}$. Note that α must be unramified by Lemma 6.2. Recall that $V_{\Sigma\alpha} = V_{\Gamma\alpha} \cong V_\alpha$ and that V_α is a 1-dimensional k_α-vector spaces, which is identified with k_α by fixing nonzero elements $X_\alpha \in V_\alpha$ (see Section 6.1). As the order of θ is 2, there are exactly 3 possibilities:

(1) $\theta(\alpha) = \alpha$ (thus $l_\alpha = m_\alpha = 1$);
(2) $\theta(\alpha) \neq \alpha$ and $\theta(\alpha) \notin \Gamma\alpha\,(= \Sigma\alpha)$ (thus $l_\alpha = m_\alpha = 2$);
(3) $\theta(\alpha) \neq \alpha$ and $\theta(\alpha) \in \Gamma\alpha\,(= \Sigma\alpha)$ (thus $l_\alpha = 2$ and $m_\alpha = 1$).

6.5.1. *The case where $\theta(\alpha) = \alpha$.* In this case, we have $[F_\alpha : F_{\pm\alpha}] = 2$ and $F_\alpha = F_{\alpha_{\mathrm{res}}}$. Let τ_α be the unique nontrivial element of $\mathrm{Gal}(F_\alpha/F_{\pm\alpha})$, hence we have $\tau_\alpha(\alpha) = -\alpha$. By noting that $[F_{\alpha_{\mathrm{res}}} : F_{\pm\alpha_{\mathrm{res}}}] \leq 2$ and $[F_{\pm\alpha} : F_{\pm\alpha_{\mathrm{res}}}] \leq 2$, we see that $[F_{\alpha_{\mathrm{res}}} : F_{\pm\alpha_{\mathrm{res}}}] = 2$ and $F_{\pm\alpha} = F_{\pm\alpha_{\mathrm{res}}}$.

$$\begin{array}{ccc} F_{\alpha_{\mathrm{res}}} & = & F_\alpha \\ \text{quad} \Big| \langle\tau_\alpha\rangle & & \text{quad} \Big| \langle\tau_\alpha\rangle \\ F_{\pm\alpha_{\mathrm{res}}} & = & F_{\pm\alpha} \end{array}$$

The action of $[\eta]$ on $V_{\Gamma\alpha}$ preserves $V_{\Gamma\alpha}$ and is k_α-linear. We let η_α be the element of $k_\alpha^\times$ such that $[\eta](X_\alpha) = \eta_\alpha X_\alpha$. Then, as an element of $\mathrm{Sp}(V_{\Gamma\alpha}) \cong \mathrm{Sp}(k_\alpha)$, $[\eta]$ is given by $x \mapsto \eta_\alpha x$. Since $[\eta]$ preserves the symplectic form described as in Section 6.1, we necessarily have $\eta_\alpha \tau_\alpha(\eta_\alpha) = 1$. We take an $\mathbb{F}_p$-rational maximal torus T of

$\mathrm{Sp}(k_\alpha)$ to be $\mathrm{Ker}(\mathrm{Nr}_{k_\alpha/k_{\alpha_{\mathrm{res}}}} \colon \mathrm{Res}_{k_\alpha/\mathbb{F}_p}\mathbb{G}_{\mathrm{m}} \to \mathrm{Res}_{k_{\alpha_{\mathrm{res}}}/\mathbb{F}_p}\mathbb{G}_{\mathrm{m}})$. Then $[\eta]$ is $\mathrm{Sp}(k_\alpha)$-conjugate to the element $t := \eta_\alpha \in k_\alpha^1$ by Lemma A.4. Now we utilize Proposition A.3 (and Lemma A.5). We have

$$l(k_\alpha, T; t) = \begin{cases} 1 & \text{if } V_{\Sigma\alpha}^{\eta} = 0, \\ 0 & \text{if } V_{\Sigma\alpha}^{\eta} \neq 0. \end{cases}$$

Since any Ω is symmetric and $q_\Omega = q_{\alpha_{\mathrm{res}}} := |k_{\alpha_{\mathrm{res}}}|$, we have

$$\chi^T(t) = \eta_\alpha^{\frac{1+q_{\alpha_{\mathrm{res}}}}{2}} = \mathrm{sgn}_{k_\alpha^1}(\eta_\alpha).$$

Thus we get

$$\Theta_{\omega_{\Sigma\alpha}}([\eta]) = |V_{\Sigma\alpha}^{\eta}|^{\frac{1}{2}} \cdot \mathrm{sgn}_{k_\alpha^1}(\eta_\alpha) \cdot \begin{cases} -1 & \text{if } V_{\Sigma\alpha}^{\eta} = 0, \\ 1 & \text{if } V_{\Sigma\alpha}^{\eta} \neq 0. \end{cases}$$

6.5.2. *The case where $\theta(\alpha) \neq \alpha$ and $\theta(\alpha) \notin \Gamma\alpha$.* In this case, we have $[F_\alpha : F_{\pm\alpha}] = 2$ and $F_\alpha = F_{\alpha_{\mathrm{res}}}$. Let τ_α be the unique nontrivial element of $\mathrm{Gal}(F_\alpha/F_{\pm\alpha})$, hence we have $\tau_\alpha(\alpha) = -\alpha$. By noting that $[F_{\alpha_{\mathrm{res}}} : F_{\pm\alpha_{\mathrm{res}}}] \leq 2$ and $[F_{\pm\alpha} : F_{\pm\alpha_{\mathrm{res}}}] \leq 2$, we see that $[F_{\alpha_{\mathrm{res}}} : F_{\pm\alpha_{\mathrm{res}}}] = 2$ and $F_{\pm\alpha} = F_{\pm\alpha_{\mathrm{res}}}$.

$$\begin{array}{ccc} F_{\alpha_{\mathrm{res}}} & = & F_\alpha \\ \text{quad}\Big| \langle\tau_\alpha\rangle & & \text{quad}\Big| \langle\tau_\alpha\rangle \\ F_{\pm\alpha_{\mathrm{res}}} & = & F_{\pm\alpha} \end{array}$$

Since $\eta^2 \in S$, the action of $[\eta]^2 = [\eta^2]$ preserves $V_{\Gamma\alpha}$ and is k_α-linear. Hence the same argument as in the previous case works. If we let η_α^2 be the element of $k_\alpha^\times$ such that $[\eta^2](X_\alpha) = \eta_\alpha^2 X_\alpha$, then we get

$$\begin{aligned} \Theta_{\omega_{\Sigma\alpha}}([\eta]^2) &= \mathrm{sgn}_{k_\alpha^1}(\eta_\alpha^2) \cdot |V_{\Sigma\alpha}^{\eta^2}|^{\frac{1}{2}} \cdot \begin{cases} -1 & \text{if } V_\alpha^{\eta^2} = 0, \\ 1 & \text{if } V_\alpha^{\eta^2} \neq 0. \end{cases} \\ &= \mathrm{sgn}_{k_\alpha^1}(\eta_\alpha^2) \cdot |V_{\Sigma\Theta(\alpha)}^{\eta}|^{\frac{1}{2}} \cdot \begin{cases} -1 & \text{if } V_\alpha^{\eta^2} = 0, \\ 1 & \text{if } V_\alpha^{\eta^2} \neq 0. \end{cases} \end{aligned}$$

6.5.3. *The case where $\theta(\alpha) \neq \alpha$ and $\theta(\alpha) \in \Gamma\alpha$.* In this case, we have $[F_\alpha : F_{\pm\alpha}] = [F_\alpha : F_{\alpha_{\mathrm{res}}}] = 2$. Let τ_α be the unique nontrivial element of $\mathrm{Gal}(F_\alpha/F_{\pm\alpha})$, hence we have $\tau_\alpha(\alpha) = -\alpha$. Let σ_α be the unique nontrivial element of $\mathrm{Gal}(F_\alpha/F_{\alpha_{\mathrm{res}}})$, hence we have $\sigma_\alpha(\alpha) = \theta(\alpha)$. Note that, as we have $\theta(\alpha) \neq -\alpha$ (recall that $\theta_{\mathbf{S}}$ preserves a Borel subgroup containing $\mathbf{S}$), we must have $\sigma_\alpha \neq \tau_\alpha$.

$$\begin{array}{ccc} F_{\alpha_{\mathrm{res}}} & \overset{\text{quad}}{\underset{\langle\sigma_\alpha\rangle}{\text{———}}} & F_\alpha \\ \text{quad}\Big| \langle\tau_\alpha\rangle & & \text{quad}\Big| \langle\tau_\alpha\rangle \\ F_{\pm\alpha_{\mathrm{res}}} & \overset{\text{quad}}{\underset{\langle\sigma_\alpha\rangle}{\text{———}}} & F_{\pm\alpha} \end{array}$$

The action of $[\eta]$ on $V_{\Gamma\alpha}$ preserves $V_{\Gamma\alpha}$. Since $[\eta](X_\alpha)$ belongs to $V_{\theta(\alpha)} = V_{\sigma_\alpha(\alpha)}$, the induced action of $[\eta]$ on V_α is σ_α-linear (note that $[\eta]$ is k_α-linear on $V_{\Gamma\alpha}$):

$$\begin{array}{ccc} V_\alpha & \xrightarrow{\cong} & V_{\Gamma\alpha} \\ & & \downarrow{\scriptstyle [\eta]} \\ V_\alpha & \xleftarrow[\cong]{} & V_{\Gamma\alpha} \end{array} \qquad \begin{array}{ccc} X_\alpha & \longmapsto & \sum_{\sigma\in\Gamma/\Gamma_\alpha} \sigma(X_\alpha) \\ & & \downarrow \\ [\eta]\circ\sigma_\alpha(X_\alpha) & \longleftarrow & \sum_{\sigma\in\Gamma/\Gamma_\alpha} [\eta]\circ\sigma(X_\alpha) \end{array}$$

More explicitly, if we let η_α be the element of $k_\alpha^\times$ such that $[\eta]\circ\sigma_\alpha(X_\alpha) = \eta_\alpha X_\alpha$, then $[\eta]$ is given by $x \mapsto \eta_\alpha\sigma_\alpha(x)$ as an element of $\mathrm{Sp}(V_{\Gamma\alpha}) \cong \mathrm{Sp}(k_\alpha)$. Since $[\eta]$ preserves the symplectic form as described in Section 6.1, we must have

$$\mathrm{Tr}_{k_\alpha/\mathbb{F}_p}(C\cdot\eta_\alpha\sigma_\alpha(x)\tau_\alpha(\eta_\alpha\sigma_\alpha(y))) = \mathrm{Tr}_{k_\alpha/\mathbb{F}_p}(C\cdot x\tau_\alpha(y)) \tag{12}$$

for any $x, y \in k_\alpha$.

Unramified case: We first consider the case where α_{res} is unramified. In this case, σ_α acts on k_α trivially. Hence we must have $\eta_\alpha\tau_\alpha(\eta_\alpha) = 1$ by (12). The multi-set of eigenvalues of $[\eta]$ on k_α is given by $\{\tau(\eta_\alpha) \mid \tau \in \mathrm{Gal}(k_\alpha/\mathbb{F}_p)\}$. We take an $\mathbb{F}_p$-rational maximal torus T of $\mathrm{Sp}(k_\alpha)$ to be

$$\mathrm{Ker}(\mathrm{Nr}_{k_\alpha/k_{\alpha_{\mathrm{res}}}}\colon \mathrm{Res}_{k_\alpha/\mathbb{F}_p}\mathbb{G}_{\mathrm{m}} \to \mathrm{Res}_{k_{\alpha_{\mathrm{res}}}/\mathbb{F}_p}\mathbb{G}_{\mathrm{m}}).$$

Then $[\eta]$ is $\mathrm{Sp}(k_\alpha)$-conjugate to the element $t := \eta_\alpha \in k_\alpha^1$ by Lemma A.4. We utilize Proposition A.3 (and Lemma A.5). We have

$$l(k_\alpha, T; t) = \begin{cases} 1 & \text{if } V_{\Sigma\alpha}^\eta = 0, \\ 0 & \text{if } V_{\Sigma\alpha}^\eta \neq 0. \end{cases}$$

Since any Ω is symmetric and $q_\Omega = q_{\alpha_{\mathrm{res}}} := |k_{\alpha_{\mathrm{res}}}|$, we have $\chi^T(t) = (\eta_\alpha)^{\frac{1+q_{\alpha_{\mathrm{res}}}}{2}} = \mathrm{sgn}_{k_\alpha^1}(\eta_\alpha)$. Hence we get

$$\Theta_{\omega_{\Sigma\alpha}}([\eta]) = |V_\alpha^\eta|^{\frac{1}{2}} \cdot \mathrm{sgn}_{k_\alpha^1}(\eta_\alpha) \cdot \begin{cases} -1 & \text{if } V_{\Sigma\alpha}^\eta = 0, \\ 1 & \text{if } V_{\Sigma\alpha}^\eta \neq 0. \end{cases}$$

Ramified case: We next consider the case where α_{res} is ramified. As σ_α and τ_α induce the same (nontrivial) action on k_α, we must have $\eta_\alpha\tau_\alpha(\eta_\alpha) = C/\tau_\alpha(C) = -1$ by (12). Note that η_α and σ_α act on $k_\alpha \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p \cong \prod_{\tau\in\mathrm{Gal}(k_\alpha/\mathbb{F}_p)} \overline{\mathbb{F}}_p$ via $(\tau(\eta_\alpha))_\tau$-multiplication and swapping the $\mathrm{Gal}(k_{\alpha_{\mathrm{res}}}/\mathbb{F}_p)$-part and $\mathrm{Gal}(k_\alpha/\mathbb{F}_p) \smallsetminus \mathrm{Gal}(k_{\alpha_{\mathrm{res}}}/\mathbb{F}_p)$-part, respectively. Thus the eigenvalues of $[\eta]$ on k_α is given by the square roots of the eigenvalues of $(\eta_\alpha\circ\sigma_\alpha)^2 = \eta_\alpha\sigma_\alpha(\eta_\alpha)$ on $k_{\alpha_{\mathrm{res}}}$. Hence, by a similar consideration to the case where $\alpha \in \Xi_{\mathrm{asym}}$ and $F_\alpha/F_{\alpha_{\mathrm{res}}}$ is ramified in Section 6.4.4, we get

$$\Theta_{\omega_{\Sigma\alpha}}([\eta]) = \begin{cases} \sqrt{-1}^{\frac{1-p}{2}[k_{\alpha_{\mathrm{res}}}:\mathbb{F}_p]} & \text{if } \sqrt{-1} \in \mathbb{F}_p, \\ (-1)^{[k_{\alpha_{\mathrm{res}}}:\mathbb{F}_p]}\sqrt{-1}^{\frac{1+p}{2}[k_{\alpha_{\mathrm{res}}}:\mathbb{F}_p]} & \text{if } \sqrt{-1} \notin \mathbb{F}_p. \end{cases}$$

Let us write $\Theta_{\omega_{\Gamma\alpha}}([\eta]) = \sqrt{-1}^{M_\alpha}$ in short.

6.6. **Twisted characters of Weil representations: summary.** Now let us summarize the computation presented in Sections 6.4 and 6.5.

By recalling that we have fixed a base point $\underline{\eta}$, we write $\eta = s\underline{\eta}$ with $s \in S$. Then we have $\eta_\alpha = \alpha(s)\underline{\eta}_\alpha$ and $\eta_{-\alpha} = \alpha(s)^{-1}\underline{\eta}_{-\alpha}$. Similarly, as $\eta^2 = s\theta_{\mathbf{S}}(s)\underline{\eta}^2$, we also have $\eta_\alpha^2 = \alpha(s)\theta(\alpha)(s)\underline{\eta}_\alpha^2$. Noting this, we introduce a sign $C_{\alpha,\underline{\eta}}$ for each $\alpha \in \Xi$ as follows. For $\alpha \in \Xi_{\mathrm{asym}}$, we put

$$C_{\alpha,\underline{\eta}} := \begin{cases} \mathrm{sgn}_{k_\alpha^\times}(\underline{\eta}_\alpha) & \text{if } \theta(\alpha) = \alpha, \\ \mathrm{sgn}_{k_\alpha^\times}(\underline{\eta}_\alpha^2) & \text{if } \theta(\alpha) \neq \alpha,\ \theta(\alpha) \notin \Sigma\alpha, \\ \mathrm{sgn}_{k_\alpha^\times}(\underline{\eta}_\alpha) \cdot \mathrm{sgn}_{k_{\alpha_{\mathrm{res}}}^\times}(-1) & \text{if } \theta(\alpha) \neq \alpha,\ \theta(\alpha) \in \Gamma\alpha,\ F_\alpha/F_{\alpha_{\mathrm{res}}}\text{: ur}, \\ \mathrm{sgn}_{k_\alpha^\times}(\underline{\eta}_\alpha) & \text{if } \theta(\alpha) \neq \alpha,\ \theta(\alpha) \in \Gamma\alpha,\ F_\alpha/F_{\alpha_{\mathrm{res}}}\text{: ram}, \\ \mathrm{sgn}_{k_\alpha^\times}(\underline{\eta}_{-\alpha}C) & \text{if } \theta(\alpha) \neq \alpha,\ \theta(\alpha) \in -\Gamma\alpha,\ \alpha_{\mathrm{res}}\text{: ur}, \\ \sqrt{-1}^{M_\alpha} & \text{if } \theta(\alpha) \neq \alpha,\ \theta(\alpha) \in -\Gamma\alpha,\ \alpha_{\mathrm{res}}\text{: ram}. \end{cases}$$

For $\alpha \in \Xi_{\mathrm{sym}}$, we put

$$C_{\alpha,\underline{\eta}} := \begin{cases} -\mathrm{sgn}_{k_\alpha^1}(\underline{\eta}_\alpha) & \text{if } \theta(\alpha) = \alpha, \\ -\mathrm{sgn}_{k_\alpha^1}(\underline{\eta}_\alpha^2) & \text{if } \theta(\alpha) \neq \alpha,\ \theta(\alpha) \notin \Gamma\alpha, \\ -\mathrm{sgn}_{k_\alpha^1}(\underline{\eta}_\alpha) & \text{if } \theta(\alpha) \neq \alpha,\ \theta(\alpha) \in \Gamma\alpha,\ \alpha_{\mathrm{res}}\text{: ur}, \\ \sqrt{-1}^{M_\alpha} & \text{if } \theta(\alpha) \neq \alpha,\ \theta(\alpha) \in \Gamma\alpha,\ \alpha_{\mathrm{res}}\text{: ram}. \end{cases}$$

We also note that, for $\alpha \in \Xi_{\mathrm{asym}}$, its restricted root α_{res} is

$$\begin{cases} \text{asym} & \text{if } \theta(\alpha) = \alpha, \\ \text{asym} & \text{if } \theta(\alpha) \neq \alpha,\ \theta(\alpha) \notin \Sigma\alpha, \\ \text{asym} & \text{if } \theta(\alpha) \neq \alpha,\ \theta(\alpha) \in \Gamma\alpha, \\ \text{ur or ram} & \text{if } \theta(\alpha) \neq \alpha,\ \theta(\alpha) \in -\Gamma\alpha, \end{cases}$$

and that, for $\alpha \in \Xi_{\mathrm{sym}}$, its restricted root α_{res} is

$$\begin{cases} \text{ur} & \text{if } \theta(\alpha) = \alpha, \\ \text{ur} & \text{if } \theta(\alpha) \neq \alpha,\ \theta(\alpha) \notin \Gamma\alpha, \\ \text{ur or ram} & \text{if } \theta(\alpha) \neq \alpha,\ \theta(\alpha) \in \Gamma\alpha. \end{cases}$$

We introduce characters $\epsilon_\alpha \colon S \to \mathbb{C}^\times$ for $\alpha \in \Phi(\mathbf{G}, \mathbf{S})$ as follows:

$$\epsilon_\alpha(s) := \begin{cases} \mathrm{sgn}_{k_\alpha^\times}(\overline{\alpha(s)}) & \text{if } \alpha \in \Phi_{\mathrm{asym}}(\mathbf{G}, \mathbf{S}), \\ \mathrm{sgn}_{k_\alpha^1}(\overline{\alpha(s)}) & \text{if } \alpha \in \Phi_{\mathrm{ur}}(\mathbf{G}, \mathbf{S}). \end{cases}$$

Then, the computation in Sections 6.4 and 6.5 are summarized as follows:

$$\Theta_{\omega_{\Sigma\alpha}}([\eta]^{m_\alpha}) = (-1)^{\bullet} \cdot C_{\alpha,\underline{\eta}} \cdot |V_{\Sigma\Theta(\alpha)}^{\eta}|^{\frac{1}{2}} \cdot \begin{cases} \prod_{i=0}^{m_\alpha - 1} \epsilon_{\theta^i(\alpha)}(s) & \text{if } \alpha_{\mathrm{res}} \text{ is asym or ur}, \\ 1 & \text{if } \alpha_{\mathrm{res}} \text{ is ram}, \end{cases}$$

where $\bullet = 1$ if $\alpha_{\mathrm{res}} \in \Xi_{\eta,\mathrm{ur}}$ and $\bullet = 0$ otherwise.

Therefore, by defining a constant $C_{\underline{\eta}}$ to be the product of $C_{\alpha,\underline{\eta}}$ over $\alpha \in \Theta_{\mathbf{S}} \backslash \ddot{\Xi}$, we get the following (recall the description of V_η in Section 6.3):

Proposition 6.7. *We have*

$$\prod_{\alpha \in \Theta_{\mathbf{S}} \backslash \ddot{\Xi}} \Theta_{\omega_{\Sigma\alpha}}([\eta]^{m_\alpha}) = (-1)^{|\ddot{\Xi}_{\eta,\mathrm{ur}}|} \cdot C_{\underline{\eta}} \cdot |V_\eta|^{\frac{1}{2}} \cdot \prod_{\substack{\alpha \in \ddot{\Xi} \\ \alpha_{\mathrm{res}}\text{: asym/ur}}} \epsilon_\alpha(s).$$

6.7. **Twisted character formula of the final form.** Now let us go back to the twisted character formula for toral supercuspidal representations. From now on, η again denotes $\delta_{<r}$ for a fixed elliptic regular semisimple element $\delta \in \tilde{S}$. Recall that $\eta = \eta_0\eta_+$ is the fixed topological Jordan decomposition.

For $s \in S$, we put

$$\tilde{\epsilon}_{\Xi}(s) := \prod_{\substack{\alpha\in\ddot{\Xi} \\ \alpha_{\mathrm{res}}\colon \mathrm{asym/ur}}} \epsilon_\alpha(s).$$

Proposition 6.8. *If we write $\eta = s\underline{\eta}$ with an element $s \in S$, then we have*

$$\Theta_{\tilde{\rho}}(\eta) = \vartheta(s)\cdot(-1)^{|\ddot{\Xi}_{\eta_0,\mathrm{ur}}|}\cdot C_{\underline{\eta}}\cdot|V_{\eta_0}|^{\frac{1}{2}}\cdot\tilde{\epsilon}_{\Xi}(s).$$

Proof. By the definition of $\tilde{\rho}$ and its twisted character (see Sections 4.2 and 5.2),

$$\Theta_{\tilde{\rho}}(\eta) = \mathrm{tr}\bigl(\rho(s)\circ I_\rho^\eta\bigr) = \mathrm{tr}\bigl(\omega([s])\circ I_\omega^\eta\bigr)\cdot\vartheta(s).$$

By Corollary 6.5 and Proposition 6.7, we have

$$\mathrm{tr}\bigl(\omega([s])\circ I_\omega^\eta\bigr) = \prod_{\alpha\in\Theta_{\mathbf{S}}\setminus\ddot{\Xi}}\Theta_{\omega_{\Sigma\alpha}}([\eta_0]^{m_\alpha}) = (-1)^{|\ddot{\Xi}_{\eta_0,\mathrm{ur}}|}\cdot C_{\underline{\eta}}\cdot|V_{\eta_0}|^{\frac{1}{2}}\cdot\tilde{\epsilon}_{\Xi}(s'),$$

where $s' \in S$ is the element satisfying $\eta_0 = s'\underline{\eta}$. Since η_0 commutes with η_+, we have that $\eta_+s'\underline{\eta}$ equals $s\underline{\eta}$, hence $\eta_+s' = s$. This implies that $\epsilon_\alpha(s) = \epsilon_\alpha(s')$ for any $\alpha \in \ddot{\Xi}$ such that α_{res} is asymmetric or symmetric unramified. Thus we get the desired identity. □

Lemma 6.9. *We have*

$$|V_{\eta_0}|^{\frac{1}{2}}\cdot|\tilde{\mathfrak{G}}_{\mathbf{G}_{\eta_0}}(\vartheta,\eta_+)| = |\mathfrak{g}_{\eta,\mathbf{x},0:0+}|^{-\frac{1}{2}}\cdot|\mathfrak{s}^\natural_{0:0+}|^{\frac{1}{2}}\cdot|D^{\mathrm{red}}_{G_\eta}(X^*)|^{\frac{1}{2}}\cdot|D^{\mathrm{red}}_{G_{\eta_0}}(\eta_+)|^{-\frac{1}{2}},$$

where D^{red} is as in [DS18, Definition 2.11].

Proof. We utilize [AS09, Proposition 5.2.12] with $(\mathbf{G},\mathbf{G}',\phi,\gamma) = (\mathbf{G}_{\eta_0},\mathbf{S}^\natural,\vartheta,\eta_+)$. As we have $[\![\eta_+;\mathbf{x},r(+)]\!]_{S^\natural} = S^\natural_{0+}$, $C^{(0+)}_{S^\natural}(\eta_+) = S^\natural$, and $C^{(0+)}_{G_{\eta_0}}(\eta_+) = G_{\eta_0}$, we get

$$\begin{aligned}&|(S^\natural,G_{\eta_0})_{\mathbf{x},(r,s):(r,s+)}|^{\frac{1}{2}}\cdot|\tilde{\mathfrak{G}}_{\mathbf{G}_{\eta_0}}(\vartheta,\eta_+)|\\ &\qquad= \bigl[[\![\eta_+;\mathbf{x},r]\!]_{G_{\eta_0}} : S^\natural_{0+}G_{\eta_0,\mathbf{x},s}\bigr]^{\frac{1}{2}}\cdot\bigl[[\![\eta_+;\mathbf{x},r+]\!]_{G_{\eta_0}} : S^\natural_{0+}G_{\eta_0,\mathbf{x},s+}\bigr]^{\frac{1}{2}}.\end{aligned}$$

By [DS18, Corollary 4.13], the right-hand side equals

$$|\mathfrak{g}_{\eta,\mathbf{x},0:0+}|^{-\frac{1}{2}}\cdot|\mathfrak{s}^\natural_{0:0+}|^{\frac{1}{2}}\cdot|D^{\mathrm{red}}_{G_\eta}(X^*)|^{\frac{1}{2}}\cdot|D^{\mathrm{red}}_{G_{\eta_0}}(\eta_+)|^{-\frac{1}{2}}.$$

Since $V_{\eta_0} = (S^\natural,G_{\eta_0})_{\mathbf{x},(r,s):(r,s+)}$ (see Section 6.3), we get the assertion. □

Now recall that the Fourier transform of the orbital integral depends on the choice of Haar measures. We let $\hat{\mu}^{\mathbf{G}_\eta}_{\mathrm{Wal},X^*}$ denote the Fourier transform of the orbital integral with respect to X^* normalized via the canonical measure of Waldspurger (see [DS18, Definition 4.6]). Then, by (the proof of) [DS18, Proposition 4.26],

$$\hat{\mu}^{\mathbf{G}_\eta}_{\mathrm{Wal},X^*} = |(\mathfrak{s}^\natural,\mathfrak{g}_\eta)_{\mathbf{x},(0,0):(0,0+)}|^{-\frac{1}{2}}\cdot\hat{\mu}^{\mathbf{G}_\eta}_{X^*} = |\mathfrak{g}_{\eta,\mathbf{x},0:0+}|^{-\frac{1}{2}}\cdot|\mathfrak{s}^\natural_{0:0+}|^{\frac{1}{2}}\cdot\hat{\mu}^{\mathbf{G}_\eta}_{X^*}.$$

Following [Kal19b, Section 4.2], we put

$$\hat{\iota}^{\mathbf{G}_\eta}_{X^*}(-) := |D^{\mathrm{red}}_{G_\eta}(X^*)|^{\frac{1}{2}}\cdot|D^{\mathrm{red}}_{G_\eta}(-)|^{\frac{1}{2}}\cdot\hat{\mu}^{\mathbf{G}_\eta}_{\mathrm{Wal},X^*}(-).$$

We put $\Phi_{\tilde{\pi}}(\delta) := \Delta_{\mathrm{IV}}^{\tilde{\mathbf{G}}}(\delta)\cdot\Theta_{\tilde{\pi}}(\delta)$, where $\Delta_{\mathrm{IV}}^{\tilde{\mathbf{G}}}(\delta)$ is the fourth absolute transfer for $\tilde{\mathbf{G}}$ in the sense of Kottwitz–Shelstad (see [KS99, Section 4.5]; this will be reviewed in Section 13.4).

Theorem 6.10. *If we write ${}^{g}\eta = s_g \cdot \underline{\eta} \in \tilde{S}$ for any $g \in G$ satisfying ${}^{g}\eta \in \tilde{S}$, then we have*

$$\Phi_{\tilde{\pi}}(\delta) = C_{\underline{\eta}} \cdot (-1)^{|\ddot{\Xi}_{\eta_0,\mathrm{ur}}|} \cdot \sum_{\substack{g\in S\backslash G/G_\eta \\ {}^{g}\eta\in\tilde{S}}} \vartheta(s_g)\cdot\tilde{\epsilon}_{\Xi}(s_g)\cdot\mathfrak{G}_{\mathbf{G}_{{}^{g}\eta_0}}(\vartheta,{}^{g}\eta_+)\cdot\hat{\iota}_{X^*}^{\mathbf{G}_{{}^{g}\eta}}(\log({}^{g}\delta_{\geq r})).$$

Proof. By Theorem 5.16, we have

$$\Theta_{\tilde{\pi}}(\delta) = \sum_{\substack{g\in S\backslash G/G_\eta \\ {}^{g}\eta\in\tilde{S}}} \Theta_{\tilde{\rho}}({}^{g}\eta)\cdot|\tilde{\mathfrak{G}}_{\mathbf{G}_{{}^{g}\eta_0}}(\vartheta,{}^{g}\eta_+)|\cdot\mathfrak{G}_{\mathbf{G}_{{}^{g}\eta_0}}(\vartheta,{}^{g}\eta_+)\cdot\hat{\mu}_{X^*}^{\mathbf{G}_{{}^{g}\eta}}(\log({}^{g}\delta_{\geq r})).$$

Let us compute each summand by fixing $g \in S\backslash G/G_\eta$ satisfying ${}^{g}\eta \in \tilde{S}$. By Proposition 6.8, we have

$$\Theta_{\tilde{\rho}}({}^{g}\eta) = \vartheta(s_g)\cdot(-1)^{|\ddot{\Xi}_{{}^{g}\eta_0,\mathrm{ur}}|}\cdot C_{\underline{\eta}}\cdot|V_{{}^{g}\eta_0}|^{\frac{1}{2}}\cdot\tilde{\epsilon}_{\Xi}(s_g).$$

Hence, with the above modification of the Fourier transform of the orbital integral, we see that the corresponding summand equals

$$\Delta_{\mathrm{IV}}^{\tilde{\mathbf{G}}}(\delta)^{-1}\cdot\vartheta(s_g)\cdot(-1)^{|\ddot{\Xi}_{{}^{g}\eta_0,\mathrm{ur}}|}\cdot C_{\underline{\eta}}\cdot\tilde{\epsilon}_{\Xi}(s_g)\cdot\mathfrak{G}_{\mathbf{G}_{{}^{g}\eta_0}}(\vartheta,{}^{g}\eta_+)\cdot\hat{\iota}_{X^*}^{\mathbf{G}_{{}^{g}\eta}}(\log({}^{g}\delta_{\geq r})).$$

by using Lemmas 6.9 and 13.9, which will be proved later. By finally noting that $|\ddot{\Xi}_{{}^{g}\eta_0,\mathrm{ur}}| = |\ddot{\Xi}_{\eta_0,\mathrm{ur}}|$ for any $g \in G$ satisfying ${}^{g}\eta \in \tilde{S}$, we get the desired formula. □

By using the notion of a-data and χ-data, which will be introduced later, we can rewrite the above formula in the following way.

Proposition 6.11. *With the notation as in Theorem 6.10, we have*

$$\begin{aligned}\Phi_{\tilde{\pi}}(\delta) = C_{\underline{\eta}} &\cdot (-1)^{|\ddot{\Xi}_{\eta_0,\mathrm{ur}}|}\cdot e(\mathbf{G}_{\eta_0})\cdot e(\mathbf{G}_\eta)\cdot\varepsilon(\mathbf{T}_{\mathbf{G}^*_{\eta_0}})\cdot\varepsilon(\mathbf{T}_{\mathbf{G}^*_{\eta}})^{-1} \\ &\sum_{\substack{g\in S\backslash G/G_\eta \\ {}^{g}\eta\in\tilde{S}}} \vartheta(s_g)\cdot\tilde{\epsilon}_{\Xi}(s_g)\cdot\Delta_{\mathrm{II}}^{\mathbf{G}_{{}^{g}\eta_0}}[a_\vartheta^\natural,\chi_\vartheta^\natural]({}^{g}\eta_+)\cdot\hat{\iota}_{X^*}^{\mathbf{G}_{{}^{g}\eta}}(\log({}^{g}\delta_{\geq r})),\end{aligned}$$

where $e(-)$ denotes the Kottwitz sign and $a_\vartheta^\natural$ and $\chi_\vartheta^\natural$ are the sets of a-data and χ-data associated to the tame elliptic toral pair $(\mathbf{S}^\natural,\vartheta|_{S^\natural})$ of $\mathbf{G}_{{}^{g}\eta_0}$ as in Section 7.1.

Proof. We investigate the summands of the right-hand side of Theorem 6.10. By the proof of [DS18, Proposition 4.21], $\mathfrak{G}_{\mathbf{G}_{\eta_0}}(\vartheta,\eta_+)$ is given by

$$\varepsilon_{\mathbf{G}_{\eta_0}}(\vartheta,\eta_+)^{-1}\cdot\varepsilon_{\mathbf{G}_{\eta_0},\mathrm{ram}}(\pi',\eta_+)\cdot\varepsilon^{\mathrm{ram}}_{\mathbf{G}_{\eta_0}}(\pi',\eta_+)\cdot\tilde{e}(\pi',\eta_+)$$

with the notation as in *loc. cit.* Since the depth-zero part of η_+ is trivial, we have $\varepsilon_{\mathbf{G}_{\eta_0}}(\vartheta,\eta_+) = 1$ (see [AS09, Proposition 3.8]) and $\varepsilon^{\mathrm{ram}}_{\mathbf{G}_{\eta_0}}(\pi',\eta_+) = 1$ (see [DS18, Notation 4.14]). On the other hand, recall that $(\mathbf{S}^\natural,\vartheta|_{S^\natural})$ is a tame elliptic toral pair of $\mathbf{G}_{\eta_0}$ by Lemma 5.4. Thus, by [Kal19b, Corollary 4.7.6], the product $\varepsilon_{\mathbf{G}_{\eta_0},\mathrm{ram}}(\pi',\eta_+)\cdot\tilde{e}(\pi',\eta_+)$ equals

$$\varepsilon_{\mathbf{S}^\natural,\mathrm{ram}}(\eta_+)\cdot e(\mathbf{G}_{\eta_0})\cdot e(\mathbf{G}_\eta)\cdot\varepsilon(\mathbf{T}_{\mathbf{G}^*_{\eta_0}})\cdot\varepsilon(\mathbf{T}_{\mathbf{G}^*_{\eta}})^{-1}\cdot\Delta_{\mathrm{II}}^{\mathbf{G}_{\eta_0}}[a_\vartheta^\natural,\chi_\vartheta^\natural](\eta_+).$$

Note that $\varepsilon_{\mathbf{S}^\natural,\mathrm{ram}}(\eta_+)$ is trivial as η_+ has no depth zero part. The same discussion can be applied to any ${}^g\eta$. Moreover, since any $g \in G$ does not change $e(\mathbf{G}_{\eta_0}) \cdot e(\mathbf{G}_\eta) \cdot \varepsilon(\mathbf{T}_{\mathbf{G}^*_{\eta_0}}) \cdot \varepsilon(\mathbf{T}_{\mathbf{G}^*_\eta})$ by conjugating η, we get the desired formula. □

7. Kaletha's LLC for regular supercuspidal representations

In this section, we review Kaletha's construction of the local Langlands correspondence for regular supercuspidal representations, mainly focusing on the case of toral supercuspidal representations.

Let $\hat{\mathbf{G}}$ be the Langlands dual group of $\mathbf{G}$. More precisely, $\hat{\mathbf{G}}$ is a connected reductive group over $\mathbb{C}$ equipped with

- a Γ-action on $\hat{\mathbf{G}}$,
- a Γ-stable splitting $\mathbf{spl}_{\hat{\mathbf{G}}} = (\hat{\mathbf{B}}, \hat{\mathbf{T}}, \{\mathcal{X}_{\alpha^\vee}\}_{\alpha^\vee})$ of $\hat{\mathbf{G}}$, and
- a Γ-equivariant isomorphism between the based root data $\Psi(\hat{\mathbf{G}})$ of $\hat{\mathbf{G}}$ and the dual $\Psi(\mathbf{G})^\vee$ of that of $\mathbf{G}$.

We put ${}^L\mathbf{G} := \hat{\mathbf{G}} \rtimes W_F$.

7.1. Kaletha's a-data and χ-data.

We first recall the notion of a set of a-data.

Definition 7.1. Let $\mathbf{S}$ be an F-rational maximal torus of $\mathbf{G}$. A family $\{a_\alpha\}_{\alpha\in\Phi(\mathbf{G},\mathbf{S})}$ of elements $a_\alpha \in F_\alpha^\times$ is called a *set of a-data (with respect to* $\mathbf{S}$*)* if the following conditions are satisfied:

- $a_{-\alpha} = a_\alpha^{-1}$ for any $\alpha \in \Phi(\mathbf{G},\mathbf{S})$, and
- $a_{\sigma(\alpha)} = \sigma(a_\alpha)$ for any $\alpha \in \Phi(\mathbf{G},\mathbf{S})$ and $\sigma \in \Gamma$.

Following [Kal19b, Section 4.7], for a tame elliptic toral pair $(\mathbf{S},\vartheta)$ of $\mathbf{G}$, we define a set $a_\vartheta = \{a_{\vartheta,\alpha}\}_{\alpha\in\Phi(\mathbf{G},\mathbf{S})}$ of a-data by the following (note that a_ϑ is simply denoted by a in [Kal19b, Section 4.7]):

$$a_{\vartheta,\alpha} = \langle H_\alpha, X^*\rangle,$$

where

- $H_\alpha := d\alpha^\vee(1) \in \mathfrak{s}(F_\alpha)$, and
- $X^* \in \mathfrak{s}^*_{-r}$ is an element associated to ϑ (see Section 4.2).

We next recall the notion of a set of (minimally) χ-data.

Definition 7.2. Let $\mathbf{S}$ be an F-rational maximal torus of $\mathbf{G}$. A family $\{\chi_\alpha\}_{\alpha\in\Phi(\mathbf{G},\mathbf{S})}$ of characters $\chi_\alpha\colon F_\alpha^\times \to \mathbb{C}^\times$ is called a *set of χ-data (with respect to* $\mathbf{S}$*)* if the following conditions are satisfied:

- $\chi_{-\alpha} = \chi_\alpha^{-1}$ for any $\alpha \in \Phi(\mathbf{G},\mathbf{S})$,
- $\chi_{\sigma(\alpha)} = \chi_\alpha \circ \sigma^{-1}$ for any $\alpha \in \Phi(\mathbf{G},\mathbf{S})$ and $\sigma \in \Gamma$, and
- $\chi_\alpha|_{F_{\pm\alpha}^\times}$ equals the quadratic character κ_α corresponding to the quadratic extension $F_\alpha/F_{\pm\alpha}$ for any $\alpha \in \Phi(\mathbf{G},\mathbf{S})_{\mathrm{sym}}$.

Definition 7.3 ([Kal19b, Definition 4.6.1])**.** Let $\mathbf{S}$ be an F-rational maximal torus of $\mathbf{G}$. A set $\{\chi_\alpha\}_{\alpha\in\Phi(\mathbf{G},\mathbf{S})}$ of χ-data with respect to $\mathbf{S}$ is said to be *minimally ramified* if the following conditions are satisfied:

- $\chi_\alpha = \mathbb{1}$ for any $\alpha \in \Phi(\mathbf{G},\mathbf{S})_{\mathrm{asym}}$,
- χ_α is unramified for any $\alpha \in \Phi(\mathbf{G},\mathbf{S})_{\mathrm{ur}}$, and
- χ_α is tamely ramified for any $\alpha \in \Phi(\mathbf{G},\mathbf{S})_{\mathrm{ram}}$.

Following [Kal19b, Section 4.7], for a tame elliptic toral pair $(\mathbf{S},\vartheta)$ of $\mathbf{G}$, we define a set $\chi_\vartheta = \{\chi_{\vartheta,\alpha}\}_{\alpha\in\Phi(\mathbf{G},\mathbf{S})}$ of minimally ramified χ-data as follows (note that χ_ϑ is denoted by χ' in [Kal19b, Section 4.7]):

- For $\alpha \in \Phi(\mathbf{G},\mathbf{S})_{\mathrm{asym}}$, let $\chi_{\vartheta,\alpha}$ be the trivial character of $F_\alpha^\times$.
- For $\alpha \in \Phi(\mathbf{G},\mathbf{S})_{\mathrm{ur}}$, let $\chi_{\vartheta,\alpha}$ be the unique unramified nontrivial quadratic character of $F_\alpha^\times$.
- For $\alpha \in \Phi(\mathbf{G},\mathbf{S})_{\mathrm{ram}}$, let $\chi_{\vartheta,\alpha}$ be the unique tamely ramified character of $F_\alpha^\times$ characterized by the following properties:
$$\chi_{\vartheta,\alpha}|_{F_{\pm\alpha}^\times} = \kappa_\alpha \quad \text{and} \quad \chi_{\vartheta,\alpha}(2a_{\vartheta,\alpha}) = \lambda_\alpha.$$

Remark 7.4. For a general tame elliptic regular pair $(\mathbf{S},\vartheta)$, Kaletha's sets of a-data a_ϑ and χ-data χ_ϑ are defined by noting the inductive structure of $\Phi(\mathbf{G},\mathbf{S})$ given by the tame twisted Levi subgroups $\vec{\mathbf{G}}$ determined by $(\mathbf{S},\vartheta)$. See [Kal19b, Section 4.7] (and also [OT21, Section 6]) for the details.

Definition 7.5. Let $\mathbf{S}$ be an F-rational maximal torus of $\mathbf{G}$. Let $a = \{a_\alpha\}_\alpha$ be a set of a-data and $\chi = \{\chi_\alpha\}_\alpha$ a set of χ-data with respect to $\mathbf{S}$. We define a function $\Delta_{\mathbf{G},\mathrm{II}}[a,\chi]\colon S \to \mathbb{C}^\times$ by
$$\Delta_{\mathbf{G},\mathrm{II}}[a,\chi](s) := \prod_{\substack{\alpha\in\dot{\Phi}(\mathbf{G},\mathbf{S})\\ \alpha(s)\neq 1}} \chi_\alpha\Big(\frac{\alpha(s)-1}{a_\alpha}\Big).$$

7.2. **DeBacker–Spice sign and Kaletha's toral invariant.** In this section, we recall two invariants which play a key role in Kaletha's construction of the local Langlands correspondence. Let $(\mathbf{S},\vartheta)$ be a tame elliptic toral pair of $\mathbf{G}$.

The first invariant is DeBacker–Spice's sign character introduced in [DS18, Section 4.3]. Recall that, in Section 6.6, we introduced a character ϵ_α of S for each $\alpha \in \Phi(\mathbf{G},\mathbf{S})_{\mathrm{asym}}$ and each $\alpha \in \Phi(\mathbf{G},\mathbf{S})_{\mathrm{ur}}$. We define characters $\epsilon_{\vartheta,\mathrm{asym}}$ and $\epsilon_{\vartheta,\mathrm{ur}}$ of S by taking their product over $\ddot{\Xi}$:
$$\epsilon_{\vartheta,\mathrm{asym}}(s) := \prod_{\alpha\in\ddot{\Xi}_{\mathrm{asym}}} \epsilon_\alpha(s) \quad \text{and} \quad \epsilon_{\vartheta,\mathrm{ur}}(s) := \prod_{\alpha\in\ddot{\Xi}_{\mathrm{ur}}} \epsilon_\alpha(s).$$

Here note that, in [DS18, Section 4.3], the products are taken over the roots satisfying the condition "$\frac{r}{2} \in \mathrm{ord}_{\mathbf{x}}(\alpha)$", where r is the depth of ϑ and $\mathrm{ord}_{\mathbf{x}}(\alpha)$ is the set defined in [DS18, Definition 3.6]. This condition is equivalent to that $\alpha \in \Xi$ (see the proof of [OT21, Proposition 5.12]).

The second invariant is the character $\epsilon_{\mathbf{S},\mathrm{ram}}$ of S defined in [Kal19b, Definition 4.7.3]. As explained in [Kal19b, Lemma 4.7.4], this can be expressed as the product of *toral invariants* $f_{(\mathbf{G},\mathbf{S})}(\alpha)$ for symmetric ramified roots α, which are introduced in [Kal15, Section 4.1]. We recall that the toral invariant $f_{(\mathbf{G},\mathbf{S})}(\alpha)$ for $\Phi(\mathbf{G},\mathbf{S})_{\mathrm{sym}}$ (not necessarily ramified) is defined as follows. We fix an element $\tau_\alpha \in \Gamma_{\pm\alpha} \smallsetminus \Gamma_\alpha$ (i.e., $\tau_\alpha \in \Gamma_{\pm\alpha}$ is an element satisfying $\tau_\alpha(\alpha) = -\alpha$). If we take an F_α-rational root vector $X_\alpha \in \mathfrak{g}_\alpha(F_\alpha)$, then $\tau_\alpha(X_\alpha)$ belongs to $\mathfrak{g}_{-\alpha}(F_\alpha)$ and the ratio of $[X_\alpha, \tau_\alpha(X_\alpha)]$ to $H_\alpha := d\alpha^\vee(1) \in \mathfrak{s}(F_\alpha)$ lies in $F_{\pm\alpha}^\times$. By noting that $\frac{[X_\alpha,\tau_\alpha(X_\alpha)]}{H_\alpha}$ is well-defined up to $\mathrm{Nr}_{F_\alpha/F_{\pm\alpha}}(F_\alpha^\times)$-multiplication, we put
$$f_{(\mathbf{G},\mathbf{S})}(\alpha) := \kappa_\alpha\Big(\frac{[X_\alpha,\tau_\alpha(X_\alpha)]}{H_\alpha}\Big) \in \{\pm 1\}.$$

Then we have

$$\epsilon_{\mathbf{S},\mathrm{ram}}(s)=\prod_{\substack{\alpha\in\dot{\Phi}(\mathbf{G},\mathbf{S})_{\mathrm{ram}}\\ \alpha(s)\neq 1\\ \mathrm{val}_F(\alpha(s)-1)=0}} f_{(\mathbf{G},\mathbf{S})}(\alpha).$$

Remark 7.6. Similarly to the definition of a_ϑ and χ_ϑ, for a general tame elliptic regular pair $(\mathbf{S},\vartheta)$, the characters $\epsilon_{\vartheta,\mathrm{asym}}$ and $\epsilon_{\vartheta,\mathrm{ur}}$ are defined by noting the inductive structure of $\Phi(\mathbf{G},\mathbf{S})$ given by the tame twisted Levi subgroups $\vec{\mathbf{G}}$ determined by $(\mathbf{S},\vartheta)$. See [Kal19b, Section 4.3] (and also [OT21, Section 6]) for the details. We also note that, in [Kal19b], the product $\epsilon_{\vartheta,\mathrm{asym}}\cdot\epsilon_{\vartheta,\mathrm{ur}}$ (resp. $\epsilon_{\mathbf{S},\mathrm{ram}}$) is denoted by ϵ^{ram} (resp. ϵ_{ram}).

7.3. **Review of Kaletha's construction.**

7.3.1. *Regular supercuspidal L-packet data.* We recall the definition of a regular supercuspidal L-packet datum. For this, we need to review several basic facts about embeddings of tori based on [Kal19b, Section 5.1].

Suppose that

- an F-rational tame torus $\mathbf{S}$ having the same rank as $\mathbf{G}$ and
- an embedding $\hat{\jmath}\colon\hat{\mathbf{S}}\hookrightarrow\hat{\mathbf{G}}$ whose $\hat{\mathbf{G}}$-conjugacy class is Γ-stable

are given. Here, by noting that $\hat{\jmath}(\hat{\mathbf{S}})$ is a maximal torus of $\hat{\mathbf{G}}$, we assume that $\hat{\jmath}(\hat{\mathbf{S}})$ itself equals $\hat{\mathbf{T}}$ by replacing $\hat{\jmath}$ with its conjugate.

Definition 7.7. Let $j\colon\mathbf{S}\hookrightarrow\mathbf{G}$ be an embedding of $\mathbf{S}$ into $\mathbf{G}$. Since its image $\mathbf{S}_j:=j(\mathbf{S})$ is a maximal torus of $\mathbf{G}$ by the rank condition, there exists an element $g\in\mathbf{G}$ such that $[g](\mathbf{S}_j)=\mathbf{T}$. We say that j is *$\hat{\jmath}$-admissible* if the inverse of the dual to the isomorphism $[g]\circ j\colon\mathbf{S}\to\mathbf{T}$ is $\hat{\mathbf{G}}$-conjugate to $\hat{\jmath}\colon\hat{\mathbf{S}}\xrightarrow{\sim}\hat{\mathbf{T}}$.

We write $\mathcal{J}^{\mathbf{G}}_{\overline{F}}$ for the $\mathbf{G}$-conjugacy class of $\hat{\jmath}$-admissible embeddings of $\mathbf{S}$ into $\mathbf{G}$. Then, since the $\hat{\mathbf{G}}$-conjugacy class of $\hat{\jmath}$ is Γ-stable, so is $\mathcal{J}^{\mathbf{G}}_{\overline{F}}$ (see [Kal19b, Section 5.1]). Thus, by Kottwitz's result on the rational conjugacy ([Kot82, Corollary 2.2]), the quasi-splitness of $\mathbf{G}$ implies that $\mathcal{J}^{\mathbf{G}}_{\overline{F}}$ has an F-rational point. In other words, there exists an F-rational $\hat{\jmath}$-admissible embedding of $\mathbf{S}$ into $\mathbf{G}$. We let $\mathcal{J}^{\mathbf{G}}$ denotes the set of F-rational points of $\mathcal{J}^{\mathbf{G}}_{\overline{F}}$. For each $j\in\mathcal{J}^{\mathbf{G}}$, we get

- an F-rational embedding $\mathbf{Z}_{\mathbf{G}}\hookrightarrow\mathbf{S}_j$,
- a Γ-stable subset $\Phi(\mathbf{G},\mathbf{S}_j)$ of $X^*(\mathbf{S}_j)$, and
- a Γ-stable subgroup $\Omega_{\mathbf{G}}(\mathbf{S}_j)$ of $\mathrm{Aut}(\mathbf{S}_j)$.

Since $j\colon\mathbf{S}\xrightarrow{\sim}\mathbf{S}_j$ is an F-rational isomorphism, by pulling back these via j, we get

- an F-rational embedding $\mathbf{Z}_{\mathbf{G}}\hookrightarrow\mathbf{S}$,
- a Γ-stable subset $j^*\Phi(\mathbf{G},\mathbf{S}_j)$ of $X^*(\mathbf{S})$, and
- a Γ-stable subgroup $j^*\Omega_{\mathbf{G}}(\mathbf{S}_j)$ of $\mathrm{Aut}(\mathbf{S})$.

By noting that all of these are independent of the choice of $j\in\mathcal{J}^{\mathbf{G}}$, we write $\Phi(\mathbf{G},\mathbf{S}_{\hat{\jmath}}):=j^*\Phi(\mathbf{G},\mathbf{S}_j)$ and $\Omega_{\mathbf{G}}(\mathbf{S}_{\hat{\jmath}}):=j^*\Omega_{\mathbf{G}}(\mathbf{S}_j)$.

Definition 7.8 ([Kal19b, Definition 5.2.4]). A *regular supercuspidal L-packet datum of* $\mathbf{G}$ is a tuple $(\mathbf{S},\hat{\jmath},\chi,\vartheta)$ consisting of

(1) an F-rational tame torus $\mathbf{S}$ having the same rank as $\mathbf{G}$,
(2) an embedding $\hat{\jmath}\colon\hat{\mathbf{S}}\hookrightarrow\hat{\mathbf{G}}$ whose $\hat{\mathbf{G}}$-conjugacy class is Γ-stable,
(3) a set χ of minimally ramified χ-data for $\Phi(\mathbf{G},\mathbf{S}_{\hat{\jmath}})$, and

(4) a character $\vartheta\colon S \to \mathbb{C}^\times$

satisfying the following conditions:

(i) $\mathbf{S}$ is elliptic in $\mathbf{G}$ (i.e., $\mathbf{S}/\mathbf{Z}_{\mathbf{G}}$ is anisotropic),
(ii) χ is $\Omega_{\mathbf{G}^0}(\mathbf{S}_{\hat{\jmath}})(F)$-invariant, and
(iii) $(\mathbf{S}, \vartheta)$ is a tame elliptic extra regular pair of $\mathbf{G}$.

We give a few more comments about the precise meanings of the conditions (ii) and (iii) in Definition 7.8 (see [Kal19b, Sections 5.1 and 5.2] for the details). Note that the condition (i) implies that $\mathbf{S}_j$ is a tame elliptic maximal torus of $\mathbf{G}$ for any $j \in \mathcal{J}^{\mathbf{G}}$. The condition (iii) means that $(\mathbf{S}_j, \vartheta_j)$ is an F-rational tame elliptic extra regular pair of $\mathbf{G}$ for any $j \in \mathcal{J}^{\mathbf{G}}$, where $\vartheta_j := \vartheta \circ j^{-1}$ (this is equivalent to that $(\mathbf{S}_j, \vartheta_j)$ is a tame elliptic extra regular pair of $\mathbf{G}$ for "some" $j \in \mathcal{J}^{\mathbf{G}}$). We define a subset $\Phi(\mathbf{G}^0, \mathbf{S}_{\hat{\jmath}})$ of $\Phi(\mathbf{G}, \mathbf{S}_{\hat{\jmath}})$ by

$$\Phi(\mathbf{G}^0, \mathbf{S}_{\hat{\jmath}}) := \{\alpha \in \Phi(\mathbf{G}, \mathbf{S}_{\hat{\jmath}}) \mid \vartheta \circ \mathrm{Nr}_{E/F} \circ \alpha^\vee(E_r^\times) = 1\},$$

where E is a tame finite extension of F splitting $\mathbf{S}$. Then $\Phi(\mathbf{G}^0, \mathbf{S}_{\hat{\jmath}})$ is a Levi subsystem of $\Phi(\mathbf{G}, \mathbf{S}_{\hat{\jmath}})$ and associates a Γ-stable subgroup $\Omega_{\mathbf{G}^0}(\mathbf{S}_{\hat{\jmath}})$ of $\Omega_{\mathbf{G}}(\mathbf{S}_{\hat{\jmath}})$ canonically.

7.3.2. *Construction of regular supercuspidal L-parameters.* We next recall the construction of regular supercuspidal L-parameters following [Kal19b, Proof of Proposition 5.2.7].

We take a regular supercuspidal L-packet datum $(\mathbf{S}, \hat{\jmath}, \chi, \vartheta)$ of $\mathbf{G}$. By applying the local Langlands correspondence for $\mathbf{S}$ to ϑ, we get an L-parameter ϕ_ϑ of $\mathbf{S}$, which is a homomorphism from W_F to ${}^L\mathbf{S}$. On the other hand, by the Langlands–Shelstad construction ([LS87, Section 2.6]), we can extend $\hat{\jmath}$ to an L-embedding ${}^Lj_\chi$ from ${}^L\mathbf{S}$ to ${}^L\mathbf{G}$ by using the set χ of χ-data. Thus, by composing these two homomorphisms, we get an L-parameter ϕ of $\mathbf{G}$:

$$\phi\colon W_F \xrightarrow{\phi_\vartheta} {}^L\mathbf{S} \xrightarrow{{}^Lj_\chi} {}^L\mathbf{G}.$$

7.3.3. *Construction of regular supercuspidal L-packets.* We finally recall the construction of regular supercuspidal L-packets following [Kal19b, Section 5.3].

For this, we need the notion of a regular supercuspidal datum:

Definition 7.9 ([Kal19b, Definition 5.3.2])**.** Let $(\mathbf{S}, \hat{\jmath}, \chi, \vartheta)$ be a regular supercuspidal L-packet datum of $\mathbf{G}$. Let $\mathcal{J}^{\mathbf{G}}$ be the $\mathbf{G}$-conjugacy classes of F-rational $\hat{\jmath}$-admissible embeddings of $\mathbf{S}$ into $\mathbf{G}$ (see Section 7.3.1). Then a *regular supercuspidal datum* (over the regular supercuspidal L-packet datum $(\mathbf{S}, \hat{\jmath}, \chi, \vartheta)$) is a tuple $(\mathbf{S}, \hat{\jmath}, \chi, \vartheta, j)$ where j is an element of $\mathcal{J}^{\mathbf{G}}$.

Remark 7.10. In the original definition given in [Kal19b, Definition 5.3.2], a regular supercuspidal datum is a tuple $(\mathbf{S}, \hat{\jmath}, \chi, \vartheta, (\mathbf{G}', \psi, z), j)$ which furthermore contains a rigid inner twist $(\mathbf{G}', \psi, z)$ of $\mathbf{G}$ (in the sense of [Kal16]). In fact, Kaletha's L-packet constructed in [Kal19b] consist not only of representations of $\mathbf{G}$ but also those of all rigid inner forms of $\mathbf{G}$. In this paper, since we focus only on the quasi-split case, we always take a rigid inner twist $(\mathbf{G}', \psi, z)$ in a regular supercuspidal datum to be the trivial twist $(\mathbf{G}, \mathrm{id}, 1)$, and omit it from the notation.

Definition 7.11 ([Kal19b, Definition 4.6.4])**.** Let $\mathbf{S}$ be an F-rational maximal torus of $\mathbf{G}$. A family $\{\zeta_\alpha\}_{\alpha \in \Phi(\mathbf{G}, \mathbf{S})}$ of characters $\zeta_\alpha\colon F_\alpha^\times \to \mathbb{C}^\times$ is called a *set of ζ-data for* $\Phi(\mathbf{G}, \mathbf{S})$ if the following conditions are satisfied:

- $\zeta_{-\alpha} = \zeta_\alpha^{-1}$ for any $\alpha \in \Phi(\mathbf{G}, \mathbf{S})$,
- $\zeta_{\sigma(\alpha)} = \zeta_\alpha \circ \sigma^{-1}$ for any $\alpha \in \Phi(\mathbf{G}, \mathbf{S})$ and $\sigma \in \Gamma$, and
- $\zeta_\alpha|_{F_{\pm\alpha}^\times} = \mathbb{1}$ for any $\alpha \in \Phi(\mathbf{G}, \mathbf{S})_{\mathrm{sym}}$.

For a set $\zeta = \{\zeta_\alpha\}$ of ζ-data for $\Phi(\mathbf{G}, \mathbf{S})$, we define a character $\zeta_S \colon S \to \mathbb{C}^\times$ by

$$\zeta_S := \prod_{\Sigma\alpha \in \ddot{\Phi}(\mathbf{G},\mathbf{S})} \zeta_{\Sigma\alpha},$$

where

- we put $\zeta_{\Sigma\alpha} := \zeta_\alpha \circ \alpha$ if $\Sigma\alpha \in \ddot{\Phi}(\mathbf{G}, \mathbf{S})_{\mathrm{asym}}$ and
- we put $\zeta_{\Sigma\alpha}$ to be the composition $S \xrightarrow{\alpha} F_\alpha^1 \cong F_\alpha^\times / F_{\pm\alpha}^\times \xrightarrow{\zeta_\alpha} \mathbb{C}^\times$ if $\Sigma\alpha \in \ddot{\Phi}(\mathbf{G}, \mathbf{S})_{\mathrm{ur}}$ (here, F_α^1 denotes the kernel of the norm map $\mathrm{Nr}_{F_\alpha/F_{\pm\alpha}}$ and the middle isomorphism is Hilbert 90th theorem)

(see [Kal19b, Definition 4.6.5]).

Remark 7.12. When we have two sets of χ-data $\chi = \{\chi_\alpha\}_{\alpha\in\Phi(\mathbf{G},\mathbf{S})}$ and $\chi' = \{\chi'_\alpha\}_{\alpha\in\Phi(\mathbf{G},\mathbf{S})}$, we can produce a set of ζ-data by taking the ratio of $\{\chi'_\alpha\}_{\alpha\in\Phi(\mathbf{G},\mathbf{S})}$ to $\{\chi_\alpha\}_{\alpha\in\Phi(\mathbf{G},\mathbf{S})}$. We let $\zeta_{\chi'/\chi}$ denote the ζ-data defined in this way:

$$\zeta_{\chi'/\chi} = \{\zeta_{\chi'/\chi,\alpha}\}_{\alpha\in\Phi(\mathbf{G},\mathbf{S})}, \quad \zeta_{\chi'/\chi,\alpha} := \chi'_\alpha \cdot \chi_\alpha^{-1}.$$

Definition 7.13 ([Kal19b, Definition 5.2.5]). An *isomorphism between two regular supercuspidal L-packet data* is a tuple

$$(\iota, g, \zeta) \colon (\mathbf{S}, \hat{\jmath}, \chi, \vartheta) \to (\mathbf{S}', \hat{\jmath}', \chi', \vartheta')$$

consisting of

1. an F-rational isomorphism $\iota \colon \mathbf{S} \to \mathbf{S}'$ of tori,
2. an element g of $\hat{\mathbf{G}}$ satisfying $\hat{\jmath} \circ \hat{\iota} = [g] \circ \hat{\jmath}'$, and
3. a set $\zeta = (\zeta_{\alpha'})_{\alpha'\in\Phi(\mathbf{G},\mathbf{S}'_{\hat{\jmath}'})}$ of ζ-data for $\Phi(\mathbf{G}, \mathbf{S}'_{\hat{\jmath}'})$ given by $\chi_{\alpha'\circ\iota} = \chi'_{\alpha'} \cdot \zeta_{\alpha'}$ satisfying the equality
$$(\zeta_{S'}^{-1} \cdot \vartheta') \circ \iota = \vartheta.$$

Remark 7.14. We give a remark on the condition (3) of Definition 7.13. Thanks to the condition (2), for any F-rational $\hat{\jmath}'$-admissible embedding $j' \colon \mathbf{S}' \hookrightarrow \mathbf{G}$, we can check that $j' \circ \iota \colon \mathbf{S} \hookrightarrow \mathbf{G}$ is an F-rational $\hat{\jmath}$-admissible embedding. This implies that we have an identification $\Phi(\mathbf{G}, \mathbf{S}'_{\hat{\jmath}'}) \cong \Phi(\mathbf{G}, \mathbf{S}'_{j'}) \cong \Phi(\mathbf{G}, \mathbf{S}_{j'\circ\iota}) \cong \Phi(\mathbf{G}, \mathbf{S}'_{\hat{\jmath}})$ given by $\alpha' \mapsto \alpha' \circ \iota$. If we transport the set of χ-data χ from $\Phi(\mathbf{G}, \mathbf{S}_{\hat{\jmath}})$ to $\Phi(\mathbf{G}, \mathbf{S}'_{\hat{\jmath}'})$ via this identification and write $\iota_*(\chi)$ for it, then the set of ζ-data ζ in the condition (3) is nothing but $\zeta_{\iota_*(\chi)/\chi'}$ with the notation as in Remark 7.12.

Definition 7.15 ([Kal19b, Definition 5.3.3]). An *isomorphism between two regular supercuspidal data* is a tuple

$$(\iota, g, \zeta, f) \colon (\mathbf{S}, \hat{\jmath}, \chi, \vartheta, j) \to (\mathbf{S}', \hat{\jmath}', \chi', \vartheta', j')$$

consisting of

1. an isomorphism of regular supercuspidal L-packet data
$$(\iota, g, \zeta) \colon (\mathbf{S}, \hat{\jmath}, \chi, \vartheta) \to (\mathbf{S}', \hat{\jmath}', \chi', \vartheta'), \quad \text{and}$$
2. an automorphism f of $\mathbf{G}$ given by a G-conjugation satisfying $j' \circ \iota = f \circ j$.

Remark 7.16. In the original definition of an isomorphism of regular supercuspidal data given in [Kal19b, Definitions 5.3.3], the fourth parameter f of a tuple (ι, g, ζ, f) is taken to be an isomorphism of rigid inner twists. As explained in Remark 7.10, in this paper we always take every rigid inner twist to be trivial. Then, since any automorphism of a rigid inner twist is given by a rational conjugation ([Kal16, Fact 5.1]), we may suppose that f is as in Definition 7.15.

Let us investigate the isomorphism classes of regular supercuspidal data over a fixed regular supercuspidal L-packet datum. Let $(\mathbf{S}, \hat{\jmath}, \chi, \vartheta)$ be a regular supercuspidal L-packet datum. If $(\iota, g, \zeta, f)\colon (\mathbf{S}, \hat{\jmath}, \chi, \vartheta, j) \to (\mathbf{S}, \hat{\jmath}, \chi, \vartheta, j')$ is an isomorphism of regular supercuspidal data ($j, j' \in \mathcal{J}^{\mathbf{G}}$), then ι is necessarily the identity map by [Kal19b, Lemma 5.2.6]. In particular, the equality $j' \circ \iota = f \circ j$ in Definition 7.15 implies that j and j' are G-conjugate. Conversely, whenever j and j' are G-conjugate, two regular supercuspidal data $(\mathbf{S}, \hat{\jmath}, \chi, \vartheta, j)$ and $(\mathbf{S}, \hat{\jmath}, \chi, \vartheta, j')$ are isomorphic. Hence, the isomorphism classes of regular supercuspidal data with a fixed regular supercuspidal L-packet datum $(\mathbf{S}, \hat{\jmath}, \chi, \vartheta)$ are parametrized by the set

$$\mathcal{J}^{\mathbf{G}}_{G} := \mathcal{J}^{\mathbf{G}}/{\sim_G} = \{\hat{\jmath}\text{-admissible } F\text{-rational embeddings } \mathbf{S} \hookrightarrow \mathbf{G}\}/{\sim_G},$$

where $\sim_G$ denotes the equivalence relation given by the G-conjugacy. In the following, we often regard $\mathcal{J}^{\mathbf{G}}_{G}$ as a subset of $\mathcal{J}^{\mathbf{G}}$ by fixing a set of representatives as long as there is no risk of confusion.

Now we explain Kaletha's construction of regular supercuspidal L-packets ([Kal19b, 1153-1154 pages]). Let $(\mathbf{S}, \hat{\jmath}, \chi, \vartheta, j)$ be a regular supercuspidal datum with $j \in \mathcal{J}^{\mathbf{G}}_{G}$. Then $(\mathbf{S}_j, \vartheta_j) := (j(\mathbf{S}), \vartheta \circ j^{-1})$ is a tame elliptic extra regular pair of $\mathbf{G}$ by the definition of a regular supercuspidal datum. We define a character ϵ_{ϑ_j} of S_j by

$$\epsilon_{\vartheta_j} := \epsilon_{\vartheta_j,\mathrm{asym}} \cdot \epsilon_{\vartheta_j,\mathrm{ur}} \cdot \epsilon_{\mathbf{S}_j,\mathrm{ram}}.$$

As in the manner of Section 7.1, we get a set χ_{ϑ_j} of χ-data for $\Phi(\mathbf{G}, \mathbf{S}_j)$. Via the identification $\Phi(\mathbf{G}, \mathbf{S}_j) \cong \Phi(\mathbf{G}, \mathbf{S}_{\hat{\jmath}})$, this induces a set $\chi_{\vartheta_{\hat{\jmath}}}$ of χ-data for $\Phi(\mathbf{G}, \mathbf{S}_{\hat{\jmath}})$, which is independent of the choice of j. By comparing $\chi_{\vartheta_{\hat{\jmath}}}$ with the set of χ-data χ contained in $(\mathbf{S}, \hat{\jmath}, \chi, \vartheta)$, we get a set of ζ-data $\zeta_{\chi_{\vartheta_{\hat{\jmath}}}/\chi}$ (Remark 7.12). We define a tame elliptic regular pair $(\mathbf{S}_j, \vartheta'_j)$ of $\mathbf{G}$ by putting the character $\vartheta'_j\colon S_j \to \mathbb{C}^\times$ to be

$$\vartheta'_j := \epsilon_{\vartheta_j} \cdot (\vartheta \cdot \zeta^{-1}_{\chi_{\vartheta_{\hat{\jmath}}}/\chi,S}) \circ j^{-1}.$$

Then we get the regular supercuspidal representation $\pi_{(\mathbf{S}_j,\vartheta'_j)}$ of G associated with the tame elliptic regular pair $(\mathbf{S}_j, \vartheta'_j)$ (see Section 4.1). Note that the G-conjugacy class of $(\mathbf{S}_j, \vartheta'_j)$ is independent of the choice of a representative of j, hence so is the isomorphism class of $\pi_{(\mathbf{S}_j,\vartheta'_j)}$. We put

$$\Pi^{\mathbf{G}}_{\phi} := \{\pi_{(\mathbf{S}_j,\vartheta'_j)} \mid j \in \mathcal{J}^{\mathbf{G}}_{G}\}.$$

7.4. **Regularity and torality on the Galois side.** In fact, the L-parameters of $\mathbf{G}$ obtained from regular supercuspidal L-packet data can be characterized in the purely Galois-theoretic language.

Definition 7.17 ([Kal19b, Definition 5.2.3])**.** We say that an L-parameter $\phi\colon W_F \rtimes \mathrm{SL}_2(\mathbb{C}) \to {}^L\mathbf{G}$ is *regular supercuspidal* if it satisfies the following:

(0) $\phi|_{\mathrm{SL}_2(\mathbb{C})}$ is trivial and ϕ is discrete, i.e., $S^\circ_\phi := \mathbf{Z}_{\hat{\mathbf{G}}}(\phi(W_F))^\circ \subset \mathbf{Z}_{\hat{\mathbf{G}}}$,
(1) $\phi(P_F)$ is contained in a torus of $\hat{\mathbf{G}}$ (note that then $\mathcal{M} := \mathbf{Z}_{\hat{\mathbf{G}}}(\phi(P_F))^\circ$ is a Levi subgroup of $\hat{\mathbf{G}}$).

(2) $\mathcal{C} := \mathbf{Z}_{\hat{\mathbf{G}}}(\phi(I_F))^\circ$ is a torus (note that then $\mathcal{T} := \mathbf{Z}_{\mathcal{M}}(\mathcal{C})$ is a maximal torus of $\mathcal{M}$. We put $\hat{\mathbf{S}}$ to be the Γ-module $\mathcal{T}$ with the Γ-action given by $[\phi(-)]$).

(3) If $n \in \mathbf{N}_{\mathcal{M}}(\mathcal{T})$ maps to a nontrivial element of $\Omega_{\mathcal{M}}(\hat{\mathbf{S}})^\Gamma$, then $n \notin \mathbf{Z}_{\hat{\mathbf{G}}}(\phi(I_F))$.

Proposition 7.18 ([Kal19b, Proposition 5.2.7])**.** *Kaletha's construction gives a bijective correspondence between the isomorphism classes of regular supercuspidal L-packet data of* $\mathbf{G}$ *and the equivalence classes of regular supercuspidal L-parameters of* $\mathbf{G}$*.*

Moreover, the torality of the regular supercuspidal representations can be also interpreted on the Galois side.

Definition 7.19 ([Kal19b, Definition 6.1.1])**.** We say that an L-parameter $\phi\colon W_F \rtimes \mathrm{SL}_2(\mathbb{C}) \to {}^L\mathbf{G}$ is *toral supercuspidal (of depth $r > 0$)* if it satisfies the following:

(0) $\phi|_{\mathrm{SL}_2(\mathbb{C})}$ is trivial and ϕ is discrete, i.e., $S_\phi^\circ := \mathbf{Z}_{\hat{\mathbf{G}}}(\phi(W_F))^\circ \subset \mathbf{Z}_{\hat{\mathbf{G}}}$,

(1) $\mathbf{Z}_{\hat{\mathbf{G}}}(\phi(I_F^r))$ is a maximal torus of $\hat{\mathbf{G}}$ containing $\phi(P_F)$, and

(2) ϕ is trivial on I_F^{r+}, that is, $\phi(\sigma) = 1 \rtimes \sigma$ for any $\sigma \in I_F^{r+}$.

Proposition 7.20 ([Kal19b, Proposition 6.1.2])**.** *Kaletha's construction gives a bijective correspondence between the isomorphism classes of regular supercuspidal L-packet data of* $\mathbf{G}$ *giving rise to toral supercuspidal representations and the equivalence classes of toral supercuspidal L-parameters of* $\mathbf{G}$*.*

8. Framework of twisted endoscopy

8.1. **Endoscopic data treated in this paper.** We introduce a structure of a twisted space on the L-group ${}^L\mathbf{G}$ following [KS99, Section 1.2] and [Wal08, Section 1.3]. The automorphism θ and the fixed splitting $\mathbf{spl}_{\hat{\mathbf{G}}}$ define an automorphism $\hat{\theta}$ of $\hat{\mathbf{G}}$; namely, $\hat{\theta}$ is the unique $\mathbf{spl}_{\hat{\mathbf{G}}}$-preserving automorphism of $\hat{\mathbf{G}}$ which is compatible with θ under the isomorphism $\Psi(\hat{\mathbf{G}}) \cong \Psi(\mathbf{G})^\vee$. Since $\hat{\theta}$ commutes with the action of Γ on $\hat{\mathbf{G}}$, we can extend it to an automorphism ${}^L\theta$ of ${}^L\mathbf{G}$ by ${}^L\theta(x, w) := (\hat{\theta}(x), w)$ for $(x, w) \in {}^L\mathbf{G} = \hat{\mathbf{G}} \rtimes W_F$. We define a twisted space on the dual side by ${}^L\tilde{\mathbf{G}} := {}^L\mathbf{G}{}^L\theta$.

We next review the notion of endoscopic data following [KS99, Section 2.1] and [Wal08, Section 1.3]. We call a quadruple $(\mathbf{H}, \mathcal{H}, s, \hat{\xi})$ *endoscopic data for the triple* $(\mathbf{G}, \theta, \mathbf{1})$ if

- $\mathbf{H}$ is a quasi-split connected reductive group over F,
- $\mathcal{H}$ is a split extension $1 \to \hat{\mathbf{H}} \to \mathcal{H} \to W_F \to 1$ such that the induced action of W_F on $\hat{\mathbf{H}}$ coincides with the action of W_F on $\hat{\mathbf{H}}$ induced from the F-rational structure of $\mathbf{H}$ up to inner automorphisms of $\hat{\mathbf{H}}$,
- $s \in \hat{\mathbf{G}}$ such that the automorphism $[s] \circ \hat{\theta}$ is quasi-semisimple, and
- $\hat{\xi}\colon \mathcal{H} \hookrightarrow {}^L\mathbf{G}$ is an L-homomorphism (i.e., continuous and commuting with projections to W_F) satisfying the following conditions:
  - $[s] \circ {}^L\theta \circ \hat{\xi} = \hat{\xi}$,
  - $\hat{\xi}|_{\hat{\mathbf{H}}}\colon \hat{\mathbf{H}} \xrightarrow{\sim} \hat{\mathbf{G}}_{s{}^L\theta} = \mathbf{Z}_{\hat{\mathbf{G}}}(s{}^L\theta)^\circ$.

When a set of endoscopic data $(\mathbf{H}, \mathcal{H}, s, \hat{\xi})$ is given, by replacing it with an equivalent data (see [KS99, Section 3.1] for the definition of the equivalence relation on endoscopic data), we may suppose that

- s belongs to $\hat{\mathbf{T}}$ which is the torus contained in $\mathbf{spl}_{\hat{\mathbf{G}}}$, and
- $(\hat{\mathbf{B}}_{\mathbf{H}}, \hat{\mathbf{T}}_{\mathbf{H}}) := \hat{\xi}^{-1}(\hat{\mathbf{B}}, \hat{\mathbf{T}})$ is a Γ-stable Borel pair of $\hat{\mathbf{H}}$.

In this paper, we assume that

$$\mathcal{H} \textbf{ in the endoscopic data } (\mathbf{H}, \mathcal{H}, s, \hat{\xi}) \textbf{ is equal to } {}^{L}\mathbf{H}.$$

Let us fix such an endoscopic data $(\mathbf{H}, {}^{L}\mathbf{H}, s, \hat{\xi})$ in the following.

We note that, the absolute Weyl group $\Omega_{\mathbf{H}}$ of $\mathbf{H}$ can be identified with a subgroup of that $\Omega_{\mathbf{G}}$ of $\mathbf{G}$ (see Section 8.2). In particular, our assumption that $p \nmid |\Omega_{\mathbf{G}}|$ implies that $p \nmid |\Omega_{\mathbf{H}}|$.

8.2. **Norm correspondence in twisted endoscopy.** Let us briefly review the notion of a norm in twisted endoscopy. (See [KS99, Section 3] for details.)

We fix a Borel pair $(\mathbf{B}_{\mathbf{H}}, \mathbf{T}_{\mathbf{H}})$ of $\mathbf{H}$ defined over F so that the Langlands dual group $\hat{\mathbf{H}}$ of $\mathbf{H}$ is equipped with an isomorphism $\Psi(\hat{\mathbf{H}}) \cong \Psi(\mathbf{H})^{\vee}$. In particular, we have isomorphisms $X^{*}(\mathbf{T}_{\mathbf{H}}) \cong X_{*}(\hat{\mathbf{T}}_{\mathbf{H}})$ and $X_{*}(\mathbf{T}_{\mathbf{H}}) \cong X^{*}(\hat{\mathbf{T}}_{\mathbf{H}})$. Since the restriction of $\hat{\xi}$ to $\hat{\mathbf{T}}_{\mathbf{H}}$ induces

$$\hat{\xi}|_{\hat{\mathbf{T}}_{\mathbf{H}}} \colon \hat{\mathbf{T}}_{\mathbf{H}} \xrightarrow{\sim} \hat{\mathbf{T}}^{\natural}\, (:= \hat{\mathbf{T}}^{\hat{\theta},\circ}),$$

by taking the dual of $\hat{\xi}|_{\hat{\mathbf{T}}_{\mathbf{H}}}$, we get an F-rational isomorphism

$$\xi \colon \mathbf{T}_{\theta} \xrightarrow{\sim} \mathbf{T}_{\mathbf{H}}.$$

By abuse of notation, we often write ξ also for the homomorphism $\mathbf{T} \twoheadrightarrow \mathbf{T}_{\theta} \xrightarrow{\xi} \mathbf{T}_{\mathbf{H}}$.

In the following, as long as there is no risk of confusion, we simply write $\Omega_{\mathbf{G}}$ and $\Omega_{\mathbf{H}}$ for the Weyl groups $\Omega_{\mathbf{G}}(\mathbf{T})$ and $\Omega_{\mathbf{H}}(\mathbf{T}_{\mathbf{H}})$, respectively. Via the isomorphism ξ, $\Omega_{\mathbf{H}}$ is identified with a subgroup of $\Omega_{\mathbf{G}}^{\theta}$ (see [KS99, Section 1.1]). Therefore ξ^{-1} induces a surjective map

$$\mathbf{T}_{\mathbf{H}}/\Omega_{\mathbf{H}} \twoheadrightarrow \mathbf{T}_{\theta}/\Omega_{\mathbf{G}}^{\theta}. \tag{13}$$

Note that $\mathbf{T}_{\mathbf{H}}/\Omega_{\mathbf{H}}$ and $\mathbf{T}_{\theta}/\Omega_{\mathbf{G}}^{\theta}$ parametrize the semisimple conjugacy classes of $\mathbf{H}$ and $\tilde{\mathbf{G}}$, respectively. Moreover, the map (13) is Γ-equivariant.

We let $\tilde{\mathbf{G}}_{\mathrm{ss}}$ (resp. $\mathbf{H}_{\mathrm{ss}}$) denote the subset of semisimple elements of $\tilde{\mathbf{G}}$ (resp. semisimple elements of $\mathbf{H}$). For $\gamma \in \mathbf{H}_{\mathrm{ss}}$ and $\delta \in \tilde{\mathbf{G}}_{\mathrm{ss}}$, we say that γ *and* δ *correspond* if the conjugacy classes of γ and δ correspond under the map (13). We say that $\gamma \in \mathbf{H}_{\mathrm{ss}}$ is $\tilde{\mathbf{G}}$-*strongly regular* if it corresponds to a strongly regular semisimple conjugacy class in $\tilde{\mathbf{G}}$. Note that if $\gamma \in \mathbf{H}_{\mathrm{ss}}$ is $\tilde{\mathbf{G}}$-strongly regular, then it is strongly regular. We let $\tilde{\mathbf{G}}_{\mathrm{srs}}$ (resp. $\mathbf{H}_{\tilde{\mathbf{G}}\text{-srs}}$) denote the subset of strongly regular semisimple elements of $\tilde{\mathbf{G}}$ (resp. $\tilde{\mathbf{G}}$-strongly regular semisimple elements of $\mathbf{H}$).

For two F-rational elements δ and δ' of $\tilde{G}_{\mathrm{srs}}$ (resp. γ and γ' of $H_{\tilde{\mathbf{G}}\text{-srs}}$), we say that they are *stably conjugate* if they are conjugate by an element of $\mathbf{G}$ (resp. $\mathbf{H}$).

When an F-rational element $\gamma \in H_{\tilde{\mathbf{G}}\text{-srs}}$ corresponds to an F-rational element $\delta \in \tilde{G}_{\mathrm{srs}}$, we say that γ *is a norm of* δ. We define $\mathcal{D}$ to be the subset of $H_{\tilde{\mathbf{G}}\text{-srs}} \times \tilde{G}_{\mathrm{srs}}$ consisting of pairs (γ, δ) such that γ is a norm of δ.

8.3. **Transfer factor.** We have a function

$$\Delta_{\mathbf{H},\tilde{\mathbf{G}}} \colon H_{\mathbf{G}\text{-srs}} \times \tilde{G}_{\mathrm{srs}} \to \mathbb{C}$$

called the *(geometric) absolute transfer factor of Langlands–Kottwitz–Shelstad* (introduced in [LS87, KS99, KS12]). When the groups $\mathbf{H}$ and $\tilde{\mathbf{G}}$ are obvious from the context, we often omit the subscript from the notation and simply write Δ for $\Delta_{\mathbf{H},\tilde{\mathbf{G}}}$. Instead of reviewing its precise definition, we give several comments on the basic properties in the following; we refer the readers to [KS99, Sections 4, 5], [Wal08, Chapitre 7], and [KS12] for the details.

(1) For any (γ,δ), $\Delta(\gamma,\delta)\neq 0$ if and only if $(\gamma,\delta)\in\mathcal{D}$, i.e., γ is a norm of δ.
(2) The transfer factor $\Delta(\gamma,\delta)$ depends on the choice of a θ-stable Whittaker datum of $\mathbf{G}$. In this paper, we choose a θ-stable Whittaker datum $\mathfrak{w}$ of $\mathbf{G}$ determined by the fixed θ-stable splitting $\mathbf{spl_G}$ of $\mathbf{G}$ (see [KS99, Section 5.3] for how to produce $\mathfrak{w}$ from $\mathbf{spl_G}$).
(3) The transfer factor $\Delta(\gamma,\delta)$ is defined to be the product of the ratio of root numbers $\varepsilon(\mathbf{T}^{\natural})\cdot\varepsilon(\mathbf{T_H})^{-1}$ and four factors $\Delta_{\mathrm{I}}(\gamma,\delta)$, $\Delta_{\mathrm{II}}(\gamma,\delta)$, $\Delta_{\mathrm{III}}(\gamma,\delta)$, and $\Delta_{\mathrm{IV}}(\gamma,\delta)$. Among these factors, $\Delta_{\mathrm{I}}(\gamma,\delta)$, $\Delta_{\mathrm{II}}(\gamma,\delta)$, and $\Delta_{\mathrm{III}}(\gamma,\delta)$ depend on the choice of a-data and χ-data for the restricted root system of $\mathbf{T}^{\diamondsuit}:=\mathbf{Z_G}(\mathbf{Z_G}(\delta))$ (this is an F-rational maximal torus in $\mathbf{G}$) although the whole product does not. For this reason, we write $\Delta_{\bullet}[a,\chi](\gamma,\delta)$ when we want to emphasize the dependence on a and χ ($\bullet\in\{\mathrm{I},\mathrm{II},\mathrm{III}\}$).
(4) Following [Kal19b], we let $\mathring{\Delta}$ denote the transfer factor Δ without the fourth factor Δ_{IV}.
(5) The ratio of absolute transfer factors
$$\Delta(\gamma,\delta;\gamma',\delta'):=\Delta(\gamma,\delta)/\Delta(\gamma',\delta')$$
is called the relative transfer factor. We also define the relative versions of $\Delta_{\bullet}$ for $\bullet\in\{\mathrm{I},\mathrm{II},\mathrm{III},\mathrm{IV}\}$ in the same way.
(6) The definition of the transfer factor given in [KS99] must be modified as announced in [KS12] (see also [Wal09, Section 2] or [Kal21b, Appendix]). We adopt the modified version "Δ'" which is compatible with the classical normalization of the local class field theory (hence consistent with, especially, [LS87],[MW16],[Kal19b]). More precisely, the factor Δ' is the defined to be the product of $\Delta_{\mathrm{I}}^{\mathrm{new},-1}$, $\Delta_{\mathrm{II}}^{\mathrm{KS}}$, $\Delta_{\mathrm{III}}^{\mathrm{KS},-1}$, and $\Delta_{\mathrm{IV}}^{\mathrm{KS}}$ (and also the epsilon factors), where $\Delta_{I}^{\mathrm{new}}$ is the factor defined in [KS12, Section 3.4] and $\Delta_{\mathrm{II}}^{\mathrm{KS}}$, $\Delta_{\mathrm{III}}^{\mathrm{KS}}$ and $\Delta_{\mathrm{IV}}^{\mathrm{KS}}$ are the factors defined in [KS99]. We note that $\Delta_{\mathrm{I}}^{\mathrm{new}}$ equals the factor $\Delta_{\mathrm{I}}^{\mathrm{KS}}$ defined in [KS99] when there is no restricted roots of type 2 or 3. in this paper, we let Δ_{I}, Δ_{II}, Δ_{III}, and Δ_{IV} denote $\Delta_{\mathrm{I}}^{\mathrm{new},-1}$, $\Delta_{\mathrm{II}}^{\mathrm{KS}}$, $\Delta_{\mathrm{III}}^{\mathrm{KS},-1}$, and $\Delta_{\mathrm{IV}}^{\mathrm{KS}}$, respectively.

9. Analysis of θ-stable regular supercuspidal L-packets

9.1. θ-twist of regular supercuspidal L-packets and L-parameters. Let $(\mathbf{S},\hat{\jmath},\chi,\vartheta)$ be a regular supercuspidal L-packet datum of $\mathbf{G}$. Let ϕ be the L-parameter of $\mathbf{G}$ associated to $(\mathbf{S},\hat{\jmath},\chi,\vartheta)$, i.e., $\phi:={}^{L}j_{\chi}\circ\phi_{\vartheta}$.

We put $\jmath\colon\mathbf{T}\to\mathbf{S}$ to be the dual isomorphism of $\hat{\jmath}$. Recall that $\mathcal{J}^{\mathbf{G}}:=\{\hat{\jmath}\text{-admissible } F\text{-rational embeddings } \mathbf{S}\hookrightarrow\mathbf{G}\}$. Let us investigate the $\hat{\jmath}$-admissibility condition. By definition, an embedding $j\colon\mathbf{S}\hookrightarrow\mathbf{G}$ is $\hat{\jmath}$-admissible if and only if there exists an element $g\in\mathbf{G}$ such that $[g]\circ j(\mathbf{S})=\mathbf{T}$ and the inverse of the dual of $[g]\circ j$ is $\hat{\mathbf{G}}$-conjugate to $\hat{\jmath}$. Since the image of $\hat{\jmath}$ is assumed to be $\hat{\mathbf{T}}$, this is equivalent to that there exists an element $\hat{w}\in\Omega_{\hat{\mathbf{G}}}:=\Omega_{\hat{\mathbf{G}}}(\hat{\mathbf{T}})$ such that $[g]\circ j$ and $[\hat{w}]\circ\hat{\jmath}$ are dual to each other. By letting $w\in\Omega_{\mathbf{G}}$ be the element corresponding to $\hat{w}\in\Omega_{\hat{\mathbf{G}}}$, this condition is equivalent to that $[g]\circ j=[w]^{-1}\circ\jmath^{-1}$. Therefore, we see that the $\hat{\jmath}$-admissibility condition simply says that j and $\jmath^{-1}$ are $\mathbf{G}$-conjugate.

The following lemma follows from this observation.

Lemma 9.1. *If we put* $\theta^{-1} \circ \mathcal{J}^{\mathbf{G}} := \{\theta^{-1} \circ j \mid j \in \mathcal{J}^{\mathbf{G}}\}$*, then we have*

$$\theta^{-1} \circ \mathcal{J}^{\mathbf{G}} = \{\hat{\theta} \circ \hat{\jmath}\text{-admissible } F\text{-rational embeddings } \mathbf{S} \hookrightarrow \mathbf{G}\}.$$

Recall that $\chi = \{\chi_\alpha\}_\alpha$ is a set of χ-data for $\Phi(\mathbf{G}, \mathbf{S}_{\hat{\jmath}}) \cong \Phi(\mathbf{G}, \mathbf{S}_j)$ (for any $j \in \mathcal{J}^{\mathbf{G}}_G$). As we have $\Phi(\mathbf{G}, \mathbf{S}_j) \xrightarrow{\sim} \Phi(\mathbf{G}, \mathbf{S}_{\theta^{-1}\circ j})$, we can transport χ to a set of χ-data for $\Phi(\mathbf{G}, \mathbf{S}_{\theta^{-1}\circ j})$, for which we write $\theta(\chi)$. Then we get a regular supercuspidal L-packet datum $(\mathbf{S}, \hat{\theta} \circ \hat{\jmath}, \theta(\chi), \vartheta)$.

The following lemma can be also found in [Zha20, Lemma 4.9].

Lemma 9.2. *The L-parameter ${}^L\theta \circ \phi$ corresponds to the regular supercuspidal L-packet datum* $(\mathbf{S}, \hat{\theta} \circ \hat{\jmath}, \theta(\chi), \vartheta)$*.*

Proof. By tracking the Langlands–Shelstad construction ([LS87, Section 2.6]) of the L-embedding ${}^Lj_\chi \colon {}^L\mathbf{S} \hookrightarrow {}^L\mathbf{G}$, we can check that ${}^L\theta \circ {}^Lj_\chi$ is nothing but the L-embedding obtained by applying the Langlands–Shelstad construction to the embedding $\hat{\theta} \circ \hat{\jmath} \colon \hat{\mathbf{S}} \hookrightarrow \hat{\mathbf{G}}$ with the χ-data $\theta(\chi)$. In other words, the L-parameter ${}^L\theta \circ {}^Lj_\chi \circ \phi_\vartheta$ is associated to the regular supercuspidal L-packet datum $(\mathbf{S}, \hat{\theta} \circ \hat{\jmath}, \theta(\chi), \vartheta)$. □

Lemma 9.3. *The L-packet $\Pi^{\mathbf{G}}_{{}^L\theta\circ\phi}$ consists of regular supercuspidal representations whose regular supercuspidal data are given by* $(\mathbf{S}, \hat{\theta} \circ \hat{\jmath}, \theta(\chi), \vartheta, \theta^{-1} \circ j)$ *for* $j \in \mathcal{J}^{\mathbf{G}}_G$*.*

Proof. This simply follows from that the equality of Lemma 9.1

$$\theta^{-1} \circ \mathcal{J}^{\mathbf{G}} = \{\hat{\theta} \circ \hat{\jmath}\text{-admissible } F\text{-rational embeddings } \mathbf{S} \hookrightarrow \mathbf{G}\}$$

induces an identification

$$\theta^{-1} \circ \mathcal{J}^{\mathbf{G}}_G \cong \{\hat{\theta} \circ \hat{\jmath}\text{-admissible } F\text{-rational embeddings } \mathbf{S} \hookrightarrow \mathbf{G}\}/{\sim_G}.$$

□

Lemma 9.4. *Let $\pi \in \Pi^{\mathbf{G}}_\phi$ be a regular supercuspidal representation whose regular supercuspidal datum is* $(\mathbf{S}, \hat{\jmath}, \chi, \vartheta, j)$*. Then its θ-twist $\pi^\theta := \pi \circ \theta$ arises from the regular supercuspidal datum* $(\mathbf{S}, \hat{\theta} \circ \hat{\jmath}, \theta(\chi), \vartheta, \theta^{-1} \circ j)$*.*

Proof. We first note that, for any tame elliptic regular pair $(\mathbf{S}_0, \vartheta_0)$ of $\mathbf{G}$, the θ-twist $\pi^\theta_{(\mathbf{S}_0,\vartheta_0)}$ of the associated regular supercuspidal representation $\pi_{(\mathbf{S}_0,\vartheta_0)}$ is equivalent to $\pi_{(\theta^{-1}(\mathbf{S}_0),\vartheta_0\circ\theta)}$. (This can be easily checked in the same way as in Section 5.2, where the toral case is treated.) Thus we have

$$\pi^\theta = \pi^\theta_{(\mathbf{S}_j,\vartheta'_j)} \cong \pi_{(\theta^{-1}(\mathbf{S}_j),\vartheta'_j\circ\theta)}.$$

Here recall that ϑ'_j is a character of $S_j = j(S)$ given by

$$\vartheta'_j := \epsilon_{\vartheta_j} \cdot (\vartheta \cdot \zeta^{-1}_{\chi_{\vartheta_{\hat{\jmath}}}/\chi,S}) \circ j^{-1}.$$

On the other hand, the regular supercuspidal representation associated to the datum $(\mathbf{S}, \hat{\theta} \circ \hat{\jmath}, \theta(\chi), \vartheta, \theta^{-1} \circ j)$ is given by $\pi_{(\theta^{-1}(\mathbf{S}_j),\vartheta'_{\theta^{-1}\circ j})}$, where

$$\vartheta'_{\theta^{-1}\circ j} = \epsilon_{\vartheta_{\theta^{-1}\circ j}} \cdot (\vartheta \cdot \zeta^{-1}_{\chi_{\vartheta_{\hat{\theta}\circ\hat{\jmath}}}/\theta(\chi),S}) \circ (\theta^{-1} \circ j)^{-1}.$$

By the definition of the character ϵ, we easily see that $\epsilon_{\vartheta_j} \circ \theta = \epsilon_{\vartheta_j\circ\theta}$. Moreover, it can be also easily checked that $\zeta_{\chi_{\vartheta_{\hat{\jmath}}}/\chi,S}$ equals $\zeta_{\chi_{\vartheta_{\hat{\theta}\circ\hat{\jmath}}}/\theta(\chi),S}$. Hence we conclude that the characters $\vartheta'_j \circ \theta$ and $\vartheta'_{\theta^{-1}\circ j}$ are equal, which implies that $\pi^\theta \cong \pi_{(\theta^{-1}(\mathbf{S}_j),\vartheta'_{\theta^{-1}\circ j})}$. □

Lemmas 9.3 and 9.4 imply the following:

Proposition 9.5. *We have* $\Pi^{\mathbf{G}}_{\phi} \circ \theta = \Pi^{\mathbf{G}}_{{}^{L}\theta\circ\phi}$.

9.2. **Structure of θ-stable L-packets.** Let us keep the notation as in Section 9.1. Thus ϕ denotes the L-parameter attached to a regular supercuspidal L-packet datum $(\mathbf{S}, \hat{\jmath}, \chi, \vartheta)$, i.e., $\phi = {}^{L}j_{\chi} \circ \phi_{\vartheta}$. In the following, we suppose that ϕ factors through the L-embedding $\hat{\xi}\colon {}^{L}\mathbf{H} \hookrightarrow {}^{L}\mathbf{G}$. As we have $[s]\circ{}^{L}\theta\circ\hat{\xi} = \hat{\xi}$, this assumption implies that we have $[s] \circ {}^{L}\theta \circ \phi = \phi$. In particular, the L-parameters ${}^{L}\theta \circ \phi$ and ϕ are $\hat{\mathbf{G}}$-conjugate and the conjugation is given by s. Thus, by Lemma 9.3 and Proposition 7.18, there exists an isomorphism between the regular supercuspidal L-packet data $(\mathbf{S}, \hat{\jmath}, \chi, \vartheta)$ and $(\mathbf{S}, \hat{\theta} \circ \hat{\jmath}, \theta(\chi), \vartheta)$. Let us investigate how such an isomorphism can be constructed explicitly.

In the following, we put $\phi' := {}^{L}\theta \circ \phi$, $\hat{\jmath}' := \hat{\theta} \circ \hat{\jmath}$, and $\chi' := \theta(\chi)$. We may and do assume that the image of $\hat{\jmath}$ is given by $\hat{\mathbf{T}}$. We define an automorphism $\hat{\mathbf{S}}$ of $\hat{\mathbf{S}}$ by $\hat{\theta}_{\mathbf{S}} := \hat{\jmath}^{-1} \circ \hat{\jmath}' = \hat{\jmath}^{-1} \circ \hat{\theta} \circ \hat{\jmath}$. Let $\theta_{\mathbf{S}}$ be the automorphism of $\mathbf{S}$ which is dual to $\hat{\theta}_{\mathbf{S}}$. Note that $\hat{\theta}_{\mathbf{S}}$ and $\theta_{\mathbf{S}}$ are involutive.

Lemma 9.6. *The automorphism $\hat{\theta}_{\mathbf{S}}$ of $\hat{\mathbf{S}}$ is Γ-equivariant, hence $\theta_{\mathbf{S}}$ is F-rational.*

Proof. As the Γ-actions on $\hat{\mathbf{S}}$ and $\hat{\mathbf{T}}$ factor through a finite quotient, we may discuss the equivariance for W_F instead of Γ. Recall that we have $\phi = {}^{L}j_{\chi} \circ \phi_{\vartheta}$. As $\phi_{\vartheta}\colon W_F \to {}^{L}\mathbf{S} = \hat{\mathbf{S}} \rtimes W_F$ is an L-parameter of $\mathbf{S}$, the W_F-action on $\hat{\mathbf{S}}$ is described by $\sigma(t) = [\phi_{\vartheta}(\sigma)](t)$ for any $\sigma \in W_F$ and $t \in \hat{\mathbf{S}}$. By noting that ${}^{L}j_{\chi}\colon {}^{L}\mathbf{S} \hookrightarrow {}^{L}\mathbf{G}$ is an L-embedding extending $\hat{\jmath}\colon \hat{\mathbf{S}} \xrightarrow{\sim} \hat{\mathbf{T}}$, this implies that $\hat{\jmath} \circ \sigma(t) = [\phi(\sigma)] \circ \hat{\jmath}(t)$ for any $\sigma \in W_F$ and $t \in \hat{\mathbf{S}}$. Similarly, we also have $\hat{\jmath}' \circ \sigma(t) = [\phi'(\sigma)] \circ \hat{\jmath}'(t)$ for any $\sigma \in W_F$ and $t \in \hat{\mathbf{S}}$. Hence, by noting that $[s] \circ \phi' = \phi$ and that $[s]|_{\hat{\mathbf{T}}} = \mathrm{id}_{\hat{\mathbf{T}}}$, we get

$$\begin{aligned}\hat{\theta}_{\mathbf{S}} \circ \sigma = \hat{\jmath}^{-1} \circ [s] \circ \hat{\jmath}' \circ \sigma &= \hat{\jmath}^{-1} \circ [s] \circ [\phi'(\sigma)] \circ \hat{\jmath}' \\ &= \hat{\jmath}^{-1} \circ [\phi(\sigma)] \circ [s] \circ \hat{\jmath}' = \sigma \circ \hat{\jmath}^{-1} \circ [s] \circ \hat{\jmath}' = \sigma \circ \hat{\theta}_{\mathbf{S}}.\end{aligned}$$

This completes the proof. □

We define a set $\zeta = (\zeta_{\alpha})_{\alpha\in\Phi(\mathbf{G},\mathbf{S})}$ of ζ-data for $\Phi(\mathbf{G}, \mathbf{S})$ by $\zeta := \zeta_{\theta_{\mathbf{S},*}(\chi)/\chi'}$ (see Definition 7.13 and also Remark 7.14).

Proposition 9.7. *The tuple $(\theta_{\mathbf{S}}, 1, \zeta)$ gives an isomorphism of regular supercuspidal L-packet data:*

$$(\theta_{\mathbf{S}}, 1, \zeta)\colon (\mathbf{S}, \hat{\jmath}, \chi, \vartheta) \xrightarrow{\sim} (\mathbf{S}, \hat{\jmath}', \chi', \vartheta).$$

To show this proposition, we recall the following property of a set of ζ-data, which is essentially discussed in [Kal19a], especially in the proof of [Kal19a, Theorem 3.16]:

Lemma 9.8. *Let χ_1 and χ_2 be sets of χ-data for $\Phi(\mathbf{G}, \mathbf{S}_{\hat{\jmath}})$. Let $c_{\chi_1/\chi_2}\colon W_F \to \hat{\mathbf{S}}$ be the L-parameter of the character $\zeta_{\chi_1/\chi_2,S}\colon S \to \mathbb{C}^{\times}$, which is regarded as a 1-cocycle. Then we have ${}^{L}j_{\chi_1} = (\hat{\jmath} \circ c_{\chi_1/\chi_2}) \cdot {}^{L}j_{\chi_2}$.*

Proof of Proposition 9.7. Our task is to check that the condition (3) of Definition 7.13 is satisfied, i.e., the equality $(\zeta_S^{-1} \cdot \vartheta) \circ \theta_{\mathbf{S}} = \vartheta$ holds. With the notation as in Lemma 9.8, the L-parameter of $\zeta_S^{-1} \cdot \vartheta$ is given by $c_{\chi'/\theta_{\mathbf{S},*}(\chi)} \cdot \phi_{\vartheta}$. Thus, by the functoriality of the local Langlands correspondence for tori, the L-parameter of $(\zeta_S^{-1} \cdot \vartheta) \circ \theta_{\mathbf{S}}$ is given by ${}^{L}\theta_{\mathbf{S}} \circ (c_{\chi'/\theta_{\mathbf{S},*}(\chi)} \cdot \phi_{\vartheta})$. We have to show that this is equivalent to ϕ_ϑ as L-parameters of $\mathbf{S}$, or equivalently, $c_{\chi'/\theta_{\mathbf{S},*}(\chi)} \cdot \phi_\vartheta$ and ${}^L\theta_{\mathbf{S}} \circ \phi_\vartheta$ are equivalent. By putting $s' := \hat{\jmath}^{-1}(s)$, let us check that $[s'] \circ (c_{\chi'/\theta_{\mathbf{S},*}(\chi)} \cdot \phi_\vartheta)$ is equal to ${}^L\theta_{\mathbf{S}} \circ \phi_\vartheta$.

Since ${}^Lj'_{\theta_{\mathbf{S},*}(\chi)}$ is injective, it suffices to show the equality after composing them with ${}^Lj'_{\theta_{\mathbf{S},*}(\chi)}$. By using Lemma 9.8, we have

$$\begin{aligned} {}^Lj'_{\theta_{\mathbf{S},*}(\chi)} \circ [s'] \circ (c_{\chi'/\theta_{\mathbf{S},*}(\chi)} \cdot \phi_\vartheta) &= [s] \circ {}^Lj'_{\theta_{\mathbf{S},*}(\chi)} \circ (c_{\chi'/\theta_{\mathbf{S},*}(\chi)} \cdot \phi_\vartheta) \\ &= [s] \circ {}^Lj'_{\chi'} \circ \phi_\vartheta = [s] \circ \phi'. \end{aligned}$$

On the other hand, by noting that $j_\chi = {}^Lj'_{\theta_{\mathbf{S},*}(\chi)} \circ {}^L\theta_{\mathbf{S}}$ (this is essentially the same identity as ${}^Lj'_{\chi'} = {}^L\theta \circ {}^Lj_\chi$, which was used in the proof of Lemma 9.2), we have

$${}^Lj'_{\theta_{\mathbf{S},*}(\chi)} \circ {}^L\theta_{\mathbf{S}} \circ \phi_\vartheta = {}^Lj_\chi \circ \phi_\vartheta = \phi.$$

As we have $[s] \circ \phi' = \phi$, we get the assertion. □

Since we have ${}^L\theta \circ \phi \cong \phi$, Proposition 9.5 implies that $\Pi_\phi^{\mathbf{G}} \circ \theta = \Pi_\phi^{\mathbf{G}}$. The effect of θ-twist on $\Pi_\phi^{\mathbf{G}}$ can be described more explicitly as follows.

Proposition 9.9. *Let π be the member of $\Pi_\phi^{\mathbf{G}}$ labeled by $(\mathbf{S}, \hat{\jmath}, \chi, \vartheta, j)$ for $j \in \mathcal{J}_G^{\mathbf{G}}$. Then its θ-twist π^θ is labeled by $(\mathbf{S}, \hat{\jmath}, \chi, \vartheta, \theta^{-1} \circ j \circ \theta_{\mathbf{S}})$.*

Proof. When π arises from $(\mathbf{S}, \hat{\jmath}, \chi, \vartheta, j)$ for $j \in \mathcal{J}_G^{\mathbf{G}}$, by Lemma 9.4, π^θ arises from the datum $(\mathbf{S}, \hat{\theta} \circ \hat{\jmath}, \theta(\chi), \vartheta, \theta^{-1} \circ j)$. On the other hand, the isomorphism of regular supercuspidal L-packet data $(\theta_{\mathbf{S}}, 1, \zeta)$ introduced above induces an isomorphism of regular supercuspidal data

$$(\theta_{\mathbf{S}}, 1, \zeta, 1) \colon (\mathbf{S}, \hat{\jmath}, \chi, \vartheta, j') \xrightarrow{\sim} (\mathbf{S}, \hat{\theta} \circ \hat{\jmath}, \theta(\chi), \vartheta, j' \circ \theta_{\mathbf{S}}^{-1})$$

for each $j' \in \mathcal{J}_G^{\mathbf{G}}$. Thus the datum $(\mathbf{S}, \hat{\theta} \circ \hat{\jmath}, \theta(\chi), \vartheta, \theta^{-1} \circ j)$ is isomorphic to $(\mathbf{S}, \hat{\jmath}, \chi, \vartheta, \theta^{-1} \circ j \circ \theta_{\mathbf{S}})$. □

Corollary 9.10. *Let π be the member of $\Pi_\phi^{\mathbf{G}}$ labeled by $(\mathbf{S}, \hat{\jmath}, \chi, \vartheta, j)$ for $j \in \mathcal{J}_G^{\mathbf{G}}$. Then the following are equivalent:*

1. *π is θ-stable, i.e., $\pi \cong \pi^\theta$,*
2. *j equals $\theta^{-1} \circ j \circ \theta_{\mathbf{S}}$ in $\mathcal{J}_G^{\mathbf{G}}$, i.e., $\theta^{-1} \circ j \circ \theta_{\mathbf{S}}$ and j are G-conjugate.*

9.3. **Embeddings of twisted tori.** We introduce several notions related to twisted maximal tori of $\tilde{\mathbf{G}}$. Suppose that $(\mathbf{S}, \tilde{\mathbf{S}})$ is a twisted space over F whose $\mathbf{S}$ is a torus. Let $\theta_{\mathbf{S}}$ be the automorphism of $\mathbf{S}$ given by $\tilde{\mathbf{S}}$, i.e., for any $s \in \mathbf{S}$ and $\eta \in \tilde{\mathbf{S}}$, we have $\theta_{\mathbf{S}}(s) = [\eta](s)$.

Definition 9.11. We say that an embedding $j \colon \mathbf{S} \hookrightarrow \mathbf{G}$ is an *F-rational embedding of a maximal torus* if j is F-rational and $\mathbf{S}_j := j(\mathbf{S})$ is a maximal torus of $\mathbf{G}$.

Definition 9.12. Let $(j, \tilde{j}) \colon (\mathbf{S}, \tilde{\mathbf{S}}) \hookrightarrow (\mathbf{G}, \tilde{\mathbf{G}})$ be an embedding of a twisted space, i.e., $j \colon \mathbf{S} \hookrightarrow \mathbf{G}$ and $\tilde{j} \colon \tilde{\mathbf{S}} \hookrightarrow \tilde{\mathbf{G}}$ are embeddings such that, for any $s_1, s_2 \in \mathbf{S}$ and $t \in \tilde{\mathbf{S}}$, we have $\tilde{j}(s_1 t s_2) = j(s_1)\tilde{j}(t)j(s_2)$. We say that $(j, \tilde{j})$ is *an F-rational embedding of a twisted maximal torus* if the following conditions are satisfied:

- j is an F-rational embedding of a maximal torus and $\tilde{j}$ is F-rational;
- $(\mathbf{S}_j, \tilde{\mathbf{S}}_j) := (j(\mathbf{S}), \tilde{j}(\tilde{\mathbf{S}}))$ is an F-rational twisted maximal torus of $\tilde{\mathbf{G}}$.

We often simply write "$(j, \tilde{j}) \colon \tilde{\mathbf{S}} \hookrightarrow \tilde{\mathbf{G}}$ is an F-rational embedding of a twisted maximal torus".

Remark 9.13. Let $(j, \tilde{j})\colon \tilde{\mathbf{S}} \hookrightarrow \tilde{\mathbf{G}}$ be an F-rational embedding of a twisted maximal torus. For any $\eta \in \tilde{\mathbf{S}}$, as we have $\theta_{\mathbf{S}} = [\eta]$, we have $j^{-1} \circ [\tilde{j}(\eta)] \circ j = \theta_{\mathbf{S}}$.

Note that if $(j, \tilde{j})\colon \tilde{\mathbf{S}} \hookrightarrow \tilde{\mathbf{G}}$ is an F-rational embedding of a twisted maximal torus, then $(j, \tilde{j}_s)\colon \tilde{\mathbf{S}} \hookrightarrow \tilde{\mathbf{G}}$ is again an F-rational embedding of a twisted maximal torus for any $s \in S$, where $\tilde{j}_s$ is defined by $\tilde{j}_s(\eta) := j(s)\tilde{j}(\eta)$ for $\eta \in \tilde{\mathbf{S}}$. The following lemma says that the converse of this fact is also true:

Lemma 9.14. *Let* $(\mathbf{S}, \tilde{\mathbf{S}})$ *be a twisted space defined over* F*. Let* $j\colon \mathbf{S} \hookrightarrow \mathbf{G}$ *be an* F*-rational embedding of a maximal torus. If* $(j, \tilde{j}_1)$ *and* $(j, \tilde{j}_2)$ *are* F*-rational embeddings of a twisted maximal torus* $\tilde{\mathbf{S}} \hookrightarrow \tilde{\mathbf{G}}$*, then there exists an element* $s \in S$ *satisfying* $\tilde{j}_2(\eta) = j(s)\tilde{j}_1(\eta)$ *for any* $\eta \in \tilde{\mathbf{S}}$*.*

Proof. If we fix an element $\eta' \in \tilde{S}$, then we have

$$j^{-1} \circ [\tilde{j}_1(\eta')] \circ j = \theta_{\mathbf{S}} = j^{-1} \circ [\tilde{j}_2(\eta')] \circ j$$

by Remark 9.13. This implies that $\tilde{j}_2(\eta')^{-1} \cdot \tilde{j}_1(\eta')$ belongs to S_j, hence there exists an $s \in S$ such that $\tilde{j}_2(\eta') = j(s)\tilde{j}_1(\eta')$. From this, we can see that $\tilde{j}_2(\eta) = j(s)\tilde{j}_1(\eta)$ for any $\eta \in \tilde{\mathbf{S}}$. $\square$

For two F-rational embeddings $(j_1, \tilde{j}_1)$ and $(j_2, \tilde{j}_2)$ of a twisted maximal torus $\tilde{\mathbf{S}} \hookrightarrow \tilde{\mathbf{G}}$, we write $(j_1, \tilde{j}_1) \sim (j_2, \tilde{j}_2)$ when $j_1 = j_2$ (this gives an equivalence relation on the set of F-rational embeddings of a twisted maximal torus). Note that, by Lemma 9.14, the image $\tilde{\mathbf{S}}_j$ of $\tilde{j}$ depends only on the equivalence class of $(j, \tilde{j})$. When $(j, \tilde{j})$ is an F-rational embedding of a twisted maximal torus, we often let j denote the equivalence classes of $(j, \tilde{j})$ by abuse of notation. Also, if we simply say "$j\colon \tilde{\mathbf{S}} \hookrightarrow \tilde{\mathbf{G}}$ is an F-rational embedding of a twisted maximal torus", then it means that we have an F-rational embedding of a twisted maximal torus $(j, \tilde{j})$ and j is its equivalence class.

As in the untwisted case, we define the stable/rational conjugacy for F-rational embeddings of a twisted maximal torus as follows:

Definition 9.15. Let $(j, \tilde{j})$ and $(j', \tilde{j}')$ be F-rational embeddings of a twisted maximal torus $\tilde{\mathbf{S}} \hookrightarrow \tilde{\mathbf{G}}$. We say that $(j, \tilde{j})$ and $(j', \tilde{j}')$ are $\mathbf{G}$-*conjugate* (resp. G-*conjugate*) if there exists an element $x \in \mathbf{G}$ (resp. $x \in G$) satisfying $j' = [x] \circ j$. In this case, we write $(j, \tilde{j}) \sim_{\mathbf{G}} (j', \tilde{j}')$ (resp. $(j, \tilde{j}) \sim_G (j', \tilde{j}')$). When j and j' are equivalence classes of F-rational embeddings of a twisted maximal torus, we say j and j' are $\mathbf{G}$-conjugate (resp. G-conjugate) if some (or, equivalently, any) representatives $(j, \tilde{j})$ and $(j', \tilde{j}')$ are $\mathbf{G}$-conjugate (resp. G-conjugate). In this case, we write $j \sim_{\mathbf{G}} j'$ (resp. $j \sim_G j'$).

9.4. **Parametrization of θ-stable members of a θ-stable packet.** Let us go back to the setting of Sections 9.1 and 9.2. Recall that $(\mathbf{S}, \hat{\jmath}, \chi, \vartheta)$ is a regular supercuspidal L-packet datum whose L-parameter satisfies $[s] \circ {}^L\theta \circ \phi = \phi$. In Section 9.2, we introduced an F-rational involutive automorphism $\theta_{\mathbf{S}}$ of $\mathbf{S}$. We consider the twisted space $\tilde{\mathbf{S}} := \mathbf{S}\theta_{\mathbf{S}}$ associated to the pair $(\mathbf{S}, \theta_{\mathbf{S}})$.

Proposition 9.16. *Let* π_j *be the member of* $\Pi_\phi^{\mathbf{G}}$ *labeled by* $(\mathbf{S}, \hat{\jmath}, \chi, \vartheta, j)$ *for* $j \in \mathcal{J}_G^{\mathbf{G}}$*. Then the following are equivalent:*

1. *π_j is θ-stable, i.e.,* $\pi_j \cong \pi_j^\theta$*,*
2. *j extends to an F-rational embedding of a twisted maximal torus* $\tilde{\mathbf{S}} \hookrightarrow \tilde{\mathbf{G}}$*.*

Proof. We first show that (2) implies (1). Suppose that j extends to an embedding $(j, \tilde{j})\colon \tilde{\mathbf{S}} \hookrightarrow \tilde{\mathbf{G}}$ of an F-rational twisted maximal torus. Then, putting $\eta := \tilde{j}(\theta_{\mathbf{S}}) \in \tilde{G}$, we get $[\eta] \circ j = j \circ \theta_{\mathbf{S}}$. In other words, if we write $\eta = \eta^\circ\theta$ with $\eta^\circ \in G$, then we have $[\eta^\circ] \circ \theta \circ j = j \circ \theta_{\mathbf{S}}$. In particular, $\theta^{-1} \circ j \circ \theta_{\mathbf{S}}$ and j are G-conjugate. This implies that π_j is θ-stable by Corollary 9.10.

We next show that (1) implies (2). Again by Corollary 9.10, we may suppose that we have an element $x \in G$ satisfying $\theta^{-1} \circ j \circ \theta_{\mathbf{S}} = [x] \circ j$. Then, by putting $\tilde{j}(\theta_{\mathbf{S}}) := \theta(x)\theta \in \tilde{G}$, the argument in the previous paragraph shows that $(j, \tilde{j})$ defines an embedding of a twisted space $(\mathbf{S}, \tilde{\mathbf{S}})$ into $(\mathbf{G}, \tilde{\mathbf{G}})$, which is defined over F. Hence our task is to show that the image $(\mathbf{S}_j, \tilde{\mathbf{S}}_j)$ is an F-rational twisted maximal torus of $(\mathbf{G}, \tilde{\mathbf{G}})$. In other words, we have to find a Borel subgroup which contains $\mathbf{S}_j$ and is preserved by $[\tilde{j}(\theta_{\mathbf{S}})] = [\theta(x)] \circ \theta$ (see Lemma 3.6).

Recall that we put $\jmath\colon \mathbf{T} \xrightarrow{\sim} \mathbf{S}$ to be the dual of $\hat{\jmath}\colon \hat{\mathbf{S}} \xrightarrow{\sim} \hat{\mathbf{T}}$ and that $\theta_{\mathbf{S}}$ is defined to be the dual of $\hat{\jmath}^{-1} \circ \hat{\theta} \circ \hat{\jmath}$. Hence we have $\theta_{\mathbf{S}} = \jmath \circ \theta \circ \jmath^{-1}$. The embedding j is given by $[y] \circ \jmath^{-1}$ for some $y \in \mathbf{G}$ by the $\hat{\jmath}$-admissibility (see the beginning of Section 9.1). Thus we get

$$j \circ \theta_{\mathbf{S}} = [y] \circ \jmath^{-1} \circ \jmath \circ \theta \circ \jmath^{-1} = [y\theta(y)^{-1}] \circ \theta \circ [y] \circ \jmath^{-1} = [y\theta(y)^{-1}] \circ \theta \circ j.$$

As we also have $\theta^{-1} \circ j \circ \theta_{\mathbf{S}} = [x] \circ j$, or equivalently, $j \circ \theta_{\mathbf{S}} = [\theta(x)] \circ \theta \circ j$, we get $[y\theta(y)^{-1}] \circ \theta \circ j = [\theta(x)] \circ \theta \circ j$. This implies that $\theta(x)^{-1}y\theta(y)^{-1} \in \theta(\mathbf{S}_j)$. Let us write $\theta(x)^{-1}y\theta(y)^{-1} = t$ with $t \in \theta(\mathbf{S}_j)$.

Since we have $\mathbf{S}_j = {}^{y}\mathbf{T}$, the Borel subgroup ${}^{y}\mathbf{B}$ contains $\mathbf{S}_j$. Let us check that ${}^{y}\mathbf{B}$ satisfies the desired condition, i.e., $[\theta(x)] \circ \theta({}^{y}\mathbf{B}) = {}^{y}\mathbf{B}$. By noting that $\mathbf{B}$ is stable under θ and that $t \in \theta(\mathbf{S}_j) \subset \theta({}^{y}\mathbf{B})$, we have

$$[\theta(x)] \circ \theta({}^{y}\mathbf{B}) = {}^{\theta(x)}\theta({}^{y}\mathbf{B}) = {}^{y\theta(y)^{-1}t^{-1}}\theta({}^{y}\mathbf{B}) = {}^{y\theta(y)^{-1}}\theta({}^{y}\mathbf{B}) = {}^{y}\theta(\mathbf{B}) = {}^{y}\mathbf{B}.$$

□

Now let us recall that

$$\mathcal{J}_G^{\mathbf{G}} = \{j\colon \mathbf{S} \hookrightarrow \mathbf{G} \mid j \text{ is } F\text{-rational and } j \sim_{\mathbf{G}} \jmath^{-1}\}/{\sim_G}$$

(see the discussion at the beginning of Section 9.1). We put

$$\tilde{\mathcal{J}}_G^{\mathbf{G}} := \{j\colon \tilde{\mathbf{S}} \hookrightarrow \tilde{\mathbf{G}} \mid j \text{ is } F\text{-rational and } j \sim_{\mathbf{G}} \jmath^{-1}\}/{\sim_G},$$

namely, $\tilde{\mathcal{J}}_G^{\mathbf{G}}$ is the set of G-conjugacy classes of equivalence classes of F-rational embeddings of a twisted maximal torus which are $\mathbf{G}$-conjugate to $\jmath^{-1}$. Note that the canonical forgetful map $\tilde{\mathcal{J}}_G^{\mathbf{G}} \hookrightarrow \mathcal{J}_G^{\mathbf{G}}\colon (j, \tilde{j}) \mapsto j$ is injective. Thus, in the following, we regard $\tilde{\mathcal{J}}_G^{\mathbf{G}}$ as a subset of $\mathcal{J}_G^{\mathbf{G}}$. By Proposition 9.16, the set $\tilde{\mathcal{J}}_G^{\mathbf{G}}$ parametrizes the θ-stable representations in $\Pi_\phi^{\mathbf{G}}$. More precisely, for each $j \in \mathcal{J}_G^{\mathbf{G}}$, the corresponding member π_j is θ-stable if and only if j belongs to $\tilde{\mathcal{J}}_G^{\mathbf{G}}$.

Remark 9.17. Recall that Shahidi's generic packet conjecture predicts that every tempered L-packet contains a unique $\mathfrak{w}$-generic member. If $\Pi_\phi^{\mathbf{G}}$ satisfies the generic packet conjecture, then the unique $\mathfrak{w}$-generic member of $\Pi_\phi^{\mathbf{G}}$ (say $\pi_{\mathfrak{w}}$) is θ-stable. Indeed, the θ-twist $\pi_{\mathfrak{w}}^\theta$ is again $\mathfrak{w}$-generic since $\mathfrak{w}$ is θ-stable. As we have $\Pi_\phi^{\mathbf{G}} \circ \theta = \Pi_\phi^{\mathbf{G}}$, both $\pi_{\mathfrak{w}}$ and $\pi_{\mathfrak{w}}^\theta$ belong to $\Pi_\phi^{\mathbf{G}}$. Hence the uniqueness part of the generic packet conjecture implies that $\pi_{\mathfrak{w}}$ and $\pi_{\mathfrak{w}}^\theta$ are isomorphic. (We note that the generic packet conjecture is proved in [Kal19b, Lemma 6.2.2] for toral regular supercuspidal L-packets, on which we will eventually focus in this paper.)

9.5. **Descended regular supercuspidal L-packet.** We keep the notation as in the previous subsections. Recall that $\phi\colon W_F \to \mathbf{G}$ factors through $\hat{\xi}\colon {}^L\mathbf{H} \hookrightarrow {}^L\mathbf{G}$. Let $\phi_{\mathbf{H}}$ be the L-parameter of $\mathbf{H}$ such that $\phi = \hat{\xi} \circ \phi_{\mathbf{H}}$.

Proposition 9.18. *The L-parameter $\phi_{\mathbf{H}}$ is regular supercuspidal.*

Proof. Let us check that the four conditions of Definition 7.17 for $\mathbf{H}$ are satisfied.

We first consider (0). Obviously $\phi|_{\mathrm{SL}_2(\mathbb{C})}$ is trivial, hence so is $\phi_{\mathbf{H}}|_{\mathrm{SL}_2(\mathbb{C})}$. Since $\mathbf{Z}_{\hat{\mathbf{H}}}(\phi(W_F))^\circ$ is contained in $\mathbf{Z}_{\hat{\mathbf{G}}}(\phi(W_F))^\circ$, we have $\mathbf{Z}_{\hat{\mathbf{H}}}(\phi(W_F))^\circ \subset \mathbf{Z}_{\hat{\mathbf{G}}} \cap \hat{\mathbf{H}} \subset \mathbf{Z}_{\hat{\mathbf{H}}}$.

We next consider (1). Since $\phi = {}^Lj_\chi \circ \phi_\vartheta$ and $\mathbf{S}$ is tamely ramified, a torus of $\hat{\mathbf{G}}$ containing $\phi(P_F)$ can be taken to be $\hat{\mathbf{T}}$. Thus $\phi_{\mathbf{H}}$ is contained in $\hat{\xi}^{-1}(\hat{\mathbf{T}}) = \hat{\mathbf{T}}_{\mathbf{H}}$.

We consider (2). Since we have $\mathbf{Z}_{\hat{\mathbf{H}}}(\phi(I_F))^\circ \subset \mathbf{Z}_{\hat{\mathbf{G}}}(\phi(I_F))^\circ \cap \hat{\mathbf{H}}$ and $\mathbf{Z}_{\hat{\mathbf{G}}}(\phi(I_F))^\circ$ is a torus, $\mathbf{Z}_{\hat{\mathbf{H}}}(\phi(I_F))^\circ$ is also a torus.

We finally consider (3). We put $\mathcal{M}_{\mathbf{H}} := \mathbf{Z}_{\hat{\mathbf{H}}}(\phi(P_F))^\circ$, $\mathcal{C}_{\mathbf{H}} := \mathbf{Z}_{\hat{\mathbf{H}}}(\phi(I_F))^\circ$, and $\mathcal{T}_{\mathbf{H}} := \mathbf{Z}_{\mathcal{M}_{\mathbf{H}}}(\mathcal{C}_{\mathbf{H}})$. Then we have $\mathbf{N}_{\mathcal{M}_{\mathbf{H}}}(\mathcal{T}_{\mathbf{H}}) \subset \mathbf{N}_{\mathcal{M}}(\mathcal{T})$ and this inclusion induces a Γ-equivariant inclusion of Weyl groups $\Omega_{\mathcal{M}_{\mathbf{H}}}(\hat{\mathbf{S}}_{\mathbf{H}}) \hookrightarrow \Omega_{\mathcal{M}}(\hat{\mathbf{S}})$. Thus, if $n \in N_{\mathcal{M}_{\mathbf{H}}}(\mathcal{T}_{\mathbf{H}})$ maps to a nontrivial element of $\Omega_{\mathcal{M}_{\mathbf{H}}}(\hat{\mathbf{S}}_{\mathbf{H}})^\Gamma$, then we have $n \notin \mathbf{Z}_{\hat{\mathbf{G}}}(\phi(I_F))$. This implies that $n \notin \mathbf{Z}_{\hat{\mathbf{H}}}(\phi(I_F))$.

□

Proposition 9.19. *If ϕ is toral supercuspidal, then so is $\phi_{\mathbf{H}}$.*

Proof. Let us check that the three conditions of Definition 7.19 for $\mathbf{H}$ are satisfied. The condition (0) is already checked in the proof of Proposition 9.18. The condition (2) for $\phi_{\mathbf{H}}$ is clearly deduced from the condition (2) for ϕ. Thus let us consider (1). By the torality of ϕ, $\mathbf{Z}_{\hat{\mathbf{G}}}(\phi(I_F^r))$ is a maximal torus of $\hat{\mathbf{G}}$ containing $\phi(P_F)$. Since we have $\phi(I_F^r) \subset \hat{\mathbf{T}}$ (by the construction of ϕ), we have $\mathbf{Z}_{\hat{\mathbf{G}}}(\phi(I_F^r)) \supset \mathbf{Z}_{\hat{\mathbf{G}}}(\hat{\mathbf{T}}) = \hat{\mathbf{T}}$. Thus we get $\mathbf{Z}_{\hat{\mathbf{G}}}(\phi(I_F^r)) = \hat{\mathbf{T}}$. This implies that $\mathcal{T}_{\mathbf{H}} := Z_{\hat{\mathbf{H}}}(\phi(I_F^r))$ is equal to $\hat{\mathbf{H}} \cap \hat{\mathbf{T}} = \hat{\mathbf{T}}_{\mathbf{H}}$, which is a maximal torus of $\hat{\mathbf{H}}$ and contains $\phi(P_F)$. □

By applying Proposition 7.18 to the descended L-parameter $\phi_{\mathbf{H}}$, we obtain a regular supercuspidal L-packet datum $(\mathbf{S}_{\mathbf{H}}, \hat{\jmath}_{\mathbf{H}}, \chi_{\mathbf{H}}, \vartheta_{\mathbf{H}})$ of $\mathbf{H}$ and hence a regular supercuspidal L-packet $\Pi^{\mathbf{H}}_{\phi_{\mathbf{H}}}$ of $\mathbf{H}$. In particular, we may and do assume $\phi_{\mathbf{H}} = {}^Lj_{\chi_{\mathbf{H}}} \circ \phi_{\vartheta_{\mathbf{H}}}$, where ${}^Lj_{\chi_{\mathbf{H}}}$ denotes the Langlands–Shelstad extension of $\hat{\jmath}_{\mathbf{H}}$ to an L-embedding via the set of χ-data $\chi_{\mathbf{H}}$:

(14)
$$\begin{array}{ccccc}
W_F & \xrightarrow{\phi_\vartheta} & \hat{\mathbf{S}} \rtimes W_F & \xhookrightarrow{{}^Lj_\chi} & \hat{\mathbf{G}} \rtimes W_F \\
& \searrow^{\phi_{\vartheta_{\mathbf{H}}}} & & & \uparrow \hat{\xi} \\
& & \hat{\mathbf{S}}_{\mathbf{H}} \rtimes W_F & \xhookrightarrow{{}^Lj_{\chi_{\mathbf{H}}}} & \hat{\mathbf{H}} \rtimes W_F
\end{array}$$

Let us investigate the relationship between $(\mathbf{S}, \hat{\jmath}, \chi, \vartheta)$ and $(\mathbf{S}_{\mathbf{H}}, \hat{\jmath}_{\mathbf{H}}, \chi_{\mathbf{H}}, \vartheta_{\mathbf{H}})$.

As we saw in the proof of Proposition 9.19, the image of $\hat{\jmath}_{\mathbf{H}}$ is given by $\hat{\jmath}_{\mathbf{H}}(\hat{\mathbf{S}}_{\mathbf{H}}) = \mathbf{Z}_{\hat{\mathbf{H}}}(\phi(I_F^r)) = \hat{\mathbf{H}} \cap \hat{\mathbf{T}} = \hat{\mathbf{T}}_{\mathbf{H}}$. Recall that $\hat{\jmath} \circ \hat{\theta}_{\mathbf{S}} = \hat{\theta} \circ \hat{\jmath}$. This implies that the embedding $\hat{\jmath}$ induces an isomorphism $\hat{\mathbf{S}}^{\hat{\theta}_{\mathbf{S}},\circ} \cong \hat{\mathbf{T}}^{\hat{\theta},\circ} = \hat{\xi}(\hat{\mathbf{T}}_{\mathbf{H}})$. Thus, by combining it with $\hat{\xi}$ and $\hat{\jmath}_{\mathbf{H}}$, we get an identification of $\hat{\mathbf{S}}_{\mathbf{H}}$ with $\hat{\mathbf{S}}^{\hat{\theta}_{\mathbf{S}},\circ}$:

$$\hat{\jmath}^{-1} \circ \hat{\xi} \circ \hat{\jmath}_{\mathbf{H}} \colon \hat{\mathbf{S}}_{\mathbf{H}} \xrightarrow{\sim} \hat{\mathbf{S}}^{\hat{\theta}_{\mathbf{S}},\circ}.$$

As discussed in the proof of Lemma 9.6, we have $\hat{\jmath} \circ \sigma(t) = [\phi(\sigma)] \circ \hat{\jmath}(t)$ for any $\sigma \in W_F$ and $t \in \hat{\mathbf{S}}$. Similarly, we also have $\hat{\jmath}_{\mathbf{H}} \circ \sigma(t) = [\phi_{\mathbf{H}}(\sigma)] \circ \hat{\jmath}_{\mathbf{H}}(t)$ for any $\sigma \in W_F$

and $t \in \hat{\mathbf{S}}_{\mathbf{H}}$. Therefore, since we have $\phi = \hat{\xi} \circ \phi_{\mathbf{H}}$, the isomorphism $\hat{\jmath}^{-1} \circ \xi \circ \hat{\jmath}_{\mathbf{H}}$ is Γ-equivariant. Thus, by taking dual, we get an F-rational isomorphism

$$\mathbf{S}_{\mathbf{H}} \cong \mathbf{S}_{\theta_{\mathbf{S}}}.$$

Proposition 9.20. *The restriction $\vartheta|_{S_{0+}}$ of ϑ to S_{0+} coincide with a pullback of the restriction $\vartheta_{\mathbf{H}}|_{S_{\mathbf{H},0+}}$ of $\vartheta_{\mathbf{H}}$ to $S_{\mathbf{H},0+}$ through the map $S_{0+} \to S_{\theta_{\mathbf{S}},0+} \cong S_{\mathbf{H},0+}$.*

Proof. By abuse of notation, we again write $\vartheta_{\mathbf{H}}$ for the pullback of $\vartheta_{\mathbf{H}}$ along the canonical map $S \to S_{\theta_{\mathbf{S}}} \cong S_{\mathbf{H}}$. Then our task is to show that the depth of the character $\vartheta^{-1} \cdot \vartheta_{\mathbf{H}}$ of S is zero. Since the local Langlands correspondence for tame tori is multiplicative and preserves the depth (see, e.g., [Yu09]), the depth of $\vartheta^{-1} \cdot \vartheta_{\mathbf{H}}$ equals that of $\phi_{\vartheta}^{-1} \cdot \phi_{\vartheta_{\mathbf{H}}}$. Here $\phi_{\vartheta}^{-1} \cdot \phi_{\vartheta_{\mathbf{H}}}$ denotes the product as 1-cocycles.

We note that the following diagram is commutative since every object is tamely ramified (more precisely, Γ-actions on $\hat{\mathbf{S}}$, $\hat{\mathbf{S}}_{\mathbf{H}}$, $\hat{\mathbf{G}}$, and $\hat{\mathbf{H}}$ are trivial on I_F^{0+} and the set of χ-data χ and $\chi_{\mathbf{H}}$ are minimally ramified):

$$\begin{array}{ccc} \hat{\mathbf{S}} \rtimes I_F^{0+} & \xhookrightarrow{{}^L j_{\chi}} & \hat{\mathbf{G}} \rtimes I_F^{0+} \\ \uparrow & & \uparrow \hat{\xi} \\ \hat{\mathbf{S}}_{\mathbf{H}} \rtimes I_F^{0+} & \xhookrightarrow{{}^L j_{\chi_{\mathbf{H}}}} & \hat{\mathbf{H}} \rtimes I_F^{0+} \end{array}$$

Thus, by taking into account the commutativity of the diagram (14), we see that the following diagram commutes:

$$\begin{array}{ccc} I_F^{0+} & \xrightarrow{\phi_{\vartheta}|_{I_F^{0+}}} & \hat{\mathbf{S}} \rtimes I_F^{0+} \\ & \searrow^{\phi_{\vartheta_{\mathbf{H}}}|_{I_F^{0+}}} & \uparrow \\ & & \hat{\mathbf{S}}_{\mathbf{H}} \rtimes I_F^{0+} \end{array}$$

This implies that depth of the L-parameter $\phi_{\vartheta}^{-1} \cdot \phi_{\vartheta_{\mathbf{H}}}$ is zero. □

Now let us suppose that ϕ is toral of depth $r > 0$. Recall that, as proved in the proof of Lemma 9.8, we have $(\zeta_S^{-1} \cdot \vartheta) \circ \theta_{\mathbf{S}} = \vartheta$. As ζ_S is tamely ramified, this implies that $\vartheta|_{S_{0+}} \circ \theta_{\mathbf{S}} = \vartheta|_{S_{0+}}$. This implies that we can take a $\theta_{\mathbf{S}}$-invariant element $X^* \in \mathfrak{s}^*_{-r}$ realizing $\vartheta|_{S_{s+:r+}}$ (see Lemma 5.3). By Proposition 9.20, we furthermore have the following (note that $\mathfrak{s}^*_{\theta_{\mathbf{S}}}$ is identified with the $\theta_{\mathbf{S}}$-fixed subspace of $\mathfrak{s}^*$):

Corollary 9.21. *We can take elements $X^* \in \mathfrak{s}^*_{-r}$ and $Y^* \in \mathfrak{s}^*_{\mathbf{H},-r}$ realizing $\vartheta|_{S_{s+:r+}}$ and $\vartheta_{\mathbf{H}}|_{S_{\mathbf{H},s+:r+}}$, respectively, so that Y^* maps to X^* under the natural map $\mathfrak{s}^*_{\mathbf{H}} \cong \mathfrak{s}^*_{\theta_{\mathbf{S}}} \hookrightarrow \mathfrak{s}^*$.*

Remark 9.22. We caution that the diagram

$$\begin{array}{ccc} \hat{\mathbf{S}} \rtimes W_F & \xhookrightarrow{{}^L j_{\chi}} & \hat{\mathbf{G}} \rtimes W_F \\ \uparrow & & \uparrow \hat{\xi} \\ \hat{\mathbf{S}}_{\mathbf{H}} \rtimes W_F & \xhookrightarrow{{}^L j_{\chi_{\mathbf{H}}}} & \hat{\mathbf{H}} \rtimes W_F \end{array}$$

is not commutative in general although it is commutative at the positive depth level as observed in the proof of Proposition 9.20. The non-commutativity of this diagram is crucially related to the computation of the transfer factor (Section 14.2.6).

10.1. **Waldspurger's diagram.** We recall the notion of a "diagram". This was introduced by Waldspurger in [Wal08, Section 3.2] first . Then, a slightly modified definition was introduced in [MW16, I.1.10]. Here we follow the latter version.

Definition 10.1. For $(\epsilon, \eta) \in H_{\mathrm{ss}} \times \tilde{G}_{\mathrm{ss}}$, *a diagram associated to* (ϵ, η) is a quadruple $D = (\mathbf{B}^{\flat}, \mathbf{T}^{\flat}, \mathbf{B}^{\diamondsuit}, \mathbf{T}^{\diamondsuit})$ satisfying the following:

- $\mathbf{T}^{\flat}$ is an F-rational maximal torus of $\mathbf{H}$,
- $(\mathbf{B}^{\flat}, \mathbf{T}^{\flat})$ is a Borel pair of $\mathbf{H}$ which is $[\epsilon]$-stable,
- $\mathbf{T}^{\diamondsuit}$ is an F-rational maximal torus of $\mathbf{G}$,
- $(\mathbf{B}^{\diamondsuit}, \mathbf{T}^{\diamondsuit})$ is a Borel pair of $\mathbf{G}$ which is $[\eta]$-stable,
- The Borel pairs $(\mathbf{B}^{\flat}, \mathbf{T}^{\flat})$ and $(\mathbf{B}_{\mathbf{H}}, \mathbf{T}_{\mathbf{H}})$ induce a unique isomorphism $\xi_{\flat} \colon \mathbf{T}^{\flat} \xrightarrow{\sim} \mathbf{T}_{\mathbf{H}}$ given by $\mathbf{H}$-conjugation. Similarly, $(\mathbf{B}^{\diamondsuit}, \mathbf{T}^{\diamondsuit})$ and $(\mathbf{B}, \mathbf{T})$ induce a unique isomorphism $\xi_{\diamondsuit} \colon \mathbf{T}^{\diamondsuit} \xrightarrow{\sim} \mathbf{T}$ given by $\mathbf{G}$-conjugation. Then the composition $\xi_{\flat}^{-1} \circ \xi \circ \xi_{\diamondsuit}$ is defined over F. (We write ξ_D for this composition.)
- If we let $g \in \mathbf{G}$ be an element such that $[g] = \xi_{\diamondsuit}$, then η belongs to $[g]^{-1}(\tilde{\mathbf{T}})$. (We put $\tilde{\mathbf{T}}^{\diamondsuit} := \mathbf{T}^{\diamondsuit}\eta$ and write $\tilde{\xi}_{\diamondsuit}$ for the map $[g] \colon \tilde{\mathbf{T}}^{\diamondsuit} \to \tilde{\mathbf{T}}$.)
- Let $\nu_D \in \mathbf{T}$ be an element such that $[g](\eta) = \nu_D\theta$. Then we have $\xi(\nu_D) = \xi_{\flat}(\epsilon)$. (We write μ_D for this element.) In other words, if we define a map $\tilde{\xi}_D \colon \tilde{\mathbf{T}}^{\diamondsuit} \to \mathbf{T}^{\flat}$ by $\tilde{\xi}_D := \xi_{\flat}^{-1} \circ \xi \circ ((-) \cdot \theta^{-1}) \circ \tilde{\xi}_{\diamondsuit}$, then we have $\tilde{\xi}_D(\eta) = \epsilon$.

$$
\begin{array}{ccccccc}
\mathbf{T}^{\flat} & \xleftarrow{\ \ \xi_D\ \ } & & & \mathbf{T}^{\diamondsuit} & & \tilde{\mathbf{T}}^{\diamondsuit} = \mathbf{T}^{\diamondsuit}\eta \\
\downarrow{\scriptstyle \xi_{\flat}} & & & & \downarrow{\scriptstyle \xi_{\diamondsuit}} & & \downarrow{\scriptstyle \tilde{\xi}_{\diamondsuit}} \\
\mathbf{T}_{\mathbf{H}} & \xleftarrow[\xi]{\cong} & \mathbf{T}_{\theta} & \twoheadleftarrow & \mathbf{T} & \xleftarrow[(-)\cdot\theta^{-1}]{} & \tilde{\mathbf{T}} = \mathbf{T}\theta
\end{array}
\qquad
\begin{array}{ccccc}
\epsilon & \xleftarrow{\ \ \tilde{\xi}_D\ \ } & & & \eta \\
\downarrow{\scriptstyle \xi_{\flat}} & & & & \downarrow{\scriptstyle \tilde{\xi}_{\diamondsuit}} \\
\mu_D & \longleftarrow & \nu_D & \longleftarrow & \nu_D\theta
\end{array}
$$

For $(\epsilon, \eta) \in H_{\mathrm{ss}} \times \tilde{G}_{\mathrm{ss}}$, let $\mathbf{D}(\epsilon, \eta)$ denote the set of diagrams associated to (ϵ, η).

Remark 10.2. (1) We often simply write ν and μ for ν_D and μ_D, respectively.

(2) The condition that $(\mathbf{B}^{\flat}, \mathbf{T}^{\flat})$ is $[\epsilon]$-stable is equivalent to that $\epsilon \in \mathbf{T}^{\flat}$, which is furthermore equivalent to that $\mathbf{T}^{\flat} \subset \mathbf{H}_{\epsilon}$.

(3) Let $D = (\mathbf{B}^{\flat}, \mathbf{T}^{\flat}, \mathbf{B}^{\diamondsuit}, \mathbf{T}^{\diamondsuit}) \in \mathbf{D}(\epsilon, \eta)$. Then $(\mathbf{T}^{\diamondsuit}, \tilde{\mathbf{T}}^{\diamondsuit})$ is an F-rational twisted maximal torus of $(\mathbf{G}, \tilde{\mathbf{G}})$ by Lemma 3.6. In particular, $\mathbf{T}^{\natural} := \mathbf{T}^{\eta,\circ}$ is a maximal torus of $\mathbf{G}_{\eta}$ and $\mathbf{T}^{\diamondsuit}$ is recovered from $\mathbf{T}^{\natural}$ by $\mathbf{T}^{\diamondsuit} = \mathbf{Z}_{\mathbf{G}}(\mathbf{T}^{\natural})$ (see Proposition 3.3).

(4) If the set $\mathbf{D}(\epsilon, \eta)$ is not empty, then the stable conjugacy classes of ϵ and η correspond in the sense of twisted endoscopy (see Section 8.2).

(5) In general, even if the stable conjugacy classes of ϵ and η correspond, the set $\mathbf{D}(\epsilon, \eta)$ might be empty. See [MW16, I.1.10].

(6) When ϵ is strongly $\tilde{\mathbf{G}}$-regular semisimple and η is strongly regular semisimple, the set $\mathbf{D}(\epsilon, \eta)$ is not empty if and only if ϵ is a norm of η by [MW16, I.1.10, Lemme]. Furthermore, a diagram associated to (ϵ, η) is essentially unique. We will investigate these facts later (Lemma 10.7) in detail.

Remark 10.3. Recall that the map $\xi \colon \mathbf{T} \twoheadrightarrow \mathbf{T}_{\theta} \xrightarrow{\sim} \mathbf{T}_{\mathbf{H}}$ induces an identification of the Weyl group $\Omega_{\mathbf{H}}(\mathbf{T}_{\mathbf{H}})$ of $\mathbf{T}_{\mathbf{H}}$ in $\mathbf{H}$ with a subgroup of $\Omega_{\mathbf{G}}(\mathbf{T})^{\theta}$. For any diagram $D = (\mathbf{B}^{\flat}, \mathbf{T}^{\flat}, \mathbf{B}^{\diamondsuit}, \mathbf{T}^{\diamondsuit}) \in \mathbf{D}(\epsilon, \eta)$, the maps $\xi_{\diamondsuit}$ and $\xi_{\flat}$ induce isomorphisms $\Omega_{\mathbf{H}}(\mathbf{T}^{\flat}) \xrightarrow{\sim} \Omega_{\mathbf{H}}(\mathbf{T}_{\mathbf{H}})$ and $\Omega_{\mathbf{G}}(\mathbf{T}^{\diamondsuit}) \xrightarrow{\sim} \Omega_{\mathbf{G}}(\mathbf{T})$, respectively. The image of

$\Omega_{\mathbf{H}_\epsilon}(\mathbf{T}^\flat) \subset \Omega_{\mathbf{H}}(\mathbf{T}^\flat)$ is contained in the image of $\Omega_{\mathbf{G}_\eta}(\mathbf{T}^\natural) \subset \Omega_{\mathbf{G}}(\mathbf{T}^\diamondsuit)$, hence we get an identification $\Omega_{\mathbf{H}_\epsilon}(\mathbf{T}^\flat) \hookrightarrow \Omega_{\mathbf{G}_\eta}(\mathbf{T}^\natural)$. Since ξ_D is F-rational, this identification is F-rational.

Lemma 10.4. *Let $D \in \mathbf{D}(\epsilon, \eta)$. Then the map $\tilde{\xi}_D$ is F-rational.*

Proof. Any element of $\tilde{\mathbf{T}}^\diamondsuit$ can be written as $t\eta$ with $t \in \mathbf{T}^\diamondsuit$. Then the image of $t\eta$ under $\tilde{\xi}_D$ is given by $\xi_D(t)\epsilon$. In other words, the map $\tilde{\xi}_D$ equals the composition $((-) \cdot \epsilon) \circ \xi_D \circ ((-) \cdot \eta^{-1})$. Since η, ϵ, and ξ_D are F-rational, so is $\tilde{\xi}_D$. □

10.2. **Equivalence relation on diagrams.** Let $(\epsilon, \eta) \in H_{\mathrm{ss}} \times \tilde{G}_{\mathrm{ss}}$. We introduce an equivalence relation on $\mathbf{D}(\epsilon, \eta)$ as follows:

Definition 10.5. We define $\sim$ to be the equivalence relation on $\mathbf{D}(\epsilon, \eta)$ generated by the following two equivalence relations: Let $D = (\mathbf{B}^\flat, \mathbf{T}^\flat, \mathbf{B}^\diamondsuit, \mathbf{T}^\diamondsuit), \bar{D} = (\bar{\mathbf{B}}^\flat, \bar{\mathbf{T}}^\flat, \bar{\mathbf{B}}^\diamondsuit, \bar{\mathbf{T}}^\diamondsuit) \in \mathbf{D}(\epsilon, \eta)$.

- **(i) $(\mathbf{H}_\epsilon, \mathbf{G}_\eta)$-conjugacy:** We say that D and $\bar{D}$ are $(\mathbf{H}_\epsilon, \mathbf{G}_\eta)$-conjugate if there exists elements $h \in \mathbf{H}_\epsilon$ and $g \in \mathbf{G}_\eta$ such that
$$\bar{D} = ({}^h\mathbf{B}^\flat, {}^h\mathbf{T}^\flat, {}^g\mathbf{B}^\diamondsuit, {}^g\mathbf{T}^\diamondsuit).$$
- **(ii) $\Omega_{\mathbf{H}}$-conjugacy:** We say that D and $\bar{D}$ are $\Omega_{\mathbf{H}}(\mathbf{T}^\flat)$-conjugate if there exists elements $w \in \Omega_{\mathbf{H}}(\mathbf{T}^\flat)$ such that
$$\bar{D} = ({}^w\mathbf{B}^\flat, \mathbf{T}^\flat, {}^w\mathbf{B}^\diamondsuit, \mathbf{T}^\diamondsuit).$$
Here, w is also regarded as an element of $\Omega_{\mathbf{G}}(\mathbf{T}^\diamondsuit)$ (see Remark 10.3).

We write $\mathbb{D}(\epsilon, \eta)$ for the set $\mathbf{D}(\epsilon, \eta)/\sim$ of equivalence classes of diagrams associated to (ϵ, η).

Remark 10.6. When $\mathbf{G}$ is untwisted (θ is trivial) and $\mathbf{H}_\epsilon$ is quasi-split, the set $\mathbb{D}(\epsilon, \eta)$ is nothing but the set $\Xi(\mathbf{H}_\epsilon, \mathbf{G}_\eta)$ used in the proof of [Kal19b, Theorem 6.3.4].

Lemma 10.7. *If $(\epsilon, \eta) \in \mathcal{D}$, then the set $\mathbb{D}(\epsilon, \eta)$ is a singleton, i.e., any two diagrams associated to (ϵ, η) are equivalent. Moreover, the maps ξ_D and $\tilde{\xi}_D$ are independent of a diagram $D \in \mathbf{D}(\epsilon, \eta)$.*

Proof. The non-emptiness of $\mathbf{D}(\epsilon, \eta)$ follows from [KS99, Lemma 3.3.B] (with the argument in the final paragraph in 29 page of [KS99]). See also [MW16, I.1.10, Lemme].

We show that any two diagrams associated to (ϵ, η) are equivalent. Let $D = (\mathbf{B}^\flat, \mathbf{T}^\flat, \mathbf{B}^\diamondsuit, \mathbf{T}^\diamondsuit), \bar{D} = (\bar{\mathbf{B}}^\flat, \bar{\mathbf{T}}^\flat, \bar{\mathbf{B}}^\diamondsuit, \bar{\mathbf{T}}^\diamondsuit) \in \mathbf{D}(\epsilon, \eta)$. Since ϵ is strongly regular semisimple, we have $\mathbf{T}^\flat = \mathbf{H}_\epsilon = \bar{\mathbf{T}}^\flat$. Similarly we have $\mathbf{T}^\natural = \mathbf{G}_\eta = \bar{\mathbf{T}}^\natural$ (recall that both $\mathbf{T}^\natural$ and $\bar{\mathbf{T}}^\natural$ are maximal tori of $\mathbf{G}_\eta$ and that $\mathbf{G}_\eta$ is a torus by the strong regularity of η). As we have $\mathbf{T}^\diamondsuit = Z_{\mathbf{G}}(\mathbf{T}^\natural)$ and $\bar{\mathbf{T}}^\diamondsuit = Z_{\mathbf{G}}(\bar{\mathbf{T}}^\natural)$, we get $\mathbf{T}^\diamondsuit = \bar{\mathbf{T}}^\diamondsuit$.

Since both $\mathbf{B}^\flat$ and $\bar{\mathbf{B}}^\flat$ are Borel subgroups of $\mathbf{H}$ containing $\mathbf{T}^\flat$, there exists an element $w \in \Omega_{\mathbf{H}}(\mathbf{T}^\flat)$ such that ${}^w\mathbf{B}^\flat = \bar{\mathbf{B}}^\flat$. Hence, by replacing D with its $\Omega_{\mathbf{H}}(\mathbf{T}^\flat)$-conjugate diagram $({}^w\mathbf{B}^\flat, \mathbf{T}^\flat, {}^w\mathbf{B}^\diamondsuit, \mathbf{T}^\diamondsuit)$, we may suppose that $\mathbf{B}^\flat = \bar{\mathbf{B}}^\flat$.

Let $g \in \mathbf{G}$ be an element satisfying $({}^g\mathbf{B}^\diamondsuit, {}^g\mathbf{T}^\diamondsuit) = (\mathbf{B}, \mathbf{T})$. Similarly, let $\bar{g} \in \mathbf{G}$ be an element satisfying $({}^{\bar{g}}\bar{\mathbf{B}}^\diamondsuit, {}^{\bar{g}}\mathbf{T}^\diamondsuit) = (\mathbf{B}, \mathbf{T})$. Since $\tilde{\xi}_D(\eta) = \epsilon = \tilde{\xi}_{\bar{D}}(\eta)$, we get
$$\xi_\flat^{-1} \circ \xi({}^g\eta \cdot \theta^{-1}) = \xi_\flat^{-1} \circ \xi({}^{\bar{g}}\eta \cdot \theta^{-1}).$$

In other words, ${}^{g}\eta\cdot\theta^{-1}, {}^{\bar{g}}\eta\cdot\theta^{-1}\in\mathbf{T}$ map to the same element of $\mathbf{T}_\theta$ under the natural quotient map $\mathbf{T}\twoheadrightarrow\mathbf{T}_\theta$. Hence we can find an element $t\in\mathbf{T}$ such that ${}^{\bar{g}}\eta\cdot\theta^{-1}=t\cdot({}^{g}\eta\cdot\theta^{-1})\cdot\theta(t)^{-1}$, equivalently, ${}^{\bar{g}}\eta=t\cdot({}^{g}\eta)\cdot t^{-1}$. Thus $\bar{g}^{-1}tg$ belongs to the (full) centralizer $\mathbf{G}^\eta$ of η in $\mathbf{G}$. By [Wal08, Section 3.1], the strong regularity of η implies $\mathbf{G}^\eta=\mathbf{Z}^\eta_{\mathbf{G}}\mathbf{G}_\eta=\mathbf{Z}^\eta_{\mathbf{G}}\mathbf{T}^\natural$. In particular, we can take an element $z\in\mathbf{Z}^\eta_{\mathbf{G}}$ such that $z\bar{g}^{-1}tg$ belongs to $\mathbf{G}_\eta$. If we put $g':=z\bar{g}^{-1}tg\in\mathbf{G}_\eta$, then we have $({}^{g'}\mathbf{B}^\diamondsuit,{}^{g'}\mathbf{T}^\diamondsuit)=(\bar{\mathbf{B}}^\diamondsuit,\bar{\mathbf{T}}^\diamondsuit)$. Hence D and $\bar{D}$ are $(\mathbf{H}_\epsilon,\mathbf{G}_\eta)$-conjugate.

Finally, noting that $\mathbf{H}_\epsilon$ and $\mathbf{G}_\eta$ are tori, we see that $(\mathbf{H}_\epsilon,\mathbf{G}_\eta)$-conjugacy does not change the maps ξ_D and $\tilde{\xi}_D$. We also see that $\Omega_{\mathbf{H}}(\mathbf{T}^\flat)$-conjugacy does not change ξ_D and $\tilde{\xi}_D$. Hence ξ_D and $\tilde{\xi}_D$ are independent of the choice of $D\in\mathbf{D}(\epsilon,\eta)$. □

Lemma 10.8. *Let $D=(\mathbf{B}^\flat,\mathbf{T}^\flat,\mathbf{B}^\diamondsuit,\mathbf{T}^\diamondsuit)\in\mathbf{D}(\epsilon,\eta)$. For any F-rational elliptic maximal torus $\bar{\mathbf{T}}^\flat$ of $\mathbf{H}_\epsilon$, there exists a diagram $(\bar{\mathbf{B}}^\flat,\bar{\mathbf{T}}^\flat,\bar{\mathbf{B}}^\diamondsuit,\bar{\mathbf{T}}^\diamondsuit)\in\mathbf{D}(\epsilon,\eta)$ which is equivalent to D.*

Proof. Let $g\in\mathbf{G}$ be an element satisfying $({}^{g}\mathbf{B}^\diamondsuit,{}^{g}\mathbf{T}^\diamondsuit)=(\mathbf{B},\mathbf{T})$ (i.e., $\xi_\diamondsuit=[g]$). Similarly, let $h\in\mathbf{H}$ be an element satisfying $({}^{h}\mathbf{B}^\flat,{}^{h}\mathbf{T}^\flat)=(\mathbf{B}_{\mathbf{H}},\mathbf{T}_{\mathbf{H}})$ (i.e., $\xi_\flat=[h]$).

Since both $\mathbf{T}^\flat$ and $\bar{\mathbf{T}}^\flat$ are maximal tori of $\mathbf{H}_\epsilon$, there exists an element $h'\in\mathbf{H}_\epsilon$ satisfying ${}^{h'}\mathbf{T}^\flat=\bar{\mathbf{T}}^\flat$. We put $\bar{h}:=hh'^{-1}$ (hence ${}^{\bar{h}}\bar{\mathbf{T}}^\flat=\mathbf{T}_{\mathbf{H}}$). We define a Borel subgroup $\bar{\mathbf{B}}^\flat$ containing $\bar{\mathbf{T}}^\flat$ by $\bar{\mathbf{B}}^\flat:={}^{\bar{h}^{-1}}\mathbf{B}_{\mathbf{H}}$ (hence ${}^{\bar{h}}\bar{\mathbf{B}}^\flat=\mathbf{B}_{\mathbf{H}}$).

Let us construct $\bar{\mathbf{T}}^\diamondsuit$. For this, we first take a quasi-split inner form $\mathbf{G}^*_\eta$ of $\mathbf{G}_\eta$ and an inner twist $\psi_\eta\colon\mathbf{G}_\eta\to\mathbf{G}^*_\eta$. Since $\mathbf{G}^*_\eta$ is quasi-split, the maximal torus $\mathbf{T}^\natural$ of $\mathbf{G}_\eta$ transfers to an F-rational maximal torus $\mathbf{T}^{\natural*}$ of $\mathbf{G}^*_\eta$ (see, e.g., [Kal19b, Lemma 3.2.2]). More precisely, by composing a $\mathbf{G}^*_\eta$-conjugation with ψ_η if necessary, we may assume that $\psi_\eta|_{\mathbf{T}^\natural}\colon\mathbf{T}^\natural\to\mathbf{T}^{\natural*}:=\psi_\eta(\mathbf{T}^\natural)$ is an F-rational isomorphism. Then the inner twist ψ_η induces a Γ-equivariant isomorphism $\Omega_{\mathbf{G}_\eta}(\mathbf{T}^\natural)\cong\Omega_{\mathbf{G}^*_\eta}(\mathbf{T}^{\natural*})$.

Since we have ${}^{h'}\mathbf{T}^\flat=\bar{\mathbf{T}}^\flat$ and $h'\in\mathbf{H}_\epsilon$, the map $\sigma\mapsto[\sigma(h')^{-1}h']$ gives a 1-cocycle of Γ valued in $\Omega_{\mathbf{H}_\epsilon}(\mathbf{T}^\flat)$. Then, by the Γ-equivariant identifications of Weyl groups $\Omega_{\mathbf{H}_\epsilon}(\mathbf{T}^\flat)\cong\Omega_{\mathbf{G}_\eta}(\mathbf{T}^\natural)$ (see Remark 10.3) and $\Omega_{\mathbf{G}_\eta}(\mathbf{T}^\natural)\cong\Omega_{\mathbf{G}^*_\eta}(\mathbf{T}^{\natural*})$, we may regard $\sigma\mapsto[\sigma(h')^{-1}h']$ as a 1-cocycle of Γ valued in $\Omega_{\mathbf{G}^*_\eta}(\mathbf{T}^{\natural*})$. By applying [Kot82, Lemma 2.1] to $(\mathbf{T}^{\natural*},\mathbf{G}^*_\eta)$, we take an element $g^*\in\mathbf{G}^*_\eta$ such that $[\sigma(g^*)^{-1}g^*]=[\sigma(h')^{-1}h']$ (note that the quasi-splitness is necessary for this fact). We put $\bar{\mathbf{T}}^{\natural*}:={}^{g^*}\mathbf{T}^{\natural*}$. Then the map

$$\bar{\mathbf{T}}^{\natural*}\xrightarrow{[g^*]^{-1}}\mathbf{T}^{\natural*}\xrightarrow{\psi_\eta^{-1}}\mathbf{T}^\natural\subset\mathbf{T}^\diamondsuit\xrightarrow{\xi_D}\mathbf{T}^\flat\xrightarrow{[h']}\bar{\mathbf{T}}^\flat$$

is defined over F.

Note that the maximal torus $\bar{\mathbf{T}}^{\natural*}$ is elliptic in $\mathbf{G}^*_\eta$. Indeed, the above homomorphism $\bar{\mathbf{T}}^{\natural*}\to\bar{\mathbf{T}}^\flat$ is locally isomorphic (isomorphic at the Lie algebra level) since $\mathbf{T}^\natural$ is the identity component of the $[\eta]$-invariant of $\mathbf{T}^\diamondsuit$ and the map $\mathbf{T}^\diamondsuit\to\mathbf{T}^\flat$ induces an isomorphism between the $[\eta]$-coinvariant of $\mathbf{T}^\diamondsuit$ and $\mathbf{T}^\flat$. Thus, since $\bar{\mathbf{T}}^\flat$ is elliptic in $\mathbf{H}$ and the center of $\mathbf{H}$ is smaller than that of $\mathbf{G}^*_\eta$, $\bar{\mathbf{T}}^{\natural*}$ is elliptic in $\mathbf{G}^*_\eta$ (later, we will review a description of the relation between these centers; see Section 11.2). Therefore $\bar{\mathbf{T}}^{\natural*}$ transfers to $\mathbf{G}_\eta$ (see [Kot86, Section 10] or [Kal19b, Lemma 3.2.1]). In other words, there exists an element $g^{*\prime}\in\mathbf{G}^*_\eta$ such that $\psi_\eta^{-1}\circ[g^{*\prime}]\colon\bar{\mathbf{T}}^{\natural*}\to\bar{\mathbf{T}}^\natural:=\psi_\eta^{-1}\circ[g^{*\prime}](\bar{\mathbf{T}}^{\natural*})$ is an F-rational isomorphism. Note that then, by putting $g':=\psi_\eta^{-1}(g^{*\prime}g^*)\in\mathbf{G}_\eta$, we have $\bar{\mathbf{T}}^\natural={}^{g'}\mathbf{T}^\natural$.

We define $\bar{\mathbf{T}}^{\diamondsuit}$ by $\bar{\mathbf{T}}^{\diamondsuit} := \mathbf{Z}_{\mathbf{G}}(\bar{\mathbf{T}}^{\natural}) = {}^{g'}\mathbf{T}^{\diamondsuit}$. We put $\bar{\mathbf{B}}^{\diamondsuit} = {}^{g'}\mathbf{B}^{\diamondsuit}$. Let $\bar{D} := (\bar{\mathbf{B}}^{\flat}, \bar{\mathbf{T}}^{\flat}, \bar{\mathbf{B}}^{\diamondsuit}, \bar{\mathbf{T}}^{\diamondsuit})$. By construction, it can be easily seen that the map $\xi_{\bar{D}}$ determined by $\bar{D}$, which is given by $[h'] \circ \xi_D \circ [g']^{-1}$, is F-rational.

$$\begin{array}{ccc} \mathbf{T}^{\natural} & \xleftarrow{\psi_\eta^{-1}} & \mathbf{T}^{\natural *} \\ \downarrow{\scriptstyle [g']} & & \downarrow{\scriptstyle [g^*]} \\ \bar{\mathbf{T}}^{\natural} & \xleftarrow{\psi_\eta^{-1}\circ[g^{*\prime}]} & \bar{\mathbf{T}}^{\natural *} \end{array} \qquad \begin{array}{ccc} \mathbf{T}^{\flat} & \xleftarrow{\xi_D} & \mathbf{T}^{\diamondsuit} \\ {\scriptstyle [h']}\downarrow & & \downarrow{\scriptstyle [g']} \\ \bar{\mathbf{T}}^{\flat} & \xleftarrow[\xi_{\bar{D}}]{} & \bar{\mathbf{T}}^{\diamondsuit} \end{array}$$

Moreover, since $h' \in \mathbf{H}_\epsilon$ and $g' \in \mathbf{G}_\eta$, we have $\tilde{\xi}_{\bar{D}}(\eta) = \epsilon$. Thus $\bar{D}$ is a diagram associated to (ϵ, η) and $(\mathbf{H}_\epsilon, \mathbf{G}_\eta)$-conjugate to D. $\square$

10.3. **Kaletha's descent lemma.** Suppose that we are in the situation of Section 9. In particular, we have the sets $\tilde{\mathcal{J}}_G^{\mathbf{G}}$ and $\mathcal{J}_H^{\mathbf{H}}$ parametrizing the (θ-stable) members of our L-packets $\Pi_\phi^{\mathbf{G}}$ and $\Pi_{\phi_{\mathbf{H}}}^{\mathbf{H}}$.

Let $\jmath\colon \mathbf{T} \to \mathbf{S}$ and $\jmath_{\mathbf{H}}\colon \mathbf{T}_{\mathbf{H}} \to \mathbf{S}_{\mathbf{H}}$ be the duals to $\hat{\jmath}\colon \hat{\mathbf{S}} \to \hat{\mathbf{T}}$ and $\hat{\jmath}_{\mathbf{H}}\colon \hat{\mathbf{S}}_{\mathbf{H}} \to \hat{\mathbf{T}}_{\mathbf{H}}$, respectively. Since both $\mathbf{T}$ and $\mathbf{S}$ are F-rational, for any $\sigma \in \Gamma$, the map $a_{\jmath,\sigma} := \sigma(\jmath)^{-1} \circ \jmath$ is an automorphism of $\mathbf{T}$. Hence we get a 1-cocycle $a_\jmath\colon \Gamma \to \mathrm{Aut}(\mathbf{T})\colon \sigma \mapsto a_{\jmath,\sigma}$. We define a 1-cocycle $a_{\jmath_{\mathbf{H}}}\colon \Gamma \to \mathrm{Aut}(\mathbf{T}_{\mathbf{H}})$ in a similar way.

Recall that any $j \in \tilde{\mathcal{J}}_G^{\mathbf{G}}$ can be written as $j = [g] \circ \jmath^{-1}$ for some $g \in \mathbf{G}$. If we define a 1-cocycle $a_j\colon \Gamma \to \Omega_{\mathbf{G}}$ by $\sigma \mapsto a_{j,\sigma} := [\sigma(g)^{-1}g]$, then this does not depend on the choice of $g \in \mathbf{G}$. Similarly, for any $j_{\mathbf{H}} \in \mathcal{J}_H^{\mathbf{H}}$, we can define a 1-cocycle $a_{j_{\mathbf{H}}}\colon \Gamma \to \Omega_{\mathbf{H}}$

Lemma 10.9. *For any $j_{\mathbf{H}} \in \mathcal{J}_H^{\mathbf{H}}$ and $j \in \tilde{\mathcal{J}}_G^{\mathbf{G}}$, we have $a_j = a_\jmath = a_{\jmath_{\mathbf{H}}} = a_{j_{\mathbf{H}}}$. Here, we naturally identify $\Omega_{\mathbf{H}}$, $\Omega_{\mathbf{G}}$, and $\mathrm{Aut}(\mathbf{T}_{\mathbf{H}})$ as a subset of $\mathrm{Aut}(\mathbf{T})$ so that the equalities make sense.*

Proof. Let $g \in \mathbf{G}$ be an element satisfying $j = [g] \circ \jmath^{-1}$. As j is defined over F, for any $\sigma \in \Gamma$, we have $\sigma([g] \circ \jmath^{-1}) = [g] \circ \jmath^{-1}$, which implies that $a_{\jmath,\sigma} = a_{j,\sigma}$, hence $a_\jmath = a_j$. Similarly, we also have $a_{\jmath_{\mathbf{H}}} = a_{j_{\mathbf{H}}}$. Thus it is enough to show that $a_\jmath = a_{\jmath_{\mathbf{H}}}$. By construction, the map $\mathbf{S} \twoheadrightarrow \mathbf{S}_{\theta_{\mathbf{S}}} \cong \mathbf{S}_{\mathbf{H}}$ (say $\xi_{\mathbf{S}}$) is the dual to $\hat{\jmath}^{-1} \circ \hat{\xi} \circ \hat{\jmath}_{\mathbf{H}}$, hence given by $\jmath_{\mathbf{H}} \circ \xi \circ \jmath^{-1}$. Since $\xi_{\mathbf{S}}$ is F-rational, for any $\sigma \in \Gamma$, we have $\sigma(\jmath_{\mathbf{H}} \circ \xi \circ \jmath^{-1}) = \jmath_{\mathbf{H}} \circ \xi \circ \jmath^{-1}$, which implies the desired assertion (recall that the identification $\Omega_{\mathbf{H}} \subset \Omega_{\mathbf{G}}^\theta$ is given through ξ). $\square$

For a semisimple element $\eta \in \tilde{G}_{\mathrm{ss}}$, we define $\tilde{\mathcal{J}}_{\mathbf{G}_\eta}^{\mathbf{G}}$ to be the set

$$\tilde{\mathcal{J}}_{\mathbf{G}_\eta}^{\mathbf{G}} := \{j\colon \tilde{\mathbf{S}} \hookrightarrow \tilde{\mathbf{G}} \mid j \text{ is } F\text{-rational, } j \sim_{\mathbf{G}} \jmath^{-1}, \text{ and } \eta \in \tilde{S}_j\}/{\sim_{\mathbf{G}_\eta}},$$

i.e., the set of $\mathbf{G}_\eta$-conjugacy classes of F-rational $\hat{\jmath}$-admissible embeddings j of a twisted maximal torus satisfying $\eta \in \tilde{S}_j$.

Similarly, for a semisimple element $\epsilon \in H_{\mathrm{ss}}$, we define $\mathcal{J}_{\mathbf{H}_\epsilon}^{\mathbf{H}}$ to be the set

$$\mathcal{J}_{\mathbf{H}_\epsilon}^{\mathbf{H}} := \{j_{\mathbf{H}}\colon \mathbf{S}_{\mathbf{H}} \hookrightarrow \mathbf{H} \mid j_{\mathbf{H}} \text{ is } F\text{-rational, } j_{\mathbf{H}} \sim_{\mathbf{H}} \jmath_{\mathbf{H}}^{-1}, \text{ and } \epsilon \in S_{j_{\mathbf{H}}}\}/{\sim_{\mathbf{H}_\epsilon}},$$

i.e., the set of $\mathbf{H}_\epsilon$-conjugacy classes of F-rational $\hat{\jmath}_{\mathbf{H}}$-admissible embeddings of $\mathbf{S}_{\mathbf{H}}$ into $\mathbf{H}$ satisfying $\epsilon \in S_{j_{\mathbf{H}}}$ (or equivalently, $\mathbf{S}_{j_{\mathbf{H}}}$ is contained in $\mathbf{H}_\epsilon$). Here, to make the notation lighter, we write $\mathbf{S}_{j_{\mathbf{H}}} := \mathbf{S}_{\mathbf{H},j_{\mathbf{H}}} = j_{\mathbf{H}}(\mathbf{S}_{\mathbf{H}})$.

In the following, we fix a semisimple element $\eta \in \tilde{G}_{\mathrm{ss}}$. Let $\mathfrak{H}_\eta \subset H_{\mathrm{ss}}$ be a set of representatives for the stable conjugacy classes of semisimple elements of H corresponding to η such that $\mathbf{H}_y$ is quasi-split for any $y \in \mathfrak{H}_\eta$.

Now we define a map

$$\underline{\mathfrak{tran}}\colon \bigsqcup_{y\in\mathfrak{H}_\eta} \mathbb{D}(y,\eta)\times\mathcal{J}^{\mathbf{H}}_{\mathbf{H}_y} \to \tilde{\mathcal{J}}^{\mathbf{G}}_{\mathbf{G}_\eta}$$

in the following manner. Let $D=(\mathbf{B}^\flat,\mathbf{T}^\flat,\mathbf{B}^\diamondsuit,\mathbf{T}^\diamondsuit)\in\mathbb{D}(y,\eta)$ for $y\in\mathfrak{H}_\eta$ and $j_{\mathbf{H}}=[h]\circ\jmath_{\mathbf{H}}^{-1}\in\mathcal{J}^{\mathbf{H}}_{\mathbf{H}_y}$ ($h\in\mathbf{H}$). Since the torus $\mathbf{S}_{j_{\mathbf{H}}}$ is elliptic in $\mathbf{H}$, we may assume that $\mathbf{T}^\flat=\mathbf{S}_{j_{\mathbf{H}}}$ by replacing D with its equivalent diagram by Lemma 10.8. We take an element $h^\flat\in\mathbf{H}$ and $g^\diamondsuit\in\mathbf{G}$ such that $\xi_\flat=[h^\flat]$ and $\xi_\diamondsuit=[g^\diamondsuit]$, respectively. Then $n_{\mathbf{H}}:=h^\flat h\in\mathbf{H}$ belongs to $\mathbf{N}_{\mathbf{H}}(\mathbf{T}_{\mathbf{H}})$. We take an element $n\in\mathbf{G}^{\theta,\circ}$ such that $[n]\in\Omega^\theta_{\mathbf{G}}$ is equal to $[n_{\mathbf{H}}]\in\Omega_{\mathbf{H}}\subset\Omega^\theta_{\mathbf{G}}$ (we can take n from $\mathbf{G}^{\theta,\circ}$; see [KS99, Section 1.1]). We define an element $\underline{\mathfrak{tran}}(D,j_{\mathbf{H}})$ of $\tilde{\mathcal{J}}^{\mathbf{G}}_{\mathbf{G}_\eta}$ to be the following embedding $(j,\tilde{j})$ of $(\mathbf{S},\tilde{\mathbf{S}})$ into $(\mathbf{G},\tilde{\mathbf{G}})$:

$$j:=[g^\diamondsuit]^{-1}\circ[n]\circ\jmath^{-1},\quad \tilde{j}:=[g^\diamondsuit]^{-1}\circ[n]\circ\tilde{\jmath}^{-1},$$

where $\tilde{\jmath}^{-1}\colon\tilde{\mathbf{S}}\to\tilde{\mathbf{T}}$ is given by $s\theta_{\mathbf{S}}\mapsto\jmath^{-1}(s)\theta$ for any $s\in\mathbf{S}$.

$$\begin{array}{ccccccc}
\mathbf{S} & \xrightarrow{\jmath^{-1}} & \mathbf{T} & \xrightarrow{[g^\diamondsuit]^{-1}\circ[n]} & \mathbf{T}^\diamondsuit & \xrightarrow{[g^\diamondsuit]} & \mathbf{T}\\
{\scriptstyle\xi_{\mathbf{S}}}\downarrow & & {\scriptstyle\xi}\downarrow & & {\scriptstyle\xi_D}\downarrow & & {\scriptstyle\xi}\downarrow\\
\mathbf{S}_{\mathbf{H}} & \xrightarrow[\jmath_{\mathbf{H}}^{-1}]{} & \mathbf{T}_{\mathbf{H}} & \xrightarrow[{[h]}]{} & \mathbf{T}^\flat & \xrightarrow[{[h^\flat]}]{} & \mathbf{T}_{\mathbf{H}}
\end{array}$$

Proposition 10.10. *The above procedure gives a well-defined map. In other words,*

1. *$(j,\tilde{j})$ is an F-rational embedding of a twisted maximal torus,*
2. *j and $\jmath^{-1}$ are $\mathbf{G}$-conjugate,*
3. *$\eta\in\tilde{S}_j$, and*
4. *the $\mathbf{G}_\eta$-conjugacy class of j is independent of the choices of auxiliary data.*

Proof. The assertion (2) is obvious by construction.

Let us check that j is F-rational. For any $\sigma\in\Gamma$, we have $\sigma(j)=j$ if and only if $[\sigma(g^\diamondsuit)]^{-1}\circ[\sigma(n)]\circ\sigma(\jmath)^{-1}=[g^\diamondsuit]^{-1}\circ[n]\circ\jmath^{-1}$, or equivalently,

$$\sigma(\jmath)^{-1}\circ\jmath=[\sigma(n)]^{-1}\circ[\sigma(g^\diamondsuit)]\circ[g^\diamondsuit]^{-1}\circ[n]. \tag{15}$$

If we put $j'_{\mathbf{H}}:=[n_{\mathbf{H}}]\circ\jmath_{\mathbf{H}}^{-1}=[h^\flat]\circ j_{\mathbf{H}}$, then we have

$$\sigma(j'_{\mathbf{H}})\circ j'^{-1}_{\mathbf{H}}=[\sigma(n_{\mathbf{H}})]\circ\sigma(\jmath_{\mathbf{H}})^{-1}\circ\jmath_{\mathbf{H}}\circ[n_{\mathbf{H}}]^{-1}=[\sigma(h^\flat)]^{-1}\circ[h^\flat],$$

hence $\sigma(\jmath_{\mathbf{H}})^{-1}\circ\jmath_{\mathbf{H}}=[\sigma(n_{\mathbf{H}})]^{-1}\circ[\sigma(h^\flat)]^{-1}\circ[h^\flat]\circ[n_{\mathbf{H}}]$. Since we have

- $\sigma(\jmath)^{-1}\circ\jmath=a_{\jmath,\sigma}=a_{\jmath_{\mathbf{H}},\sigma}=\sigma(\jmath_{\mathbf{H}})^{-1}\circ\jmath_{\mathbf{H}}$ (Lemma 10.9),
- $[n]^{-1}=[n_{\mathbf{H}}]^{-1}$ and $[\sigma(n)]^{-1}=[\sigma(n_{\mathbf{H}})]^{-1}$, and
- $[\sigma(g^\diamondsuit)]\circ[g^\diamondsuit]^{-1}=[\sigma(h^\flat)]\circ[h^\flat]^{-1}$ (by the F-rationality of ξ_D),

(all the equalities are considered in $\Omega_{\mathbf{H}}\subset\Omega^\theta_{\mathbf{G}}$), we get the equality (15).

By noting that n is θ-invariant and D is a diagram associated to (y,η), we see that $\tilde{\mathbf{S}}_j=\tilde{j}(\tilde{\mathbf{S}})$ contains $\eta\in\tilde{G}$. Combined with the F-rationality of j, this shows that $\tilde{\mathbf{S}}_j$ is F-rational and $(\mathbf{S}_j,\tilde{\mathbf{S}}_j)$ is an F-rational twisted maximal torus of $(\mathbf{G},\tilde{\mathbf{G}})$. Hence we get the assertions (1) and also (3).

We consider (4). As long as D and $\jmath_{\mathbf{H}}$ are fixed, the embedding $(j,\tilde{j})$ is obviously independent of the choices of $n_{\mathbf{H}}$, n, $h^\flat$, and $g^\diamondsuit$. Moreover, it is also easy to see that $(j,\tilde{j})$ does not change even if we replace D with a $\Omega_{\mathbf{H}}(\mathbf{T}^\flat)$-equivalent diagram. Thus our task is to show that, if we take

- another embedding $\bar{\jmath}_{\mathbf{H}} \in \mathcal{J}^{\mathbf{H}}_{\mathbf{H}_y}$ which is $\mathbf{H}_y$-conjugate to $\jmath_{\mathbf{H}}$ and
- another diagram $\bar{D} = (\bar{\mathbf{B}}^\flat, \bar{\mathbf{T}}^\flat, \bar{\mathbf{B}}^\diamondsuit, \bar{\mathbf{T}}^\diamondsuit) \in \mathbf{D}(y,\eta)$ which is $(\mathbf{H}_y, \mathbf{G}_\eta)$-conjugate to D and satisfies $\bar{\mathbf{T}}^\flat = \mathbf{S}_{\bar{\jmath}_{\mathbf{H}}}$,

then j and $\bar{j}$ (which is constructed from $\bar{D}$ and $\bar{\jmath}_{\mathbf{H}}$) are $\mathbf{G}_\eta$-conjugate.

We take $h_y \in \mathbf{H}_y$ such that $\bar{\jmath}_{\mathbf{H}} = [h_y] \circ \jmath_{\mathbf{H}}$ (hence $\bar{\jmath}_{\mathbf{H}} = [\bar{h}] \circ \jmath_{\mathbf{H}}^{-1}$, where $\bar{h} = h_y h$). Let $h' \in \mathbf{H}_y$ and $g' \in \mathbf{G}_\eta$ be elements such that

$$({}^{h'}\bar{\mathbf{B}}^\flat, {}^{h'}\bar{\mathbf{T}}^\flat, {}^{g'}\bar{\mathbf{B}}^\diamondsuit, {}^{g'}\bar{\mathbf{T}}^\diamondsuit) = (\mathbf{B}^\flat, \mathbf{T}^\flat, \mathbf{B}^\diamondsuit, \mathbf{T}^\diamondsuit).$$

Then the element $\bar{h}^\flat := h^\flat h' \in \mathbf{H}$ satisfies $\bar{\xi}_\flat = [\bar{h}^\flat]$. Similarly, the element $\bar{g}^\diamondsuit := g^\diamondsuit g' \in \mathbf{G}$ satisfies $\bar{\xi}_\diamondsuit = [\bar{g}^\diamondsuit]$. We take an element $\bar{n} \in \mathbf{G}^{\theta,\circ}$ such that $[\bar{n}] \in \Omega^\theta_{\mathbf{G}}$ is equal to $[\bar{n}_{\mathbf{H}}] \in \Omega_{\mathbf{H}} \subset \Omega^\theta_{\mathbf{G}}$, where $\bar{n}_{\mathbf{H}} := \bar{h}^\flat \bar{h} \in \mathbf{N}_{\mathbf{H}}(\mathbf{T}_{\mathbf{H}})$. Then, by construction, $\bar{j}$ is given by $[\bar{g}^{\diamondsuit -1}] \circ [\bar{n}] \circ \jmath^{-1}$.

In the following, we simply write ν and μ for ν_D and μ_D associated to D, respectively (see Definition 10.1). As we have $\bar{g}^\diamondsuit := g^\diamondsuit g'$ and $g' \in \mathbf{G}_\eta$, $\bar{j}$ is $\mathbf{G}_\eta$-conjugate to $[g^\diamondsuit]^{-1} \circ [\bar{n}] \circ \jmath^{-1}$. Since $j = [g^\diamondsuit]^{-1} \circ [n] \circ \jmath^{-1}$, it suffices to show that $[g^\diamondsuit]^{-1} \circ [\bar{n}] \circ [n]^{-1} \circ [g^\diamondsuit] \in \mathrm{Aut}(\mathbf{T}^\diamondsuit)$ is realized by an element of $\Omega_{\mathbf{G}_\eta}(\mathbf{T}^\natural) \subset \Omega_{\mathbf{G}}(\mathbf{T}^\diamondsuit)$. Since $\xi_\diamondsuit = [g^\diamondsuit]$ induces an identification $\Omega_{\mathbf{G}_\eta}(\mathbf{T}^\natural) \cong \Omega_{\mathbf{G}_{\nu\theta}}(\mathbf{T}^{\theta,\circ})$, it is equivalent to showing that $[\bar{n}] \circ [n]^{-1} \in \mathrm{Aut}(\mathbf{T})$ is realized by an element of $\Omega_{\mathbf{G}_{\nu\theta}}(\mathbf{T}^{\theta,\circ}) \subset \Omega_{\mathbf{G}}(\mathbf{T})$. By noting that $\Omega_{\mathbf{H}_\mu}(\mathbf{T}_{\mathbf{H}})$ is identified with a subgroup of $\Omega_{\mathbf{G}_{\nu\theta}}(\mathbf{T}^{\theta,\circ})$ (both regarded as subgroups of $\Omega_{\mathbf{G}}(\mathbf{T})$), let us show a slightly stronger statement that $[\bar{n}] \circ [n]^{-1} \in \mathrm{Aut}(\mathbf{T})$ is realized by an element of $\Omega_{\mathbf{H}_\mu}(\mathbf{T}_{\mathbf{H}})$. By construction, we have $[n] = [n_{\mathbf{H}}] = [h^\flat h]$ and $[\bar{n}] = [\bar{n}_{\mathbf{H}}] = [\bar{h}^\flat \bar{h}] = [h^\flat h' h_y h]$. Thus we get $[\bar{n}] \circ [n]^{-1} = [h^\flat] \circ [h' h_y] \circ [h^\flat]^{-1}$. Since $\xi_\flat = [h^\flat]$ induces an identification $\Omega_{\mathbf{H}_y}(\mathbf{T}^\flat) \cong \Omega_{\mathbf{H}_\mu}(\mathbf{T}_{\mathbf{H}})$ and $[h' h_y]$ belongs to $\Omega_{\mathbf{H}_y}(\mathbf{T}^\flat)$, we get the assertion. □

The following is the twisted version of Kaletha's "descent lemma" [Kal15, Lemma 6.5]:

Proposition 10.11. *For each $y \in \mathfrak{H}_\eta$, the restriction of $\underline{\mathfrak{tran}}$ to $\mathbb{D}(y,\eta) \times \mathcal{J}^{\mathbf{H}}_{\mathbf{H}_y}$ is a $\pi_0(\mathbf{H}^y)(F)$-torsor onto its image. Furthermore, $\underline{\mathfrak{tran}}$ induces a bijective map*

$$\mathfrak{tran}\colon \bigsqcup_{y \in \mathfrak{H}_\eta} \bigl(\mathbb{D}(y,\eta) \times \mathcal{J}^{\mathbf{H}}_{\mathbf{H}_y}\bigr)/\pi_0(\mathbf{H}^y)(F) \to \tilde{\mathcal{J}}^{\mathbf{G}}_{\mathbf{G}_\eta}.$$

Proof. We first show the surjectivity. Suppose that an element $j = [g] \circ \jmath^{-1}$ of $\tilde{\mathcal{J}}^{\mathbf{G}}_{\mathbf{G}_\eta}$ is given, where $g \in \mathbf{G}$. We take an(y) element $\jmath_{\mathbf{H}} = [h] \circ \jmath_{\mathbf{H}}^{-1}$ of $\mathcal{J}^{\mathbf{H}}_{H}$, where $h \in \mathbf{H}$. We put $\mathbf{T}^\diamondsuit := \mathbf{S}_j = {}^g\mathbf{T}$ and $\mathbf{B}^\diamondsuit := {}^g\mathbf{B}$. Then, by putting $[g]^{-1}(\eta) = \nu\theta \in \tilde{\mathbf{T}}$, $\mu := \xi(\nu)$, and $\epsilon := [h](\mu)$, we can check that $\epsilon \in H_{\mathrm{ss}}$ and that $D' := ({}^h\mathbf{B}_{\mathbf{H}}, {}^h\mathbf{T}_{\mathbf{H}}, \mathbf{B}^\diamondsuit, \mathbf{T}^\diamondsuit)$ is a diagram associated to (ϵ, η) (note that $\mathbf{S}_{\jmath_{\mathbf{H}}} = {}^h\mathbf{T}_{\mathbf{H}}$ and use Lemma 10.9 to check the F-rationality of $\xi_{D'}$). By the definition of the set $\mathfrak{H}_\eta$, there exists a unique element $y \in \mathfrak{H}_\eta$ which is stably $\mathbf{H}$-conjugate to ϵ. Since $\mathbf{H}_y$ is the quasi-split inner form of $\mathbf{H}_\epsilon$, the maximal torus ${}^h\mathbf{T}_{\mathbf{H}}$ of $\mathbf{H}_\epsilon$ transfers to $\mathbf{H}_y$ (see, e.g., [Kal19b, Lemma 3.2.2]). More precisely, we can find an element $h' \in \mathbf{H}$ such that $[h'](\epsilon) = y$ and $[h']$ gives an F-rational isomorphism from ${}^h\mathbf{T}_{\mathbf{H}}$ to ${}^{h'h}\mathbf{T}_{\mathbf{H}}$. Hence, by putting $\mathbf{T}^\flat := {}^{h'h}\mathbf{T}_{\mathbf{H}}$ and $\mathbf{B}^\flat := {}^{h'h}\mathbf{B}_{\mathbf{H}}$, we get a diagram $D := (\mathbf{B}^\flat, \mathbf{T}^\flat, \mathbf{B}^\diamondsuit, \mathbf{T}^\diamondsuit)$ associated to (y,η). If we put $\jmath'_{\mathbf{H}} := [h'] \circ \jmath_{\mathbf{H}}$, then $\jmath'_{\mathbf{H}}$ belongs to $\mathcal{J}^{\mathbf{H}}_{\mathbf{H}_y}$. Furthermore, by going back to the construction of the map $\underline{\mathfrak{tran}}$, we can easily check that $\underline{\mathfrak{tran}}(D, \jmath'_{\mathbf{H}}) = j$.

We next investigate the fibers of $\underline{\mathfrak{tran}}$. For this, let us take two diagrams $D = (\mathbf{B}^\flat, \mathbf{T}^\flat, \mathbf{B}^\diamondsuit, \mathbf{T}^\diamondsuit) \in \mathbf{D}(y,\eta)$, $\bar{D} = (\bar{\mathbf{B}}^\flat, \bar{\mathbf{T}}^\flat, \bar{\mathbf{B}}^\diamondsuit, \bar{\mathbf{T}}^\diamondsuit) \in \mathbf{D}(\bar{y},\eta)$ for $y, \bar{y} \in \mathfrak{H}_\eta$ and two embeddings $j_{\mathbf{H}} \in \mathcal{J}^{\mathbf{H}}_{\mathbf{H}_y}$, $\bar{j}_{\mathbf{H}} \in \mathcal{J}^{\mathbf{H}}_{\mathbf{H}_{\bar{y}}}$ satisfying $\underline{\mathfrak{tran}}(D, j_{\mathbf{H}}) = \underline{\mathfrak{tran}}(\bar{D}, \bar{j}_{\mathbf{H}})$. We may suppose that $\mathbf{T}^\flat = \mathbf{S}_{j_{\mathbf{H}}}$ and $\bar{\mathbf{T}}^\flat = \mathbf{S}_{\bar{j}_{\mathbf{H}}}$ by Lemma 10.8. Let $h \in \mathbf{H}$ and $\bar{h} \in \mathbf{H}$ be elements satisfying $j_{\mathbf{H}} = [h] \circ \jmath^{-1}_{\mathbf{H}}$ and $\bar{j}_{\mathbf{H}} = [\bar{h}] \circ \jmath^{-1}_{\mathbf{H}}$, respectively. By replacing $\bar{D}$ with its $\mathbf{G}_\eta$-equivalent diagram if necessary, we may suppose that $(D, j_{\mathbf{H}})$ and $(\bar{D}, \bar{j}_{\mathbf{H}})$ produce exactly the same embedding j. (Note that then $\mathbf{T}^\diamondsuit = \mathbf{S}_j = \bar{\mathbf{T}}^\diamondsuit$.) We take $h^\flat \in \mathbf{H}$, $g^\diamondsuit \in \mathbf{G}$, and $n \in \mathbf{G}^{\theta,\circ}$ (which corresponds to $n_{\mathbf{H}} := h^\flat h$) for D as in the definition of $\underline{\mathfrak{tran}}$. Similarly, for $\bar{D}$, we take $\bar{h}^\flat \in \mathbf{H}$, $\bar{g}^\diamondsuit \in \mathbf{G}$, and $\bar{n} \in \mathbf{G}^{\theta,\circ}$ (which corresponds to $\bar{n}_{\mathbf{H}} := \bar{h}^\flat \bar{h}$) for $\bar{D}$ as in the definition of $\underline{\mathfrak{tran}}$. Then we have $[g^\diamondsuit]^{-1} \circ [n] \circ \jmath^{-1} = [\bar{g}^\diamondsuit]^{-1} \circ [\bar{n}] \circ \jmath^{-1}$.

Thus we have $[n\bar{n}^{-1}] = [g^\diamondsuit \bar{g}^{\diamondsuit -1}]$, which is an equality as elements of the Weyl group $\Omega_{\mathbf{H}} \subset \Omega_{\mathbf{G}}$. We write w for this element. Recall that $[\bar{h}^\flat]$ and $[\bar{g}^\diamondsuit]$ induce an identifications $\Omega_{\mathbf{H}}(\bar{\mathbf{T}}^\flat) \cong \Omega_{\mathbf{H}}$ and $\Omega_{\mathbf{G}}(\bar{\mathbf{T}}^\diamondsuit) \cong \Omega_{\mathbf{G}}$. If we put $w^\flat \in \Omega_{\mathbf{H}}(\bar{\mathbf{T}}^\flat)$ and $w^\diamondsuit \in \Omega_{\mathbf{G}}(\bar{\mathbf{T}}^\diamondsuit)$ to be the images of $w \in \Omega_{\mathbf{H}}$ under these identifications, respectively, then $w^\flat$ and $w^\diamondsuit$ are identified through $\xi_{\bar{D}}$ (see Remark 10.3). By replacing the diagram $\bar{D}$ with its $\Omega_{\mathbf{H}}(\bar{\mathbf{T}}^\flat)$-equivalent diagram $({}^{w^\flat}\bar{\mathbf{B}}^\flat, \bar{\mathbf{T}}^\flat, {}^{w^\diamondsuit}\bar{\mathbf{B}}^\diamondsuit, \bar{\mathbf{T}}^\diamondsuit)$, we may assume that $(\bar{\mathbf{B}}^\diamondsuit, \bar{\mathbf{T}}^\diamondsuit) = (\mathbf{B}^\diamondsuit, \mathbf{T}^\diamondsuit)$. Note that then we have $g^\diamondsuit = \bar{g}^\diamondsuit$ and $[n] = [\bar{n}]$.

Recall that $\nu_D \in \mathbf{T}$ (resp. $\nu_{\bar{D}} \in \mathbf{T}$) is the element such that $[g^\diamondsuit](\eta) = \nu_D\theta$ (resp. $[\bar{g}^\diamondsuit](\eta) = \nu_{\bar{D}}\theta$), hence we have $\nu_{\bar{D}} = \nu_D$. This implies that $\mu_{\bar{D}} = \mu_D$. As we have $[h^\flat](y) = \mu_D$ and $[\bar{h}^\flat](\bar{y}) = \mu_{\bar{D}}$, we get $[h^{\flat -1}\bar{h}^\flat](\bar{y}) = y$. Note that the equality $[n] = [\bar{n}]$ is equivalent to the equality $[h^{\flat -1}\bar{h}^\flat] = [h\bar{h}^{-1}]$. Since $[h\bar{h}^{-1}] = j_{\mathbf{H}} \circ \bar{j}^{-1}_{\mathbf{H}}$ gives an F-rational isomorphism from $\bar{\mathbf{T}}^\flat$ to $\mathbf{T}^\flat$ (i.e., stable conjugacy between $\bar{\mathbf{T}}^\flat$ and $\mathbf{T}^\flat$), this implies that y and $\bar{y}$ are stably conjugate. Thus the definition of the set $\mathfrak{H}_\eta$ implies that $y = \bar{y}$. We also get $h\bar{h}^{-1} \in \mathbf{H}^y$. Therefore, by putting $h_y := h\bar{h}^{-1} \in \mathbf{H}^y$, we get $(\mathbf{B}^\flat, \mathbf{T}^\flat, \mathbf{B}^\diamondsuit, \mathbf{T}^\diamondsuit) = ({}^{h_y}\bar{\mathbf{B}}^\flat, {}^{h_y}\bar{\mathbf{T}}^\flat, \bar{\mathbf{B}}^\diamondsuit, \bar{\mathbf{T}}^\diamondsuit)$ and $j_{\mathbf{H}} = [h_y] \circ \bar{j}_{\mathbf{H}}$.

Thus the remaining task is to show that, by replacing $\bar{D}$ with its $\mathbf{H}_y$-equivalent one if necessary, we can take h_y to be F-rational. This follows from [Kal15, Lemma 6.3] (cf. the proof of the descent lemma in the untwisted case; [Kal15, Lemma 6.5]). ☐

Remark 10.12. Note that $\mathcal{J}^{\mathbf{H}}_{\mathbf{H}_y}$ is not empty for any $y \in \mathfrak{H}_\eta$. Hence, in particular, Proposition 10.11 implies the following: $\tilde{\mathcal{J}}^{\mathbf{G}}_{\mathbf{G}_\eta}$ is empty if and only if $\mathbb{D}(y, \eta)$ is empty for any $y \in \mathfrak{H}_\eta$.

11. Waldspurger's descent theorems on twisted endoscopy

In this section, we review a part of Waldspurger's framework "l'endoscopie tordue n'est pas si tordue".

Note that, in the following of this paper, we need to require that our exponential map is invariant under conjugation. However, this property might not be satisfied by a mock exponential map in the sense of [AS09, Appendix A]. So, from now on, we furthermore assume that

$$p \geq (2 + e_F)n,$$

where e_F is the ramification index of $F/\mathbb{Q}_p$ and n is the minimum dimension of a faithful representation of $\mathbf{G}$. It is known that the "traditional" exponential map converges on the topologically nilpotent loci under this assumption, thus we can choose it as our exponential map (see [DR09, Appendix B] and also [Wal08, Appendice B]).

11.1. **Non-standard endoscopy.** Let us start with recalling the formalism of non-standard endoscopy following [Wal08, Sections 1.7, 1.8].

Let $\mathbf{G}_1$ and $\mathbf{G}_2$ be quasi-split semisimple simply-connected groups over F. For each $\mathbf{G}_i$, we fix a Borel pair $(\mathbf{B}_i, \mathbf{T}_i)$ defined over F. Let Ω_i denote the Weyl group of $\mathbf{T}_i$ in $\mathbf{G}_i$. We write Φ_i and $\Phi_i^\vee$ for the set of roots and coroots of $\mathbf{T}_i$ in $\mathbf{G}_i$, respectively. Suppose that we have an isomorphism $j_*\colon X_*(\mathbf{T}_1)_\mathbb{Q} \xrightarrow{\sim} X_*(\mathbf{T}_2)_\mathbb{Q}$. Let $j^*\colon X^*(\mathbf{T}_2)_\mathbb{Q} \xrightarrow{\sim} X^*(\mathbf{T}_1)_\mathbb{Q}$ denote the dual to j_*.

Then the triple $(\mathbf{G}_1, \mathbf{G}_2, j_*)$ is called a *non-standard endoscopic triple* if the following conditions are satisfied:

(1) There exist bijections $\tau^\vee\colon \Phi_1^\vee \xrightarrow{\sim} \Phi_2^\vee$ and $\tau\colon \Phi_2 \xrightarrow{\sim} \Phi_1$ and functions $b^\vee\colon \Phi_1^\vee \to \mathbb{Q}_{>0}$ and $b\colon \Phi_2 \to \mathbb{Q}_{>0}$ such that
  (a) $\alpha_2^\vee = \tau^\vee(\tau(\alpha_2)^\vee)$ for any $\alpha_2 \in \Phi_2$;
  (b) we have $j_*(\alpha_1^\vee) = b^\vee(\alpha_1^\vee)\cdot\tau^\vee(\alpha_1^\vee)$ for any $\alpha_1^\vee \in \Phi_1^\vee$ and $j^*(\alpha_2) = b(\alpha_2)\cdot\tau(\alpha_2)$ for any $\alpha_2 \in \Phi_2$.
(2) The isomorphisms j_* and j^* are Γ-equivariant.

For a non-standard endoscopic triple $(\mathbf{G}_1, \mathbf{G}_2, j^*)$, the isomorphism j_* induces an isomorphism between the Lie algebras $\mathfrak{t}_1 := \operatorname{Lie}\mathbf{T}_1$ and $\mathfrak{t}_2 := \operatorname{Lie}\mathbf{T}_2$:

$$\mathfrak{t}_1 \cong X_*(\mathbf{T}_1)\otimes_{\overline{F}} \xrightarrow{j_*} X_*(\mathbf{T}_2)\otimes_{\overline{F}} \cong \mathfrak{t}_2,$$

which induces a bijection

$$(\mathfrak{t}_1/\Omega_1)^\Gamma \cong (\mathfrak{t}_2/\Omega_2)^\Gamma.$$

Thus, through this bijection, we can define a bijective correspondence between the sets of stable conjugacy classes of semisimple elements of $\mathfrak{g}_1$ and $\mathfrak{g}_2$, which preserves the regular semisimplicity.

11.2. **Decomposition of twisted endoscopy.** We next briefly review Waldspurger's decomposition result on twisted endoscopy established in [Wal08, Section 3].

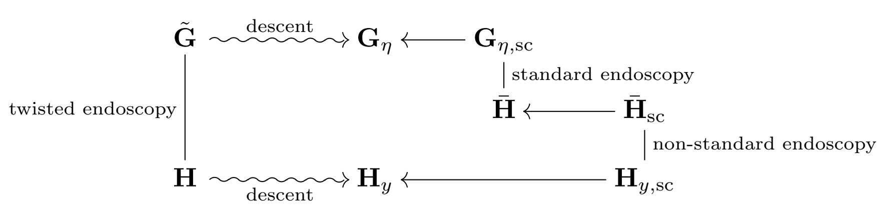
Let $(y, \eta) \in H_{\mathrm{ss}} \times \tilde{G}_{\mathrm{ss}}$. In the following, we assume that

- the connected centralizer $\mathbf{H}_y$ of y in $\mathbf{H}$ is quasi-split, and
- the set $\mathbf{D}(y, \eta)$ of diagrams associated to (y, η) is not empty.

We fix a diagram $D = (\mathbf{B}^\flat, \mathbf{T}^\flat, \mathbf{B}^\diamondsuit, \mathbf{T}^\diamondsuit) \in \mathbf{D}(y, \eta)$. In [Wal08, Sections 3.5 and 3.6], Waldspurger constructed a quasi-split connected reductive group $\bar{\mathbf{H}}$ over F equipped with

- standard endoscopic data $(\bar{\mathbf{H}}, \bar{\mathcal{H}}, \bar{s}, \hat{\bar{\xi}})$ of $\mathbf{G}_{\eta,\mathrm{sc}}$, and
- a non-standard endoscopic triple $(\mathbf{H}_{y,\mathrm{sc}}, \bar{\mathbf{H}}_{\mathrm{sc}}, j_*)$,

where the subscript "sc" denotes the simply-connected cover of the derived subgroup. Here we emphasize that the construction of these objects depends on the choice of $D \in \mathbf{D}(y, \eta)$.



Let us review how the stable conjugacy classes correspond under this picture ([Wal08, Section 3.8]). We first consider the decompositions of the Lie algebras

$$\begin{aligned}\mathfrak{g}_\eta &= \mathfrak{g}_{\eta,\mathrm{sc}} \oplus \mathfrak{z}_{\mathbf{G}_\eta} = \operatorname{Lie}\mathbf{G}_{\eta,\mathrm{sc}} \oplus \operatorname{Lie}\mathbf{Z}_{\mathbf{G}_\eta},\\ \bar{\mathfrak{h}} &= \bar{\mathfrak{h}}_{\mathrm{sc}} \oplus \mathfrak{z}_{\bar{\mathbf{H}}} = \operatorname{Lie}\bar{\mathbf{H}}_{\mathrm{sc}} \oplus \operatorname{Lie}\mathbf{Z}_{\bar{\mathbf{H}}},\\ \mathfrak{h}_y &= \mathfrak{h}_{y,\mathrm{sc}} \oplus \mathfrak{z}_{\mathbf{H}_y} = \operatorname{Lie}\mathbf{H}_{y,\mathrm{sc}} \oplus \operatorname{Lie}\mathbf{Z}_{\mathbf{H}_y}.\end{aligned}$$

For any $X \in \mathfrak{g}_\eta$, $\bar{Y} \in \bar{\mathfrak{h}}$, and $Y \in \mathfrak{h}_y$, we write $X = X_{\mathrm{sc}} + X_Z$, $\bar{Y} = \bar{Y}_{\mathrm{sc}} + \bar{Y}_Z$, and $Y = Y_{\mathrm{sc}} + Y_Z$ for their decompositions according to the above direct sum decompositions, respectively. We note that we have an F-rational isomorphism $\mathfrak{z}_{\mathbf{H}_y} \cong \mathfrak{z}_{\bar{\mathbf{H}}} \oplus \mathfrak{z}_{\mathbf{G}_\eta}$ (see [Wal08, Section 3.8]).

For our convenience, let us introduce the following terminology:

Definition 11.1. . We say that $(Y, X) \in \mathfrak{h}_{y,0+} \times \mathfrak{g}_{\eta,0+}$ is a *D-norm pair* if

- $\eta \exp(X) \in \tilde{G}$ is strongly regular semisimple,
- $y \exp(Y) \in H$ is strongly $\tilde{\mathbf{G}}$-regular semisimple,

and there exists an element $\bar{Y} \in \bar{\mathfrak{h}}$ satisfying the following:

- $\bar{Y} \in \bar{\mathfrak{h}}$ is a norm of $X_{\mathrm{sc}} \in \mathfrak{g}_{\eta,\mathrm{sc}}$ in the sense of standard endoscopy,
- the stable conjugacy classes of $\bar{Y}_{\mathrm{sc}} \in \bar{\mathfrak{h}}_{\mathrm{sc}}$ and $Y_{\mathrm{sc}} \in \mathfrak{h}_{y,\mathrm{sc}}$ correspond in the sense of non-standard endoscopy (see Section 11.1),
- $Y_Z \in \mathfrak{z}_{\mathbf{H}_y}$ corresponds to $\bar{Y}_Z + X_Z \in \mathfrak{z}_{\bar{\mathbf{H}}} \oplus \mathfrak{z}_{\mathbf{G}_\eta}$ under the identification $\mathfrak{z}_{\mathbf{H}_y} \cong \mathfrak{z}_{\bar{\mathbf{H}}} \oplus \mathfrak{z}_{\mathbf{G}_\eta}$.

The following is a part of [Wal08, Section 3.8, Lemme]:

Proposition 11.2. *For any D-norm pair (Y, X), $(y \exp(Y), \eta \exp(X)) \in \mathcal{D}$.*

11.3. **Descent of transfer factor.** We write Δ^D for the (absolute or relative) Lie algebra transfer factor for the pair $(\bar{\mathbf{H}}, \mathbf{G}_{\eta,\mathrm{sc}})$. Note that we put the symbol D on the exponent in order to emphasize that the endoscopic structure of $(\bar{\mathbf{H}}, \mathbf{G}_{\eta,\mathrm{sc}})$ depends on the choice of a diagram $D \in \mathbf{D}(y, \eta)$.

Theorem 11.3 ([Wal08, Section 3.9, Théorème])**.** *There exists a neighborhood $\mathfrak{V}$ of 0 in $\mathfrak{h}_{y,0+}$ such that, for any D-norm pairs $(Y, X), (\underline{Y}, \underline{X}) \in \mathfrak{V} \times \mathfrak{g}_{\eta,0+}$, we have*

$$\Delta\bigl(y \exp(Y), \eta \exp(X); y \exp(\underline{Y}), \eta \exp(\underline{X})\bigr) = \Delta^D(\bar{Y}, X_{\mathrm{sc}}; \underline{\bar{Y}}, \underline{X}_{\mathrm{sc}}),$$

where $\bar{Y}$ and $\underline{\bar{Y}}$ are the elements of $\bar{\mathfrak{h}}$ associated to (Y, X) and $(\underline{Y}, \underline{X})$ as in Definition 11.1, respectively.

Corollary 11.4. *The absolute Lie algebra transfer factor $\Delta^D(-,-)$ can be normalized so that there exists a neighborhood $\mathfrak{V}$ of 0 in $\mathfrak{h}_{y,0+}$ such that, for any D-norm pair $(Y, X) \in \mathfrak{V} \times \mathfrak{g}_{\eta,0+}$, we have*

$$\Delta\bigl(y \exp(Y), \eta \exp(X)\bigr) = \Delta^D(\bar{Y}, X_{\mathrm{sc}}).$$

11.4. **Transfer of Fourier transforms of orbital integrals.** In this section, we summarize the results on the transfer of the Fourier transforms of orbital integrals on Lie algebras, which were established by Waldspurger and Ngô.

For any connected reductive group $\mathbf{J}$ over F equipped with an invariant symmetric non-degenerate bilinear form $B_{\mathfrak{j}}$ on $\mathfrak{j} = \operatorname{Lie}\mathbf{J}(F)$, we let $\gamma(\mathfrak{j})$ denote the Weil constant of $(\mathfrak{j}, B_{\mathfrak{j}})$ with respect to the fixed non-trivial additive character ψ_F of F (see [Wal97, Section 3.1]). We note that hence $\gamma(\mathfrak{j})$ depends on the choices of $B_{\mathfrak{j}}$ and

ψ_F although the notation does not contain these symbols. For regular semisimple elements $X \in \mathfrak{j}$ and $X^* \in \mathfrak{j}$, we put

$$D^{\mathbf{J}}_{X^*}(X) := \gamma(\mathfrak{j}) \cdot \hat{\iota}^{\mathbf{J}}_{X^*}(X).$$

Here, $\hat{\iota}^{\mathbf{J}}_{X^*}(X)$ is the normalized Fourier transform of the orbital integral of X^* (see Section 6.7; note that X^* is regarded as an element of $\mathfrak{j}^*$ via $B_{\mathfrak{j}}$). We also put

$$D^{\mathbf{J},\mathrm{st}}_{X^*}(X) := \sum_{X^{*\prime} \sim_{\mathbf{J}} X^*/\sim_J} D^{\mathbf{J}}_{X^{*\prime}}(X),$$

where the index set is over the J-conjugacy classes within the stable conjugacy class of X^* in $\mathfrak{j}$.

11.4.1. *The case of standard endoscopy.* Let $\mathbf{J}$ be a connected reductive group over F and $\mathbf{J}'$ a standard endoscopic group of $\mathbf{J}$. We fix an invariant symmetric non-degenerate bilinear form $B_{\mathfrak{j}}$ on $\mathfrak{j}$. Then $B_{\mathfrak{j}}$ induces an invariant symmetric non-degenerate bilinear form on $\mathfrak{j}'$ (see [Wal95, Section VIII.6]). Let us write $B_{\mathfrak{j}'}$ for this bilinear form. We remark that these bilinear forms satisfy the following consistency property on the maximal tori. Let $\mathbf{T}_{\mathbf{J}}$ and $\mathbf{T}_{\mathbf{J}'}$ be maximal tori of $\mathbf{J}$ and $\mathbf{J}'$ belonging to the (implicitly fixed) pinnings of $\mathbf{J}$ and $\mathbf{J}'$, respectively. Then the endoscopic structure of $\mathbf{J}'$ in $\mathbf{J}$ gives an isomorphism $\xi_{\mathbf{J}} \colon \mathbf{T}_{\mathbf{J}} \cong \mathbf{T}_{\mathbf{J}'}$, which induces an isomorphism $\xi_{\mathbf{J}} \colon \mathfrak{t}_{\mathbf{J}} \cong \mathfrak{t}_{\mathbf{J}'}$ on the Lie algebras. With these notation, for any $X, X' \in \mathfrak{t}_{\mathbf{J}}$, we have $B_{\mathfrak{j}}(X, X') = B_{\mathfrak{j}'}(\xi_{\mathbf{J}}(X), \xi_{\mathbf{J}}(X'))$.

For a strongly $\mathbf{J}$-regular semisimple element $Y^* \in \mathfrak{j}'$ and a strongly regular semisimple element $X \in \mathfrak{j}$, we put

$$D_{\mathbf{J}',\mathbf{J}}(Y^*, X) := \sum_{X^* \leftrightarrow Y^*/\sim_J} \mathring{\Delta}_{\mathbf{J}',\mathbf{J}}(Y^*, X^*) D^{\mathbf{J}}_{X^*}(X),$$

where the index set is over the J-conjugacy classes of strongly regular semisimple elements of $\mathfrak{j}$ which correspond to Y^*, and $\mathring{\Delta}_{\mathbf{J}',\mathbf{J}}(Y^*, X^*)$ denotes the Lie algebra transfer factor without the fourth factor. We also put

$$\tilde{D}_{\mathbf{J}',\mathbf{J}}(Y^*, X) := \sum_{Y' \leftrightarrow X/\sim_{\mathbf{J}'}} \mathring{\Delta}_{\mathbf{J}',\mathbf{J}}(Y', X) D^{\mathbf{J}',\mathrm{st}}_{Y^*}(Y'),$$

where the index set is over the stable conjugacy classes of the elements of $\mathfrak{j}'$ which correspond to X.

With these notation, the following holds:

Theorem 11.5 ([Wal97, 1.2. Conjecture]; [Wal06],[Ngô10])**.** *We have*

$$\tilde{D}_{\mathbf{J}',\mathbf{J}}(Y^*, X) = D_{\mathbf{J}',\mathbf{J}}(Y^*, X).$$

11.4.2. *The case of non-standard endoscopy.* Let $(\mathbf{G}_1, \mathbf{G}_2, j^*)$ be a non-standard endoscopic triple. We fix an invariant symmetric non-degenerate bilinear form B_i on $\mathfrak{g}_i$ for each i such that we have $B_1(X, X') = B_2\bigl(j_*(X), j_*(X')\bigr)$ for any $X, X' \in \mathfrak{t}_1$.

Theorem 11.6 ([Wal08, Proposition 1.8])**.** *For any regular semisimple elements $Y_1 \in \mathfrak{g}_1$ and $Y_2 \in \mathfrak{g}_2$ which correspond (resp. $X_1^* \in \mathfrak{g}_1$ and $X_2 \in \mathfrak{g}_2$ which correspond), we have*

$$D^{\mathbf{G}_1,\mathrm{st}}_{X_1^*}(Y_1) = D^{\mathbf{G}_2,\mathrm{st}}_{X_2^*}(Y_2).$$

11.4.3. *The case of isogeny.* Let $\mathbf{J}$ be a connected reductive group over F. We fix a J-invariant symmetric non-degenerate bilinear form $B_{\mathfrak{j}}$ on $\mathfrak{j}$. Then we get an identification $\mathfrak{j} \cong \mathfrak{j}^*$ which also induces identifications $\mathfrak{j}_{\mathrm{sc}} \cong \mathfrak{j}_{\mathrm{sc}}^*$ and $\mathfrak{z}_{\mathbf{J}} \cong \mathfrak{z}_{\mathbf{J}}^*$.

Lemma 11.7. *For a strongly regular semisimple element $X \in \mathfrak{j}$ with decomposition $X = X_{\mathrm{sc}} + X_Z \in \mathfrak{j}_{\mathrm{sc}} \oplus \mathfrak{z}_{\mathbf{J}}$ and a strongly regular semisimple element $X^* \in \mathfrak{j}$ with decomposition $X^* = X_{\mathrm{sc}}^* + X_Z^* \in \mathfrak{j}_{\mathrm{sc}} \oplus \mathfrak{z}_{\mathbf{J}}$, we have*

$$D_{X^*}^{\mathbf{J},\mathrm{st}}(X) = \gamma(\mathfrak{z}_{\mathbf{J}}) \cdot \psi_F(B_{\mathfrak{j}}(X_Z^*, X_Z)) \cdot D_{X_{\mathrm{sc}}^*}^{\mathbf{J}_{\mathrm{sc}},\mathrm{st}}(X_{\mathrm{sc}}).$$

Proof. According to [Wal97, 4.4 (1)], we have

$$\hat{\iota}_{X^*}^{\mathbf{J}}(X) = \psi_F(B_{\mathfrak{j}}(X_Z^*, X_Z)) \cdot \hat{\iota}_{X_{\mathrm{sc}}^*}^{\mathbf{J}_{\mathrm{sc}}}(X_{\mathrm{sc}}).$$

In general, for the orthogonal sum $V_1 \oplus V_2$ of any finite-dimensional quadratic spaces V_1 and V_2, we have $\gamma(V_1 \oplus V_2) = \gamma(V_1) \cdot \gamma(V_2)$. Hence we have $\gamma(\mathfrak{j}) = \gamma(\mathfrak{j}_{\mathrm{sc}}) \cdot \gamma(\mathfrak{z}_{\mathbf{J}})$. This implies that

$$D_{X^*}^{\mathbf{J}}(X) = \gamma(\mathfrak{z}_{\mathbf{J}}) \cdot \psi_F(B_{\mathfrak{j}}(X_Z^*, X_Z)) \cdot D_{X_{\mathrm{sc}}^*}^{\mathbf{J}_{\mathrm{sc}}}(X_{\mathrm{sc}}).$$

For any $X'^* \in \mathfrak{j}$ with decomposition $X_{\mathrm{sc}}'^* + X_Z'^* \in \mathfrak{j}_{\mathrm{sc}} \oplus \mathfrak{z}_{\mathbf{J}}$, X'^* is stably $\mathbf{J}$-conjugate (resp. J-conjugate) to X^* if and only if $X_{\mathrm{sc}}'^*$ is stably $\mathbf{J}$-conjugate (resp. J-conjugate) to X_{sc}^* and $X_Z'^* = X_Z^*$. Thus we get the assertion. □

11.4.4. *Combined form.* Now let us go back to our situation; $\mathbf{H}$ is a twisted endoscopic group of $\tilde{\mathbf{G}}$. Suppose that we have $(y, \eta) \in \tilde{G}_{\mathrm{ss}} \times H_{\mathrm{ss}}$ satisfying $y \in \mathfrak{H}_\eta$ (see Section 10.3) and that we have a diagram $D \in \mathbf{D}(y, \eta)$. Then we get the associated group $\bar{\mathbf{H}}$ as in Section 11.2. We fix invariant symmetric non-degenerate bilinear forms $B_{\mathfrak{g}_\eta}$ on $\mathfrak{g}_\eta$, $B_{\bar{\mathfrak{h}}}$ on $\bar{\mathfrak{h}}$, and $B_{\mathfrak{h}_y}$ on $\mathfrak{g}_\eta$ such that the restriction of $B_{\mathfrak{g}_\eta}$ to $\mathfrak{z}_{\mathbf{H}_y}$ is identified with the orthogonal sum of the restrictions of $B_{\mathfrak{g}_\eta}$ to $\mathfrak{z}_{\mathbf{G}_\eta}$ and $B_{\bar{\mathfrak{h}}}$ to $\mathfrak{z}_{\bar{\mathbf{H}}}$ through the isomorphism $\mathfrak{z}_{\mathbf{H}_y} \cong \mathfrak{z}_{\mathbf{G}_\eta} \oplus \mathfrak{z}_{\bar{\mathbf{H}}}$.

We take

- a strongly regular semisimple element $Y^* \in \mathfrak{h}_{y,0+}$ with decomposition $Y^* = Y_{\mathrm{sc}}^* + Y_Z^* \in \mathfrak{h}_{y,\mathrm{sc}} \oplus \mathfrak{z}_{\mathbf{H}_y}$,
- a strongly regular semisimple element $\bar{Y}_{\mathrm{sc}}^* \in \bar{\mathfrak{h}}_{\mathrm{sc},0+}$ whose stable conjugacy class corresponds to that of Y_{sc}^*, and
- a strongly regular semisimple element $X \in \mathfrak{g}_{\eta,0+}$ with decomposition $X = X_{\mathrm{sc}} + X_Z \in \mathfrak{g}_{\eta,\mathrm{sc}} \oplus \mathfrak{z}_{\mathbf{G}_\eta}$.

Let $Y_Z^* = \bar{Y}_Z^* + X_Z^* \in \mathfrak{z}_{\bar{\mathbf{H}}} \oplus \mathfrak{z}_{\mathbf{G}_\eta}$ be the decomposition of the center part $Y_Z^* \in \mathfrak{z}_{\mathbf{H}_y}$. We put $\bar{Y}^* := \bar{Y}_{\mathrm{sc}}^* + \bar{Y}_Z^*$.

Proposition 11.8. *With the above notation, we have*

$$\sum_{Y \overset{D}{\leftrightarrow} X/\sim_{\mathbf{H}_y}} \mathring{\Delta}^D(\bar{Y}, X_{\mathrm{sc}}) D_{Y^*}^{\mathbf{H}_y,\mathrm{st}}(Y) = \sum_{X^* \overset{D}{\leftrightarrow} Y^*/\sim_{G_\eta}} \mathring{\Delta}^D(\bar{Y}^*, X_{\mathrm{sc}}^*) D_{X^*}^{\mathbf{G}_\eta}(X),$$

where

- *the left sum is over the stable conjugacy classes of strongly regular semisimple elements Y of $\mathfrak{h}_{y,0+}$ such that (Y, X) is a D-norm pair ($\bar{Y}$ is the element associated to (Y, X) as in Definition 11.1), and*
- *the right sum is over the G_η-conjugacy classes of strongly regular semisimple elements X^* of $\mathfrak{g}_{\eta,0+}$ such that (Y^*, X^*) is a D-norm pair.*

Proof. Let $Y \in \mathfrak{h}_{y,0+}$ be a strongly regular semisimple element with decomposition $Y_{\mathrm{sc}} + Y_Z \in \mathfrak{h}_{y,\mathrm{sc}} \oplus \mathfrak{z}_{\mathbf{H}_y}$. Suppose that Y_{sc} corresponds to the stable conjugacy class of a strongly regular semisimple element $\bar{Y}_{\mathrm{sc}} \in \bar{\mathfrak{h}}_{\mathrm{sc}}$. Also suppose that Y_Z equals $\bar{Y}_Z + X'_Z$ under the isomorphism $\mathfrak{z}_{\mathbf{H}_y} \cong \mathfrak{z}_{\bar{\mathbf{H}}} \oplus \mathfrak{z}_{\mathbf{G}_\eta}$. Then, by definition, (Y, X) is a D-norm pair if and only if $\bar{Y} := \bar{Y}_{\mathrm{sc}} + \bar{Y}_Z$ is a norm of X_{sc} and $X'_Z = X_Z$. Hence, by noting that two strongly regular semisimple elements $Y_1, Y_2 \in \mathfrak{h}_y$ are stably conjugate if and only if $Y_{1,\mathrm{sc}}, Y_{2,\mathrm{sc}} \in \mathfrak{h}_{y,\mathrm{sc}}$ are stably conjugate and $Y_{1,Z} = Y_{2,Z}$, we see that the left-hand side of the desired identity equals

$$(16) \qquad \sum_{\bar{Y}_Z \in \mathfrak{z}_{\bar{\mathbf{H}}}} \gamma(\mathfrak{z}_{\mathbf{H}_y}) \cdot \psi_F(B_{\mathfrak{h}_y}(Y_Z^*, Y_Z)) \sum_{Y_{\mathrm{sc}} \leftrightarrow X_{\mathrm{sc}}/\sim_{\mathbf{H}_{y,\mathrm{sc}}}} \mathring{\Delta}^D(\bar{Y}, X_{\mathrm{sc}}) D^{\mathbf{H}_{y,\mathrm{sc}},\mathrm{st}}_{Y^*_{\mathrm{sc}}}(Y_{\mathrm{sc}})$$

by Lemma 11.7 (transfer for isogeny) for $\mathbf{H}_y$. Here, the second sum is over the stable conjugacy classes of strongly regular semisimple elements of $\mathfrak{h}_{y,\mathrm{sc}}$ such that $\bar{Y}_{\mathrm{sc}} + \bar{Y}_Z$ is a norm of X_{sc}, where $\bar{Y}_{\mathrm{sc}} \in \bar{\mathfrak{h}}_{\mathrm{sc}}$ is an element whose stable conjugacy class corresponds to Y_{sc}. Note that the index set $\{\bar{Y}_Z \in \mathfrak{z}_{\bar{\mathbf{H}}}\}$ of the first sum is infinite, but only finite of them have a nontrivial contribution because of the second sum.

By Theorem 11.6 (transfer for non-standard endoscopy), (16) equals

$$(17) \qquad \sum_{\bar{Y}_Z \in \mathfrak{z}_{\bar{\mathbf{H}}}} \gamma(\mathfrak{z}_{\mathbf{H}_y}) \cdot \psi_F(B_{\mathfrak{h}_y}(Y_Z^*, Y_Z)) \sum_{\bar{Y}_{\mathrm{sc}} \leftrightarrow X_{\mathrm{sc}}/\sim_{\bar{\mathbf{H}}_{\mathrm{sc}}}} \mathring{\Delta}^D(\bar{Y}, X_{\mathrm{sc}}) D^{\bar{\mathbf{H}}_{\mathrm{sc}},\mathrm{st}}_{\bar{Y}^*_{\mathrm{sc}}}(\bar{Y}_{\mathrm{sc}}),$$

where the second sum is over the stable conjugacy classes of strongly regular semisimple elements $\bar{Y}_{\mathrm{sc}} \in \bar{\mathfrak{h}}_{\mathrm{sc}}$ such that $\bar{Y}_{\mathrm{sc}} + \bar{Y}_Z$ is a norm of $X_{\mathrm{sc}} \in \mathfrak{g}_{\eta,\mathrm{sc}}$. By rearranging the sums, we see that (17) equals

$$(18) \qquad \sum_{\bar{Y} \leftrightarrow X_{\mathrm{sc}}/\sim_{\bar{\mathbf{H}}}} \gamma(\mathfrak{z}_{\mathbf{H}_y}) \cdot \psi_F(B_{\mathfrak{h}_y}(Y_Z^*, Y_Z)) \cdot \mathring{\Delta}^D(\bar{Y}, X_{\mathrm{sc}}) D^{\bar{\mathbf{H}}_{\mathrm{sc}},\mathrm{st}}_{\bar{Y}^*_{\mathrm{sc}}}(\bar{Y}_{\mathrm{sc}}),$$

where the sum is over the set of stable conjugacy classes of strongly regular semisimple elements of $\bar{\mathfrak{h}}$ which are norms of $X_{\mathrm{sc}} \in \mathfrak{g}_{\eta,\mathrm{sc}}$. By noting that $\gamma(\mathfrak{z}_{\mathbf{H}_y}) = \gamma(\mathfrak{z}_{\mathbf{G}_\eta}) \cdot \gamma(\mathfrak{z}_{\bar{\mathbf{H}}})$ and that $\psi_F(B_{\mathfrak{h}_y}(Y_Z^*, Y_Z)) = \psi_F(B_{\bar{\mathfrak{h}}}(\bar{Y}_Z^*, \bar{Y}_Z)) \cdot \psi_F(B_{\mathfrak{g}_\eta}(X_Z^*, X_Z))$, Lemma 11.7 (transfer for isogeny) for $\bar{\mathbf{H}}$ implies that (18) equals

$$(19) \qquad \gamma(\mathfrak{z}_{\mathbf{G}_\eta}) \cdot \psi_F(B_{\mathfrak{g}_\eta}(X_Z^*, X_Z)) \sum_{\bar{Y} \leftrightarrow X_{\mathrm{sc}}/\sim_{\bar{\mathbf{H}}}} \mathring{\Delta}^D(\bar{Y}, X_{\mathrm{sc}}) D^{\bar{\mathbf{H}},\mathrm{st}}_{\bar{Y}^*}(\bar{Y}).$$

Finally, by Theorem 11.5 (transfer for standard endoscopy), (19) equals

$$(20) \qquad \gamma(\mathfrak{z}_{\mathbf{G}_\eta}) \cdot \psi_F(B_{\mathfrak{g}_\eta}(X_Z^*, X_Z)) \sum_{X^*_{\mathrm{sc}} \leftrightarrow \bar{Y}^*/\sim_{G_{\eta,\mathrm{sc}}}} \mathring{\Delta}^D(\bar{Y}^*, X^*_{\mathrm{sc}}) D^{\mathbf{G}_{\eta,\mathrm{sc}}}_{X^*_{\mathrm{sc}}}(X_{\mathrm{sc}}),$$

where the index set is over the $G_{\eta,\mathrm{sc}}$-conjugacy classes of strongly regular semisimple elements of $\mathfrak{g}_{\eta,\mathrm{sc}}$ which correspond to $\bar{Y}^*$. Then the same argument as in the proof of Lemma 11.7 implies that (20) equals

$$(21) \qquad \sum_{X^*_{\mathrm{sc}} \leftrightarrow \bar{Y}^*/\sim_{G_\eta}} \mathring{\Delta}^D(\bar{Y}^*, X^*_{\mathrm{sc}}) D^{\mathbf{G}_\eta}_{X^*}(X).$$

We recall that (Y^*, X^*) is a D-norm pair if and only if X^*_{sc} corresponds to $\bar{Y}^*$ and the center part of X^* is given by X^*_Z, which determined by Y^*. Thus we see that the index set of the sum in (21) is nothing but that of the sum on the right-hand side of the desired identity. □

12. Toral invariants for restricted roots

12.1. **Root systems.** Let $\eta \in \tilde{G}_{\mathrm{ss}}$ and $y \in \mathfrak{H}_\eta$ (see Section 10.3) such that $\mathbf{D}(y,\eta)$ is not empty. Note that y and η correspond in the sense of twisted endoscopy. Let us recall how $\Phi(\mathbf{G}_\eta, \mathbf{T}^\natural)$ and $\Phi(\mathbf{H}_y, \mathbf{T}^\flat)$ are described in terms of the restricted roots following [Wal08, Section 3.3]. In the following, we fix a diagram $D = (\mathbf{B}^\flat, \mathbf{T}^\flat, \mathbf{B}^\diamondsuit, \mathbf{T}^\diamondsuit) \in \mathbf{D}(y,\eta)$ and simply write ν for ν_D (resp. μ for μ_D).

Recall that, with the notation as in Section 3.3, we have

$$\Phi(\mathbf{G}_{\nu\theta}, \mathbf{T}^\natural) = \{p^*(\alpha) \mid \alpha \in \Phi(\mathbf{G}, \mathbf{T}); N(\alpha)(\nu) = 1\} \subset \Phi_{\mathrm{res}}(\mathbf{G}, \mathbf{T}),$$
$$\Phi^\vee(\mathbf{G}_{\nu\theta}, \mathbf{T}^\natural) = \{N(\alpha^\vee) \mid \alpha^\vee \in \Phi^\vee(\mathbf{G}, \mathbf{T}); N(\alpha)(\nu) = 1\} \subset \Phi^\vee_{\mathrm{res}}(\mathbf{G}, \mathbf{T})$$

(note that now we assume that $\Phi_{\mathrm{res}}(\mathbf{G}, \mathbf{T})$ does not contain any restricted root of type 2 or 3). The sets $\Phi(\mathbf{H}_\mu, \mathbf{T}_\mathbf{H})$ and $\Phi^\vee(\mathbf{H}_\mu, \mathbf{T}_\mathbf{H})$ are given as follows:

$$\Phi(\mathbf{H}_\mu, \mathbf{T}_\mathbf{H}) = \{N(\alpha) \mid \alpha \in \Phi(\mathbf{G}, \mathbf{T}); N(\alpha^\vee)(s) = 1, N(\alpha)(\nu) = 1\} \subset X^*(\mathbf{T})^\theta \cong X^*(\mathbf{T}_\mathbf{H}),$$
$$\Phi^\vee(\mathbf{H}_\mu, \mathbf{T}_\mathbf{H}) = \{p_*(\alpha^\vee) \mid \alpha^\vee \in \Phi^\vee(\mathbf{G}, \mathbf{T}); N(\alpha^\vee)(s) = 1, N(\alpha)(\nu) = 1\} \subset Y_*(\mathbf{T}) \cong X_*(\mathbf{T}_\mathbf{H}).$$

We define an injective map $i^\vee \colon \Phi^\vee(\mathbf{H}_\mu, \mathbf{T}_\mathbf{H}) \to \Phi^\vee(\mathbf{G}_{\nu\theta}, \mathbf{T}^\natural)$ by $i^\vee(p_*(\alpha^\vee)) := N(\alpha^\vee)$. Then we can regard $\Phi^\vee(\mathbf{H}_\mu, \mathbf{T}_\mathbf{H})$ as a subset of $\Phi^\vee(\mathbf{G}_{\nu\theta}, \mathbf{T}^\natural)$ via $i^\vee$.

Now we transfer this discussion to $\Phi(\mathbf{G}, \mathbf{T}^\diamondsuit)$ by using the fixed diagram D. Via the map $\xi_\diamondsuit$, $\Phi^{(\vee)}_{\mathrm{res}}(\mathbf{G}, \mathbf{T}^\diamondsuit)$ is identified with $\Phi^{(\vee)}_{\mathrm{res}}(\mathbf{G}, \mathbf{T})$. Moreover, $\Phi^{(\vee)}(\mathbf{G}_\eta, \mathbf{T}^\natural)$ is mapped to $\Phi^{(\vee)}(\mathbf{G}_{\nu\theta}, \mathbf{T}^\natural)$ by this identification. Similarly, via the map $\xi_\flat$, $\Phi^{(\vee)}(\mathbf{H}, \mathbf{T}^\flat)$ is identified with $\Phi^{(\vee)}(\mathbf{H}, \mathbf{T}_\mathbf{H})$, and $\Phi^{(\vee)}(\mathbf{H}_y, \mathbf{T}^\flat)$ is mapped to $\Phi^{(\vee)}(\mathbf{H}_\mu, \mathbf{T}_\mathbf{H})$. By combining these bijective maps with the previous injective map $i^\vee$, we may identify $\Phi^\vee(\mathbf{H}_y, \mathbf{T}^\flat)$ as a subset of $\Phi^\vee(\mathbf{G}_\eta, \mathbf{T}^\natural)$. Accordingly, we may also may identify $\Phi(\mathbf{H}_y, \mathbf{T}^\flat)$ as a subset of $\Phi(\mathbf{G}_\eta, \mathbf{T}^\natural)$.

Let $(Y, X) \in \mathfrak{h}_{y,0+} \times \mathfrak{g}_{\eta,0+}$ be a D-norm pair. Let us fix bilinear forms $B_{\mathfrak{g}_\eta}$ on $\mathfrak{g}_\eta$, $B_{\bar{\mathfrak{h}}}$ on $\bar{\mathfrak{h}}$, and $B_{\mathfrak{h}_y}$ on $\mathfrak{h}_y$ as in Section 11.4. Then $X \in \mathfrak{g}_\eta$ (resp. $Y \in \mathfrak{h}_y$) can be identified with an element $X^* \in \mathfrak{g}_\eta^*$ (resp. $Y^* \in \mathfrak{h}_y^*$).

Lemma 12.1. *Let $\alpha_y^\vee$ be an element of $\Phi^\vee(\mathbf{H}_y, \mathbf{T}^\flat)$ which is identified with an element $\alpha_\eta^\vee$ of $\Phi^\vee(\mathbf{G}_\eta, \mathbf{T}^\natural)$. Let $\alpha^\vee \in \Phi^\vee(\mathbf{G}, \mathbf{T}^\diamondsuit)$ be the coroot satisfying $\alpha_y^\vee = p_*(\alpha^\vee) \in X_*(\mathbf{T}^\flat)$ and $\alpha_\eta^\vee = N(\alpha^\vee) \in X_*(\mathbf{T}^\natural)$. Then we have*

$$l_\alpha \cdot \langle d\alpha_y^\vee(1), Y^* \rangle = \langle d\alpha_\eta^\vee(1), X^* \rangle.$$

Proof. Since (Y, X) is a norm pair with respect to $D = (\mathbf{B}^\flat, \mathbf{T}^\flat, \mathbf{B}^\diamondsuit, \mathbf{T}^\diamondsuit)$, we may suppose that $X \in \mathfrak{t}^\natural$ and $Y \in \mathfrak{t}^\flat$ and $\xi_D \colon \mathfrak{t}^\diamondsuit \twoheadrightarrow \mathfrak{t}^\flat$ maps X to Y (cf. the argument in the proof of Lemma 10.8). Hence, by our choice of bilinear forms, X^* is identified with Y^* under the dual isomorphism $\mathfrak{t}^{\natural *} \cong \mathfrak{t}^{\flat *}$ to $\mathfrak{t}^\natural \cong \mathfrak{t}^\flat$. Thus we can see that, under the identification $\xi_D \colon \mathfrak{t}^\natural \cong \mathfrak{t}^\flat$, we have $\langle d\alpha_\eta^\vee(1), X^* \rangle = \sum_{i=0}^{l_\alpha} \langle d[\eta]^i(\alpha)^\vee(1), X^* \rangle = l_\alpha \langle d\alpha^\vee(1), X^* \rangle = l_\alpha \langle d\alpha_y^\vee(1), Y^* \rangle$. □

12.2. **Analysis of ramified restricted roots.** Let us next suppose that we have an F-rational twisted maximal torus $\tilde{\mathbf{T}}^\diamondsuit$ of $\tilde{\mathbf{G}}$ and topologically semisimple elements $\eta, \eta' \in \tilde{T}^\diamondsuit$. We investigate the relation between the ramified roots of $\Phi(\mathbf{G}_\eta, \mathbf{T}^\natural)$ and those of $\Phi(\mathbf{G}_{\eta'}, \mathbf{T}^\natural)$. For convenience, let us introduce the following notation:

$$\Phi_{\mathrm{res}}(\mathbf{G}, \mathbf{T}^\diamondsuit)_\bullet^{(\star)} := \{\alpha_{\mathrm{res}} \in \Phi_{\mathrm{res}}(\mathbf{G}, \mathbf{T}^\diamondsuit)_\bullet \mid \alpha \in \Phi(\mathbf{G}, \mathbf{T}^\diamondsuit)_\star\},$$

where $\bullet, \star \in \{\mathrm{asym}, \mathrm{ur}, \mathrm{ram}\}$. By fixing a Borel subgroup $\mathbf{B}^{\diamondsuit}$ containing $\mathbf{T}^{\diamondsuit}$ and stabilized by the action of $\tilde{\mathbf{T}}^{\diamondsuit}$, we take an element $g^{\diamondsuit} \in \mathbf{G}$ satisfying $[g^{\diamondsuit}](\mathbf{B}^{\diamondsuit}, \mathbf{T}^{\diamondsuit}) = (\mathbf{B}, \mathbf{T})$. Let $\nu := [g^{\diamondsuit}](\eta)$ and $\nu' := [g^{\diamondsuit}](\eta')$. We simply write $\theta_{\diamondsuit}$ for the involution $\theta_{\mathbf{T}^{\diamondsuit}}$ of $\mathbf{T}^{\diamondsuit}$ determined by its twisted structure.

Lemma 12.2. *For any $\alpha_{\mathrm{res}} \in \Phi_{\mathrm{res}}(\mathbf{G}, \mathbf{T}^{\diamondsuit})^{(\mathrm{asym})}_{\mathrm{ram}}$, we have $\alpha_{\mathrm{res}} \in \Phi(\mathbf{G}_{\eta}, \mathbf{T}^{\natural})$ if and only if $\alpha_{\mathrm{res}} \in \Phi(\mathbf{G}_{\eta'}, \mathbf{T}^{\natural})$.*

Proof. For any $\alpha_{\mathrm{res}} \in \Phi_{\mathrm{res}}(\mathbf{G}, \mathbf{T}^{\diamondsuit})^{(\mathrm{asym})}_{\mathrm{ram}}$, the following hold:

- $F_{\alpha} = F_{\pm\alpha}$ and $F_{\alpha} = F_{\alpha_{\mathrm{res}}}$,
- $F_{\alpha_{\mathrm{res}}}/F_{\pm\alpha_{\mathrm{res}}}$ and $F_{\pm\alpha}/F_{\pm\alpha_{\mathrm{res}}}$ are quadratic ramified, and
- $\theta_{\diamondsuit}(\alpha) \in -\Gamma\alpha$; let $\tau_{\alpha} \in \Gamma$ be an element satisfying $\tau_{\alpha}(\alpha) = -\theta_{\diamondsuit}(\alpha)$.

$$\begin{array}{ccc} F_{\alpha_{\mathrm{res}}} & = & F_{\alpha} \\ {\scriptstyle \mathrm{quad}}\big| {\scriptstyle \mathrm{ram}} & & \big\| \\ F_{\pm\alpha_{\mathrm{res}}} & \overset{\mathrm{quad}}{\underset{\mathrm{ram}}{\text{———}}} & F_{\pm\alpha} \end{array}$$

By the description explained in Section 12.1, it suffices to show that $N(\alpha)(\nu) = N(\alpha)(\nu')$, which is equivalent to $\alpha(\eta^2) = \alpha(\eta'^2)$. (Here, in the first equality, we regard α as a root of $\mathbf{T}$ and again write α for it.) Since both $\alpha(\eta^2)$ and $\alpha(\eta'^2)$ are of finite prime-to-p order, it is enough to show that $\alpha(\eta^2) \equiv \alpha(\eta'^2) \pmod{\mathfrak{p}_{\overline{F}}}$.

Let $t \in T^{\diamondsuit}$ be an element satisfying $\eta' = t\eta$. Then we have $\eta'^2 = (t\eta)^2 = t \cdot \theta_{\diamondsuit}(t) \cdot \eta^2$. Since t is F-rational and $\tau_{\alpha}(\alpha) = -\theta_{\diamondsuit}(\alpha)$, we have

$$\alpha(t \cdot \theta_{\diamondsuit}(t)) = \alpha(t) \cdot \theta_{\diamondsuit}(\alpha)(t) = \alpha(t) \cdot \tau_{\alpha}(\alpha(t))^{-1}.$$

By noting that $F_{\alpha_{\mathrm{res}}}/F_{\pm\alpha_{\mathrm{res}}}$ is ramified, we have $\alpha(t) \cdot \tau_{\alpha}(\alpha(t))^{-1} \equiv 1 \pmod{\mathfrak{p}_{\overline{F}}}$. □

Lemma 12.3. *For any $\alpha_{\mathrm{res}} \in \Phi_{\mathrm{res}}(\mathbf{G}, \mathbf{T}^{\diamondsuit})^{(\mathrm{ur})}_{\mathrm{ram}}$, we have $\alpha_{\mathrm{res}} \in \Phi(\mathbf{G}_{\eta}, \mathbf{T}^{\natural})$ if and only if $\alpha_{\mathrm{res}} \in \Phi(\mathbf{G}_{\eta'}, \mathbf{T}^{\natural})$.*

Proof. For any $\alpha_{\mathrm{res}} \in \Phi_{\mathrm{res}}(\mathbf{G}, \mathbf{T}^{\diamondsuit})^{(\mathrm{ur})}_{\mathrm{ram}}$, the following hold:

- $F_{\alpha}/F_{\pm\alpha}$ and $F_{\alpha}/F_{\alpha_{\mathrm{res}}}$ are quadratic unramified, and
- $F_{\alpha_{\mathrm{res}}}/F_{\pm\alpha_{\mathrm{res}}}$ and $F_{\pm\alpha}/F_{\pm\alpha_{\mathrm{res}}}$ are quadratic ramified.

$$\begin{array}{ccc} F_{\alpha_{\mathrm{res}}} & \overset{\mathrm{quad}}{\underset{\mathrm{ur}}{\text{———}}} & F_{\alpha} \\ {\scriptstyle \mathrm{quad}}\big| {\scriptstyle \mathrm{ram}} & & {\scriptstyle \mathrm{quad}}\big| {\scriptstyle \mathrm{ur}} \\ F_{\pm\alpha_{\mathrm{res}}} & \overset{\mathrm{quad}}{\underset{\mathrm{ram}}{\text{———}}} & F_{\pm\alpha} \end{array}$$

We let $\sigma_{\alpha}, \tau_{\alpha} \in \Gamma$ be elements satisfying $\sigma_{\alpha}(\alpha) = \theta_{\diamondsuit}(\alpha)$ and $\tau_{\alpha}(\alpha) = -\alpha$, respectively. With the same notation and arguments as in the proof of Lemma 12.2, it suffices to show that $\alpha(t) \cdot \theta_{\diamondsuit}(\alpha)(t) \equiv 1 \pmod{\mathfrak{p}_{\overline{F}}}$ for any $t \in T^{\diamondsuit}$. Since t is F-rational and $\tau_{\alpha}(\alpha) = -\alpha$, we have

$$\mathrm{Nr}_{F_{\alpha}/F_{\pm\alpha}}(\alpha(t)) = \alpha(t) \cdot \tau_{\alpha}(\alpha(t)) = \alpha(t) \cdot \alpha(t)^{-1} = 1.$$

Similarly, we have

$$\mathrm{Nr}_{F_{\alpha}/F_{\alpha_{\mathrm{res}}}}(\alpha(t)) = \alpha(t) \cdot \sigma_{\alpha}(\alpha(t)) = \alpha(t) \cdot \theta_{\diamondsuit}(\alpha)(t).$$

Since both $F_{\alpha}/F_{\alpha_{\mathrm{res}}}$ and $F_{\alpha}/F_{\pm\alpha}$ are unramified quadratic extensions, we get

$$\alpha(t) \cdot \theta_{\diamondsuit}(\alpha)(t) = \mathrm{Nr}_{F_{\alpha}/F_{\alpha_{\mathrm{res}}}}(\alpha(t)) \equiv \mathrm{Nr}_{F_{\alpha}/F_{\pm\alpha}}(\alpha(t)) = 1 \pmod{\mathfrak{p}_{\overline{F}}}.$$

□

Lemma 12.4. *For any $\alpha_{\mathrm{res}} \in \Phi_{\mathrm{res}}(\mathbf{G}, \mathbf{T}^{\diamondsuit})^{(\mathrm{ram})}_{\mathrm{ram}}$ with $l_\alpha = 2$, we have $\alpha_{\mathrm{res}} \in \Phi(\mathbf{G}_\eta, \mathbf{T}^{\natural})$ if and only if $\alpha_{\mathrm{res}} \in \Phi(\mathbf{G}_{\eta'}, \mathbf{T}^{\natural})$.*

Proof. Suppose that $\alpha \in \Phi(\mathbf{G}, \mathbf{T}^{\diamondsuit})$ is a symmetric ramified root with $l_\alpha = 2$. In this case, the following hold:

- $F_\alpha/F_{\pm\alpha}$ and $F_{\alpha_{\mathrm{res}}}/F_{\pm\alpha_{\mathrm{res}}}$ are quadratic ramified, and
- $F_\alpha/F_{\alpha_{\mathrm{res}}}$ and $F_{\alpha_{\mathrm{res}}}/F_{\pm\alpha_{\mathrm{res}}}$ are quadratic unramified.

$$\begin{array}{ccc} F_{\alpha_{\mathrm{res}}} & \overset{\mathrm{quad}}{\underset{\mathrm{ur}}{\text{———}}} & F_\alpha \\ \mathrm{quad}\,\Big|\,\mathrm{ram} & & \mathrm{ram}\,\Big|\,\mathrm{quad} \\ F_{\pm\alpha_{\mathrm{res}}} & \overset{\mathrm{ur}}{\underset{\mathrm{quad}}{\text{———}}} & F_{\pm\alpha} \end{array}$$

(Indeed, since $\theta_{\diamondsuit}(\alpha) \neq -\alpha$, we cannot have $F_{\alpha_{\mathrm{res}}} = F_{\pm\alpha}$. If $F_{\pm\alpha}/F_{\pm\alpha_{\mathrm{res}}}$ is a quadratic ramified extension different from $F_{\alpha_{\mathrm{res}}}$, then F_α must contain a quadratic unramified extension of $F_{\pm\alpha_{\mathrm{res}}}$, hence we get a contradiction. Thus $F_{\pm\alpha}/F_{\pm\alpha_{\mathrm{res}}}$ must be quadratic unramified.) Let $\tau_\alpha \in \Gamma$ be an element satisfying $\tau_\alpha(\alpha) = -\alpha$. Then τ_α restricts to the nontrivial element of $\mathrm{Gal}(F_{\alpha_{\mathrm{res}}}/F_{\pm\alpha_{\mathrm{res}}})$. Let $\sigma_\alpha \in \Gamma$ be an element satisfying $\sigma_\alpha(\alpha) = \theta_{\diamondsuit}(\alpha)$.

With the same notation and arguments as in the proof of Lemma 12.2, it suffices to show that $\alpha(t) \cdot \theta_{\diamondsuit}(\alpha)(t) \equiv 1 \pmod{\mathfrak{p}_{\overline{F}}}$. Since t is F-rational and $\tau_\alpha(\alpha) = -\alpha$,

$$\mathrm{Nr}_{F_\alpha/F_{\pm\alpha}}(\alpha(t)) = \alpha(t) \cdot \tau_\alpha(\alpha(t)) = \alpha(t) \cdot \alpha(t)^{-1} = 1.$$

As $F_\alpha/F_{\pm\alpha}$ is ramified, this implies that $\alpha(t) \equiv \pm 1 \pmod{\mathfrak{p}_{\overline{F}}}$. Thus we get

$$\alpha(t) \cdot \theta_{\diamondsuit}(\alpha)(t) = \alpha(t) \cdot \sigma_\alpha(\alpha(t)) = \mathrm{Nr}_{F_\alpha/F_{\alpha_{\mathrm{res}}}}(\alpha(t)) \equiv \mathrm{Nr}_{F_\alpha/F_{\alpha_{\mathrm{res}}}}(\pm 1) = 1 \pmod{\mathfrak{p}_{\overline{F}}}.$$

□

12.3. **Descent of toral invariants.** Let us keep the notation as in Section 12.2. We next investigate the relation between the toral invariants for $(\mathbf{G}_\eta, \mathbf{T}^{\natural})$ and those for $(\mathbf{G}_{\eta'}, \mathbf{T}^{\natural})$. Before we start our discussion, we note that the roots in the Θ-orbits $\Theta\alpha$ of $\alpha \in \Phi(\mathbf{G}, \mathbf{T})$ are orthogonal to each other and that, for any root vector $X_\alpha \in \mathfrak{g}_\alpha$, we have $\theta^{l_\alpha}(X_\alpha) = X_\alpha$ (these are true since we are assuming that there is no restricted root of type 2 or 3; see [KS99, (1.3.5-1.3.7)]).

Let $t \in T^{\diamondsuit}$ be the element satisfying $\eta' = t\eta$.

Proposition 12.5. *Let $\alpha_{\mathrm{res}} \in \Phi_{\mathrm{res}}(\mathbf{G}, \mathbf{T}^{\diamondsuit})^{(\mathrm{asym})}_{\mathrm{ram}}$. Suppose that $\alpha_{\mathrm{res}} \in \Phi(\mathbf{G}_\eta, \mathbf{T}^{\natural})$, which is equivalent to $\alpha_{\mathrm{res}} \in \Phi(\mathbf{G}_{\eta'}, \mathbf{T}^{\natural})$ by Lemma 12.2. Then we have*

$$f_{(\mathbf{G}_\eta, \mathbf{T}^{\natural})}(\alpha_{\mathrm{res}}) = f_{(\mathbf{G}_{\eta'}, \mathbf{T}^{\natural})}(\alpha_{\mathrm{res}}) \cdot \epsilon_\alpha(t).$$

Proof. We use the notation as in the proof of Lemma 12.2. We take an element X_α of $\mathfrak{g}_\alpha(F_\alpha)$. Then $X_{\alpha_{\mathrm{res}},\eta} := X_\alpha + [\eta](X_\alpha)$ belongs to $\mathfrak{g}_{\eta,\alpha_{\mathrm{res}}}(F_{\alpha_{\mathrm{res}}})$ (note that $F_{\alpha_{\mathrm{res}}} = F_\alpha$). Thus, by the definition of the toral invariant (see Section 7.2),

$$\begin{aligned} f_{(\mathbf{G}_\eta, \mathbf{T}^{\natural})}(\alpha_{\mathrm{res}}) &= \kappa_{\alpha_{\mathrm{res}}}\left(\frac{[X_{\alpha_{\mathrm{res}},\eta}, \tau_\alpha(X_{\alpha_{\mathrm{res}},\eta})]}{H_{\alpha_{\mathrm{res}}}}\right) \\ &= \kappa_{\alpha_{\mathrm{res}}}\left(\frac{[X_\alpha, [\eta](\tau_\alpha(X_\alpha))] + [[\eta](X_\alpha), \tau_\alpha(X_\alpha)]}{H_{\alpha_{\mathrm{res}}}}\right) \end{aligned}$$

(note that $\tau_\alpha(\alpha) = -\theta_\diamondsuit(\alpha)$). Since we have $[\eta]([X_\alpha, [\eta](\tau_\alpha(X_\alpha))]) = [[\eta](X_\alpha), \tau_\alpha(X_\alpha)]$ and $H_{\alpha_{\mathrm{res}}} = H_\alpha + [\eta](H_\alpha)$, we get

$$f_{(\mathbf{G}_\eta, \mathbf{T}^\natural)}(\alpha_{\mathrm{res}}) = \kappa_{\alpha_{\mathrm{res}}}\biggl(\frac{[X_\alpha, [\eta](\tau_\alpha(X_\alpha))]}{H_\alpha}\biggr).$$

By the same computation, we get

$$f_{(\mathbf{G}_{\eta'}, \mathbf{T}^\natural)}(\alpha_{\mathrm{res}}) = \kappa_{\alpha_{\mathrm{res}}}\biggl(\frac{[X_\alpha, [\eta'](\tau_\alpha(X_\alpha))]}{H_\alpha}\biggr).$$

Hence, we get

$$f_{(\mathbf{G}_{\eta'}, \mathbf{T}^\natural)}(\alpha_{\mathrm{res}}) = f_{(\mathbf{G}_\eta, \mathbf{T}^\natural)}(\alpha_{\mathrm{res}}) \cdot \kappa_{\alpha_{\mathrm{res}}}(\tau_\alpha(\alpha(t))).$$

Since $\kappa_{\alpha_{\mathrm{res}}}$ is the quadratic character of $F_{\pm\alpha_{\mathrm{res}}}$ corresponding to the extension $F_{\alpha_{\mathrm{res}}}/F_{\pm\alpha_{\mathrm{res}}}$, by noting that $\alpha(t)$ belongs to $\mathcal{O}^\times_{F_{\pm\alpha_{\mathrm{res}}}}$, we conclude that

$$\kappa_{\alpha_{\mathrm{res}}}(\tau_\alpha(\alpha(t))) = \mathrm{sgn}_{k_\alpha^\times}(\overline{\alpha(t)}) = \epsilon_\alpha(t).$$

□

Proposition 12.6. *Let* $\alpha_{\mathrm{res}} \in \Phi_{\mathrm{res}}(\mathbf{G}, \mathbf{T}^\diamondsuit)^{(\mathrm{ur})}_{\mathrm{ram}}$. *Suppose that* $\alpha_{\mathrm{res}} \in \Phi(\mathbf{G}_\eta, \mathbf{T}^\natural)$, *which is equivalent to* $\alpha_{\mathrm{res}} \in \Phi(\mathbf{G}_{\eta'}, \mathbf{T}^\natural)$ *by Lemma 12.3. Then we have*

$$f_{(\mathbf{G}_\eta, \mathbf{T}^\natural)}(\alpha_{\mathrm{res}}) = f_{(\mathbf{G}_{\eta'}, \mathbf{T}^\natural)}(\alpha_{\mathrm{res}}) \cdot \epsilon_\alpha(t).$$

Proof. We use the notation as in the proof of Lemma 12.3. We take an element X_α of $\mathfrak{g}_\alpha(F_\alpha)$. Then $X_{\alpha_{\mathrm{res}},\eta} := X_\alpha + [\eta](X_\alpha)$ belongs to $\mathfrak{g}_{\eta,\alpha_{\mathrm{res}}}(F_\alpha)$. To compute the toral invariant $f_{(\mathbf{G}_\eta, \mathbf{T}^\natural)}(\alpha_{\mathrm{res}})$, let us scale $X_{\alpha_{\mathrm{res}},\eta}$ so that it is $F_{\alpha_{\mathrm{res}}}$-rational. Let $C_\eta \in F_\alpha^\times$ be the constant determined by $\sigma_\alpha(X_\alpha) = C_\eta \cdot [\eta](X_\alpha)$. Note that then $\sigma_\alpha([\eta](X_\alpha)) = C_\eta \cdot X_\alpha$. Indeed, since η is F-rational, we have

$$\sigma_\alpha([\eta](X_\alpha)) = [\eta](\sigma_\alpha(X_\alpha)) = [\eta](C_\eta \cdot [\eta](X_\alpha)) = C_\eta \cdot \alpha(\eta^2) \cdot X_\alpha = C_\eta \cdot X_\alpha.$$

Thus we have

$$\sigma_\alpha^2(X_\alpha) = \sigma_\alpha(C_\eta \cdot [\eta](X_\alpha)) = \sigma_\alpha(C_\eta) \cdot C_\eta \cdot X_\alpha = \mathrm{Nr}_{F_\alpha/F_{\alpha_{\mathrm{res}}}}(C_\eta) \cdot X_\alpha.$$

On the other hand, since σ_α^2 belongs to Γ_α, σ_α^2 fixes X_α. This implies that $\mathrm{Nr}_{F_\alpha/F_{\alpha_{\mathrm{res}}}}(C_\eta) = 1$. Hence, by the Hilbert 90th theorem, we can find an element $x_\eta \in F_\alpha^\times$ satisfying $C_\eta = x_\eta/\sigma_\alpha(x_\eta)$. By putting $\tilde{X}_{\alpha_{\mathrm{res}},\eta} := x_\eta \cdot X_{\alpha_{\mathrm{res}},\eta}$, we get an $F_{\alpha_{\mathrm{res}}}$-rational root vector $\tilde{X}_{\alpha_{\mathrm{res}},\eta} \in \mathfrak{g}_{\eta,\alpha_{\mathrm{res}}}(F_{\alpha_{\mathrm{res}}})$.

Now let us compute the toral invariant using $\tilde{X}_{\alpha_{\mathrm{res}},\eta}$:

$$\begin{aligned} f_{(\mathbf{G}_\eta, \mathbf{T}^\natural)}(\alpha_{\mathrm{res}}) &= \kappa_{\alpha_{\mathrm{res}}}\biggl(\frac{[\tilde{X}_{\alpha_{\mathrm{res}},\eta}, \tau_\alpha(\tilde{X}_{\alpha_{\mathrm{res}},\eta})]}{H_{\alpha_{\mathrm{res}}}}\biggr) \\ &= \kappa_{\alpha_{\mathrm{res}}}\biggl(\frac{[x_\eta X_\alpha, \tau_\alpha(x_\eta X_\alpha)] + [x_\eta[\eta](X_\alpha), \tau_\alpha(x_\eta[\eta](X_\alpha))]}{H_{\alpha_{\mathrm{res}}}}\biggr). \end{aligned}$$

By the same argument as in the proof of Proposition 12.5, this equals

$$\kappa_{\alpha_{\mathrm{res}}}\biggl(\frac{[x_\eta X_\alpha, \tau_\alpha(x_\eta X_\alpha)]}{H_\alpha}\biggr).$$

By the same computation, putting $C_{\eta'} \in F_\alpha^\times$ and $x_{\eta'} \in F_\alpha^\times$ in the same manner,

$$f_{(\mathbf{G}_{\eta'}, \mathbf{T}^\natural)}(\alpha_{\mathrm{res}}) = \kappa_{\alpha_{\mathrm{res}}}\biggl(\frac{[x_{\eta'} X_\alpha, \tau_\alpha(x_{\eta'} X_\alpha)]}{H_\alpha}\biggr).$$

Thus we get

$$f_{(\mathbf{G}_{\eta'},\mathbf{T}^{\natural})}(\alpha_{\mathrm{res}}) = f_{(\mathbf{G}_{\eta},\mathbf{T}^{\natural})}(\alpha_{\mathrm{res}}) \cdot \kappa_{\alpha_{\mathrm{res}}}\big((x_{\eta'}x_{\eta}^{-1}) \cdot \tau_{\alpha}(x_{\eta'}x_{\eta}^{-1})\big).$$

We put $y := x_{\eta'}x_{\eta}^{-1} \in F_{\alpha}^{\times}$, hence $\kappa_{\alpha_{\mathrm{res}}}((x_{\eta'}x_{\eta}^{-1}) \cdot \tau_{\alpha}(x_{\eta'}x_{\eta}^{-1})) = \kappa_{\alpha_{\mathrm{res}}}(y \cdot \tau_{\alpha}(y))$. As $C_{\eta} \cdot [\eta](X_{\alpha}) = \sigma_{\alpha}(X_{\alpha}) = C_{\eta'} \cdot [\eta'](X_{\alpha})$, we have $C_{\eta'} = \alpha(t)^{-1}C_{\eta}$. Hence we have $y/\sigma_{\alpha}(y) = \alpha(t)^{-1}$. Here we note that, since $F_{\alpha}/F_{\alpha_{\mathrm{res}}}$ is unramified, we can choose x_{η} and $x_{\eta'}$ to be elements of $\mathcal{O}_{F_{\alpha}}^{\times}$, hence $y \in \mathcal{O}_{F_{\alpha}}^{\times}$. We note that the composition

$$k_{\alpha}^{1} \overset{\cong}{\longleftarrow} k_{\alpha}^{\times}/k_{\pm\alpha}^{\times} \xrightarrow{\mathrm{Nr}_{k_{\alpha}/k_{\pm\alpha}}} k_{\pm\alpha}^{\times}/k_{\pm\alpha}^{\times 2} \cong \mu_{2}: \quad y/\sigma_{\alpha}(y) \mapsto y \mapsto y \cdot \tau_{\alpha}(y)$$

defines the unique nontrivial quadratic character of k_{α}^{1}. Hence we get

$$\kappa_{\alpha_{\mathrm{res}}}(y \cdot \tau_{\alpha}(y)) = \mathrm{sgn}_{k_{\alpha}^{1}}(\overline{\alpha(t)})^{-1} = \epsilon_{\alpha}(t).$$

□

Proposition 12.7. *Let $\alpha_{\mathrm{res}} \in \Phi_{\mathrm{res}}(\mathbf{G}, \mathbf{T}^{\diamondsuit})_{\mathrm{ram}}^{(\mathrm{ram})}$. Suppose that $\alpha_{\mathrm{res}} \in \Phi(\mathbf{G}_{\eta}, \mathbf{T}^{\natural})$ and $\alpha_{\mathrm{res}} \in \Phi(\mathbf{G}_{\eta'}, \mathbf{T}^{\natural})$. Then we have*

$$f_{(\mathbf{G}_{\eta},\mathbf{T}^{\natural})}(\alpha_{\mathrm{res}}) = \begin{cases} f_{(\mathbf{G}_{\eta'},\mathbf{T}^{\natural})}(\alpha_{\mathrm{res}}) & \text{if } F_{\alpha} = F_{\alpha_{\mathrm{res}}}, \\ f_{(\mathbf{G}_{\eta'},\mathbf{T}^{\natural})}(\alpha_{\mathrm{res}}) \cdot \alpha(t) & \text{if } F_{\alpha} \neq F_{\alpha_{\mathrm{res}}}. \end{cases}$$

Here, in the latter case, we have $\alpha(t) = \pm 1$, hence regard $\alpha(t) \in \{\pm 1\} \subset \mathbb{C}^{\times}$.

Proof. We first consider the case where $F_{\alpha} = F_{\alpha_{\mathrm{res}}}$. In this case,

- $F_{\alpha}/F_{\pm\alpha}$ and $F_{\alpha_{\mathrm{res}}}/F_{\pm\alpha_{\mathrm{res}}}$ are quadratic unramified, and
- $F_{\pm\alpha} = F_{\pm\alpha_{\mathrm{res}}}$.

$$\begin{array}{ccc} F_{\alpha_{\mathrm{res}}} & = & F_{\alpha} \\ \text{quad}\ \Big|\ \text{ram} & & \text{ram}\ \Big|\ \text{quad} \\ F_{\pm\alpha_{\mathrm{res}}} & = & F_{\pm\alpha} \end{array}$$

Let $\tau_{\alpha} \in \Gamma$ be an element satisfying $\tau_{\alpha}(\alpha) = -\alpha$. Then τ_{α} restricts to the nontrivial element of $\mathrm{Gal}(F_{\alpha_{\mathrm{res}}}/F_{\pm\alpha_{\mathrm{res}}})$.

When $\theta_{\diamondsuit}(\alpha) = \alpha$, we get $f_{(\mathbf{G}_{\eta},\mathbf{T}^{\natural})}(\alpha_{\mathrm{res}}) = f_{(\mathbf{G},\mathbf{T}^{\diamondsuit})}(\alpha)$ simply because we can use the same root vector X_{α} both for computing $f_{(\mathbf{G}_{\eta},\mathbf{T}^{\natural})}(\alpha_{\mathrm{res}})$ and $f_{(\mathbf{G},\mathbf{T}^{\diamondsuit})}(\alpha)$. When $\theta_{\diamondsuit}(\alpha) \neq \alpha$, by taking X_{α} and $X_{\alpha_{\mathrm{res}},\eta}$ in the same way and applying the same argument as in the case where $\alpha_{\mathrm{res}} \in \Phi(\mathbf{G}, \mathbf{S})_{\mathrm{res},\mathrm{ram}}^{(\mathrm{ur})}$, we have

$$\begin{aligned} f_{(\mathbf{G}_{\eta},\mathbf{T}^{\natural})}(\alpha_{\mathrm{res}}) &= \kappa_{\alpha_{\mathrm{res}}}\left(\frac{[X_{\alpha_{\mathrm{res}},\eta}, \tau_{\alpha}(X_{\alpha_{\mathrm{res}},\eta})]}{H_{\alpha_{\mathrm{res}}}}\right) \\ &= \kappa_{\alpha_{\mathrm{res}}}\left(\frac{[X_{\alpha}, \tau_{\alpha}(X_{\alpha})] + [[\eta](X_{\alpha}), \tau_{\alpha}([\eta](X_{\alpha}))]}{H_{\alpha_{\mathrm{res}}}}\right) \\ &= \kappa_{\alpha}\left(\frac{[X_{\alpha}, \tau_{\alpha}(X_{\alpha})]}{H_{\alpha}}\right) = f_{(\mathbf{G},\mathbf{T}^{\diamondsuit})}(\alpha). \end{aligned}$$

Hence we get $f_{(\mathbf{G}_{\eta},\mathbf{T}^{\natural})}(\alpha_{\mathrm{res}}) = f_{(\mathbf{G},\mathbf{T}^{\diamondsuit})}(\alpha)$ regardless of whether $\theta_{\diamondsuit}(\alpha) = \alpha$ or not.

Since $f_{(\mathbf{G}_{\eta'},\mathbf{T}^{\natural})}(\alpha_{\mathrm{res}}) = f_{(\mathbf{G},\mathbf{T}^{\diamondsuit})}(\alpha)$ for the same reason, we get $f_{(\mathbf{G}_{\eta},\mathbf{T}^{\natural})}(\alpha_{\mathrm{res}}) = f_{(\mathbf{G}_{\eta'},\mathbf{T}^{\natural})}(\alpha_{\mathrm{res}})$.

We next consider the case where $F_\alpha/F_{\alpha_{\mathrm{res}}}$ is quadratic. We use the same notation as in the proof of Lemma 12.4. By the same usage of notation and arguments as in the case where $\alpha \in \Phi(\mathbf{G},\mathbf{S})^{(\mathrm{ur})}_{\mathrm{res,ram}}$, we get

$$f_{(\mathbf{G}_{\eta'},\mathbf{T}^\natural)}(\alpha_{\mathrm{res}}) = f_{(\mathbf{G}_{\eta},\mathbf{T}^\natural)}(\alpha_{\mathrm{res}}) \cdot \kappa_{\alpha_{\mathrm{res}}}(y \cdot \tau_\alpha(y)).$$

Recall that $y \in \mathcal{O}_{F_\alpha^\times}$ is an element such that $y/\sigma_\alpha(y) = \alpha(t)^{-1}$. In the present case (where $F_\alpha/F_{\pm\alpha}$ is ramified), since $\alpha(t) \in \mathrm{Ker}(\mathrm{Nr}_{F_\alpha/F_{\pm\alpha}})$, we have $\alpha(t) \equiv \pm 1 \pmod{\mathfrak{p}_{\overline{F}}}$. Then we can check that $\kappa_{\alpha_{\mathrm{res}}}(y \cdot \tau_\alpha(y))$ equals $+1$ or -1 according to $\alpha(t) \equiv +1$ or $\alpha(t) \equiv -1$. Hence we get

$$f_{(\mathbf{G}_{\eta'},\mathbf{T}^\natural)}(\alpha_{\mathrm{res}}) = f_{(\mathbf{G}_{\eta},\mathbf{T}^\natural)}(\alpha_{\mathrm{res}}) \cdot \alpha(t).$$

□

13. Some computation of transfer factors

In this section, we establish some formulas on transfer factors which will be needed in our proof of the twisted endoscopic character relation.

Let $(\gamma, \delta) \in \mathcal{D}$ and we fix $D = (\mathbf{B}^\flat, \mathbf{T}^\flat, \mathbf{B}^\diamondsuit, \mathbf{T}^\diamondsuit) \in \mathbf{D}(\gamma, \delta)$, which is unique up to equivalence by Lemma 10.7. Note that, in particular, we have $\gamma \in T^\flat$ and $\delta \in \tilde{T}^\diamondsuit$. We also fix a set $a = \{a_{\alpha_{\mathrm{res}}}\}_{\alpha_{\mathrm{res}}}$ of a-data and a set $\chi = \{\chi_{\alpha_{\mathrm{res}}}\}_{\alpha_{\mathrm{res}}}$ of minimally ramified χ-data for $\Phi_{\mathrm{res}}(\mathbf{G}, \mathbf{T}^\diamondsuit)$ (in the sense of Kaletha; Definition 7.3).

Let us suppose that $\delta \in \tilde{T}^\diamondsuit$ is elliptic strongly regular semisimple with a normal r-approximation $\delta = \delta_0\delta^+_{<r}\delta_{\geq r}$ (recall Definition 3.15). Then, by using the maps $\tilde{\xi}_D$ and ξ_D, we can associate a decomposition $\gamma = \gamma_0\gamma^+_{<r}\gamma_{\geq r}$ to γ by transferring the decomposition $\delta = \delta_0\delta^+_{<r}\delta_{\geq r}$.

Lemma 13.1. *The decomposition $\gamma = \gamma_0\gamma^+_{<r}\gamma_{\geq r}$ gives a normal r-approximation.*

Proof. This follows from that $\Phi(\mathbf{H}, \mathbf{T}^\flat)$ is identified with a subset of $\Phi_{\mathrm{res}}(\mathbf{G}, \mathbf{T}^\diamondsuit)$ (Section 12.1) and that $p \neq 2$. □

We put $\nu_0 := \tilde{\xi}_\diamondsuit(\delta_0) \cdot \theta^{-1}$ and $\nu_+ := \xi_\diamondsuit(\delta_+)$. We also put $\nu^+_{<r} := \xi_\diamondsuit(\delta^+_{<r})$ and $\nu_{\geq r} := \xi_\diamondsuit(\delta_{\geq r})$. Thus we have $\nu\theta = (\nu_0\theta) \cdot \nu^+_{<r} \cdot \nu_{\geq r}$.

13.1. First factor Δ_{I}.

Lemma 13.2. *For any $(\bar{\gamma}, \bar{\delta}) \in \mathcal{D}$ satisfying $\bar{\gamma} \in T^\flat$ and $\bar{\delta} \in \tilde{T}^\diamondsuit$, we have*

$$\Delta_{\mathrm{I}}[a,\chi](\gamma,\delta) = \Delta_{\mathrm{I}}[a,\chi](\bar{\gamma},\bar{\delta}).$$

Proof. By definition (see [KS99, Section 4.2]), the first factor $\Delta_{\mathrm{I}}(\bar{\gamma}, \bar{\delta})$ depends only on the F-rational (twisted) maximal tori of $\mathbf{H}$ and $\tilde{\mathbf{G}}$ containing $\bar{\gamma}$ and $\bar{\delta}$, respectively. Thus we get the assertion. □

13.2. Second factor Δ_{II}.

For any element $\delta' \in \tilde{T}^\diamondsuit$, we put

$$\Delta_{\mathrm{II}}^{\tilde{\mathbf{G}}}[a,\chi](\delta') := \prod_{\substack{\alpha_{\mathrm{res}} \in \dot{\Phi}_{\mathrm{res}}(\mathbf{G},\mathbf{T}^\diamondsuit) \\ N(\alpha)(\nu') \neq 1}} \chi_{\alpha_{\mathrm{res}}}\biggl(\frac{N(\alpha)(\nu')-1}{a_{\alpha_{\mathrm{res}}}}\biggr),$$

where $\nu' \in \mathbf{T}$ is the element such that $\tilde{\xi}_\diamondsuit(\delta') = \nu'\theta$.

The following is the twisted version of [Kal19b, Lemma 4.6.7].

Lemma 13.3. *If we put*

$$\chi(\delta_0) := \prod_{\alpha_{\mathrm{res}} \in \dot{\Phi}(\mathbf{G}_{\delta_0}, \mathbf{T}^{\natural})} \chi_{\alpha_{\mathrm{res}}}(l_\alpha),$$

then we have

$$\Delta_{\mathrm{II}}^{\tilde{\mathbf{G}}}[a,\chi](\delta) = \Delta_{\mathrm{II}}^{\tilde{\mathbf{G}}}[a,\chi](\delta_0) \cdot \Delta_{\mathrm{II}}^{\mathbf{G}_{\delta_0}}[a,\chi](\delta_+) \cdot \chi(\delta_0).$$

Proof. By definition, we have

$$\Delta_{\mathrm{II}}^{\tilde{\mathbf{G}}}[a,\chi](\delta_0) := \prod_{\substack{\alpha_{\mathrm{res}} \in \dot{\Phi}_{\mathrm{res}}(\mathbf{G}, \mathbf{T}^{\diamondsuit}) \\ N(\alpha)(\nu_0) \neq 1}} \chi_{\alpha_{\mathrm{res}}}\Bigl(\frac{N(\alpha)(\nu_0) - 1}{a_{\alpha_{\mathrm{res}}}}\Bigr),$$

and

$$\Delta_{\mathrm{II}}^{\mathbf{G}_{\delta_0}}[a,\chi](\delta_+) = \prod_{\substack{\alpha_{\mathrm{res}} \in \dot{\Phi}(\mathbf{G}_{\delta_0}, \mathbf{T}^{\natural}) \\ \alpha_{\mathrm{res}}(\nu_+) \neq 1}} \chi_{\alpha_{\mathrm{res}}}\Bigl(\frac{\alpha_{\mathrm{res}}(\nu_+) - 1}{a_{\alpha_{\mathrm{res}}}}\Bigr).$$

Let $\alpha_{\mathrm{res}} \in \Phi_{\mathrm{res}}(\mathbf{G}, \mathbf{T}^{\diamondsuit})$. Note that we have

$$N(\alpha)(\nu_0)^{\frac{2}{l_\alpha}} = \Bigl(\prod_{i=0}^{l_\alpha - 1} \theta^i(\alpha)(\nu_0)\Bigr)^{\frac{2}{l_\alpha}} = \prod_{i=0}^{1} \theta^i(\alpha)(\nu_0) = \alpha\bigl((\nu_0\theta)^2\bigr).$$

As $\nu_0\theta$ is of finite prime-to-p order modulo the center and $p \neq 2$, we see that $N(\alpha)(\nu_0)$ is a root of unity of prime-to-p-order in $\overline{F}$. Since we have

$$N(\alpha)(\nu) = N(\alpha)(\nu_0) \cdot N(\alpha)(\nu_+) = N(\alpha)(\nu_0) \cdot \alpha(\nu_+)^{l_\alpha},$$

and ν_+ is pro-unipotent, $N(\alpha)(\nu) \neq 1$ holds if and only if exactly one of the following holds:

- $N(\alpha)(\nu_0) \neq 1$, or
- $N(\alpha)(\nu_0) = 1$ and $\alpha(\nu_+)^{l_\alpha} \neq 1$ (the latter condition is furthermore equivalent to $\alpha(\nu_+) \neq 1$ as l_α is prime to p).

When $N(\alpha)(\nu_0) \neq 1$, by noting that $\chi_{\alpha_{\mathrm{res}}}$ is tamely ramified, we get

$$\chi_{\alpha_{\mathrm{res}}}\Bigl(\frac{N(\alpha)(\nu) - 1}{a_{\alpha_{\mathrm{res}}}}\Bigr) = \chi_{\alpha_{\mathrm{res}}}\Bigl(\frac{N(\alpha)(\nu_0) - 1}{a_{\alpha_{\mathrm{res}}}}\Bigr).$$

When $N(\alpha)(\nu_0) = 1$ and $\alpha(\nu_+) \neq 1$, we have

$$\chi_{\alpha_{\mathrm{res}}}\Bigl(\frac{N(\alpha)(\nu) - 1}{a_{\alpha_{\mathrm{res}}}}\Bigr) = \chi_{\alpha_{\mathrm{res}}}\Bigl(\frac{N(\alpha)(\nu_+) - 1}{a_{\alpha_{\mathrm{res}}}}\Bigr).$$

As we have $N(\alpha)(\nu_+) = \alpha(\nu_+)^{l_\alpha}$ and l_α is prime to p, we have

$$N(\alpha)(\nu_+) - 1 = \bigl(\alpha(\nu_+) - 1\bigr)\bigl(\alpha(\nu_+)^{l_\alpha - 1} + \cdots + 1\bigr) \in \bigl(\alpha(\nu_+) - 1\bigr) \cdot l_\alpha \cdot (1 + \mathfrak{p}_{F_\alpha}).$$

Hence again the tamely-ramifiedness of $\chi_{\alpha_{\mathrm{res}}}$ implies that

$$\chi_{\alpha_{\mathrm{res}}}\Bigl(\frac{N(\alpha)(\nu_+) - 1}{a_{\alpha_{\mathrm{res}}}}\Bigr) = \chi_{\alpha_{\mathrm{res}}}\Bigl(\frac{\alpha(\nu_+) - 1}{a_{\alpha_{\mathrm{res}}}}\Bigr) \cdot \chi_{\alpha_{\mathrm{res}}}(l_\alpha).$$

Recall that $\Phi(\mathbf{G}_{\delta_0}, \mathbf{T}^{\natural})$ is identified with the subset of $\Phi_{\mathrm{res}}(\mathbf{G}, \mathbf{T}^{\diamondsuit})$ consisting of restricted roots α_{res} satisfying $N(\alpha)(\nu_0) = 1$ (see Section 3.3). Thus, by noting that δ is strongly regular semisimple, hence no root $\alpha \in \Phi(\mathbf{G}_{\delta_0}, \mathbf{T}^{\natural})$ satisfies $\alpha(\nu_+) = 1$, we get the assertion (see Lemma 3.16). $\square$

The following will be a crucially important ingredient in our proof of the twisted endoscopic character relation.

Lemma 13.4. *The constant $\chi(\delta_0)$ in Lemma 13.3 does not depend on $\delta_0 \in \tilde{T}^{\diamondsuit}$.*

Proof. Recall that $\chi(\delta_0)$ is defined to be the product of $\chi_{\alpha_{\mathrm{res}}}(l_\alpha)$ over the set $\alpha_{\mathrm{res}} \in \dot{\Phi}(\mathbf{G}_{\delta_0}, \mathbf{T}^\natural)$. Since χ is minimally ramified, $\chi_{\alpha_{\mathrm{res}}}(l_\alpha)$ can be nontrivial only when α_{res} is ramified and $l_\alpha \neq 1$. However, the set of such restricted roots does not depend on δ_0 by Lemmas 12.2, 12.3, and 12.4. □

Lemma 13.5. *For any sufficiently large positive integer $m \in \mathbb{Z}_{>0}$, we have*

$$\Delta_{\mathrm{II}}[a,\chi](\gamma_{<r}\cdot\gamma_{\geq r}^{p^{2m}}, \delta_{<r}\cdot\delta_{\geq r}^{p^{2m}}) = \Delta_{\mathrm{II}}[a,\chi](\gamma_{<r}\cdot\gamma_{\geq r}, \delta_{<r}\cdot\delta_{\geq r}).$$

Proof. By Lemma 13.3, we have

$$\Delta_{\mathrm{II}}^{\tilde{\mathbf{G}}}[a,\chi](\delta_{<r}\cdot\delta_{\geq r}) = \Delta_{\mathrm{II}}^{\tilde{\mathbf{G}}}[a,\chi](\delta_0)\cdot\Delta_{\mathrm{II}}^{\mathbf{G}_{\delta_0}}[a,\chi](\delta_{<r}^{+}\cdot\delta_{\geq r})\cdot\chi(\delta_0),$$
$$\Delta_{\mathrm{II}}^{\tilde{\mathbf{G}}}[a,\chi](\delta_{<r}\cdot\delta_{\geq r}^{p^{2m}}) = \Delta_{\mathrm{II}}^{\tilde{\mathbf{G}}}[a,\chi](\delta_0)\cdot\Delta_{\mathrm{II}}^{\mathbf{G}_{\delta_0}}[a,\chi](\delta_{<r}^{+}\cdot\delta_{\geq r}^{p^{2m}})\cdot\chi(\delta_0).$$

According to the proof of [Kal19b, Lemma 6.3.3], we have

$$\Delta_{\mathrm{II}}^{\mathbf{G}_{\delta_0}}[a,\chi](\delta_{<r}^{+}\cdot\delta_{\geq r}) = \Delta_{\mathrm{II}}^{\mathbf{G}_{\delta_0}}[a,\chi](\delta_{<r}^{+}\cdot\delta_{\geq r}^{p^{2m}})$$

for any sufficiently large positive integer $m \in \mathbb{Z}_{>0}$, hence get $\Delta_{\mathrm{II}}^{\tilde{\mathbf{G}}}[a,\chi](\delta_{<r}\cdot\delta_{\geq r}) = \Delta_{\mathrm{II}}^{\tilde{\mathbf{G}}}[a,\chi](\delta_{<r}\cdot\delta_{\geq r}^{p^{2m}})$. Similarly, we have $\Delta_{\mathrm{II}}^{\mathbf{H}}[a,\chi](\gamma_{<r}\cdot\gamma_{\geq r}) = \Delta_{\mathrm{II}}^{\mathbf{H}}[a,\chi](\gamma_{<r}\cdot\gamma_{\geq r}^{p^{2m}})$. Since the second factor $\Delta_{\mathrm{II}}[a,\chi](\gamma,\delta)$ is defined to be the ratio of $\Delta_{\mathrm{II}}^{\tilde{\mathbf{G}}}[a,\chi](\delta)$ to $\Delta_{\mathrm{II}}^{\mathbf{H}}[a,\chi](\gamma)$, we get the assertion. □

13.3. **Third factor Δ_{III}.** Since we assume that $\mathbf{G}$ is quasi-split (and also fix a θ-stable splitting of $\mathbf{G}$), we have the absolute third factor $\Delta_{\mathrm{III}}[a,\chi](\gamma,\delta)$ given according to the manner of [KS99, Section 5.3], which satisfies

$$\Delta_{\mathrm{III}}[a,\chi](\gamma,\delta;\bar{\gamma},\bar{\delta}) = \Delta_{\mathrm{III}}[a,\chi](\gamma,\delta)/\Delta_{\mathrm{III}}[a,\chi](\bar{\gamma},\bar{\delta})$$

for any $(\bar{\gamma},\bar{\delta}) \in \mathcal{D}$. We review the construction of $\Delta_{\mathrm{III}}[a,\chi](\gamma,\delta;\bar{\gamma},\bar{\delta})$ following [KS99, Section 4.4]. We fix a diagram $\bar{D} = (\bar{\mathbf{B}}^\flat, \bar{\mathbf{T}}^\flat, \bar{\mathbf{B}}^\diamondsuit, \bar{\mathbf{T}}^\diamondsuit) \in \mathbf{D}(\bar{\gamma},\bar{\delta})$.

The relative third factor is given by using the following Take–Nakayama pairing for hyper-cohomology of tori (see [KS99, Appendix A]):

$$(22)\qquad \langle -,-\rangle_{\mathrm{TN}}\colon H^1(F, \mathbf{U}_0 \xrightarrow{1-\theta} \mathbf{S}_0) \times H^1(W_F, \hat{\mathbf{S}}_0 \xrightarrow{1-\hat{\theta}} \hat{\mathbf{U}}_0) \to \mathbb{C}^\times.$$

Let us recall the definitions of the tori $\mathbf{U}_0$ and $\mathbf{S}_0$.

We first note the following lemma, which is a rephrase of [KS99, Lemma 3.3.B]:

Lemma 13.6. *There exists a θ-stable F-rational maximal torus $\mathbf{T}_0$ and a θ-stable Borel subgroup $\mathbf{B}_0$ containing $\mathbf{T}_0$ such that the isomorphism $\mathbf{T}^\diamondsuit \to \mathbf{T}_0$ given by the Borel pairs $(\mathbf{B}^\diamondsuit, \mathbf{T}^\diamondsuit)$ and $(\mathbf{B}_0, \mathbf{T}_0)$ is F-rational.*

In the following, we fix $(\mathbf{B}_0, \mathbf{T}_0)$ and $(\bar{\mathbf{B}}_0, \bar{\mathbf{T}}_0)$ as in this lemma for $(\mathbf{B}^\diamondsuit, \mathbf{T}^\diamondsuit)$ and $(\bar{\mathbf{B}}^\diamondsuit, \bar{\mathbf{T}}^\diamondsuit)$. We then get canonical isomorphisms $\hat{\mathbf{T}}_0 \cong \hat{\mathbf{T}}$ and $\hat{\bar{\mathbf{T}}}_0 \cong \hat{\mathbf{T}}$ (not necessarily Γ-equivariant). We identify $\hat{\mathbf{T}}_0$ and $\hat{\bar{\mathbf{T}}}_0$ with $\hat{\mathbf{T}}$ via these isomorphisms but keep using the symbols $\hat{\mathbf{T}}_0$ and $\hat{\bar{\mathbf{T}}}_0$ in order to emphasize that their Galois actions are not the one coming from the Γ-action on $\hat{\mathbf{T}} \subset \hat{\mathbf{G}}$. We take $g_1 \in \mathbf{G}_{\mathrm{sc}}$ such that $[g_1](\mathbf{B}^\diamondsuit, \mathbf{T}^\diamondsuit) = (\mathbf{B}_0, \mathbf{T}_0)$, i.e., $[g_1]\colon \mathbf{T}^\diamondsuit \to \mathbf{T}_0$ is the F-rational isomorphism as in Lemma 13.6. Similarly, also for $(\bar{\mathbf{B}}^{\diamondsuit}, \bar{\mathbf{T}}^{\diamondsuit})$, we take $\bar{g}_1 \in \mathbf{G}_{\mathrm{sc}}$ realizing the F-rational isomorphism $[\bar{g}_1]\colon \bar{\mathbf{T}}^{\diamondsuit} \to \bar{\mathbf{T}}_0$.

In the following, the subscript "sc" denotes the preimage in the simply-connected cover $\mathbf{G}_{\mathrm{sc}}$ of the derived group of $\mathbf{G}$ and the subscript "ad" denotes the image in the adjoint group $\mathbf{G}_{\mathrm{ad}}$ of $\mathbf{G}$. We use a similar notation also for $\hat{\mathbf{G}}$. We define F-rational tori $\mathbf{S}_0$ and $\mathbf{U}_0$ by

$$\mathbf{S}_0 = (\mathbf{T}_0 \times \bar{\mathbf{T}}_0)/\Delta_{-}\mathbf{Z}_{\mathbf{G}} \cong \mathbf{T}_0 \times \bar{\mathbf{T}}_{0,\mathrm{ad}}\colon (t, \bar{t}) \mapsto (t\bar{t}, \bar{t}_{\mathrm{ad}}) \tag{23}$$

$$\mathbf{U}_0 = (\mathbf{T}_{0,\mathrm{sc}} \times \bar{\mathbf{T}}_{0,\mathrm{sc}})/\Delta_{-}\mathbf{Z}_{\mathbf{G}_{\mathrm{sc}}} \cong \mathbf{T}_{0,\mathrm{sc}} \times \bar{\mathbf{T}}_{0,\mathrm{ad}}\colon (t, \bar{t}) \mapsto (t\bar{t}, \bar{t}_{\mathrm{ad}}). \tag{24}$$

Here, $\Delta_{-}\mathbf{Z}_{\mathbf{G}_{(\mathrm{sc})}} := \{(z, z^{-1}) \mid z \in \mathbf{Z}_{\mathbf{G}_{(\mathrm{sc})}}\}$. Thus the dual tori are given by

$$\hat{\mathbf{S}}_0 \cong \hat{\mathbf{T}}_0 \times \hat{\bar{\mathbf{T}}}_{0,\mathrm{sc}} \quad \text{and} \quad \hat{\mathbf{U}}_0 \cong \hat{\mathbf{T}}_{0,\mathrm{ad}} \times \hat{\bar{\mathbf{T}}}_{0,\mathrm{sc}}.$$

We consider the homomorphism $1-\theta\colon \mathbf{T}_0 \to \mathbf{T}_0\colon t \mapsto t/\theta(t)$ and also its lift to $\mathbf{T}_{0,\mathrm{sc}}$. We define homomorphisms of $\mathbf{S}_0$ and $\mathbf{U}_0$, for which we again write $1-\theta$, to be the one induced by $(t, \bar{t}) \mapsto ((1-\theta)(t), (1-\theta)(\bar{t}))$. Then we get homomorphisms $1-\hat{\theta}$ on $\hat{\mathbf{S}}_0$ and $\hat{\mathbf{U}}_0$. Accordingly, the hyper-cohomology and the Tate–Nakayama pairing as in (22) makes sense. The relative third factor is defined by

$$\Delta_{\mathrm{III}}[a, \chi](\gamma, \delta; \bar{\gamma}, \bar{\delta}) := \langle \mathrm{inv}(\gamma, \delta; \bar{\gamma}, \bar{\delta}), \mathbf{A}\rangle_{\mathrm{TN}}^{-1}.$$

(Note that the right-hand side is inverted according to [KS12].) Thus let us next explain the constructions of $\mathrm{inv}(\gamma, \delta; \bar{\gamma}, \bar{\delta})$ and $\mathbf{A}$.

We first consider $\mathrm{inv}(\gamma, \delta; \bar{\gamma}, \bar{\delta})$. By putting $v_\sigma := g_1 \cdot \sigma(g_1)^{-1}$ and $\bar{v}_\sigma := \bar{g}_1 \cdot \sigma(\bar{g}_1)^{-1}$, we get a 1-cocycle $V\colon \Gamma \to \mathbf{S}_0$ which maps σ to the image of $(v_\sigma^{-1}, \bar{v}_\sigma) \in \mathbf{T}_0 \times \bar{\mathbf{T}}_0$ in $\mathbf{S}_0$. On the other hand, we put $\delta_0 \rtimes \theta := [g_1](\delta)$ and $\bar{\delta}_0 \rtimes \theta := [g_1](\bar{\delta})$ (thus $\delta_0, \bar{\delta}_0 \in \mathbf{T}_0$). We define an element $D \in \mathbf{S}_0$ to be the image of $(\delta_0, \bar{\delta}_0^{-1}) \in \mathbf{T}_0 \times \bar{\mathbf{T}}_0$ in $\mathbf{S}_0$. Then (V, D) forms a 1-hyper-cocycle. We let $\mathrm{inv}(\gamma, \delta; \bar{\gamma}, \bar{\delta})$ be the hyper-cohomology class of (V, D).

We next consider $\mathbf{A}$. We introduce two kinds of L-embeddings ${}^{L}j_\chi^1$ and ${}^{L}j_\chi^{\mathbf{H}}$.

(1) Let ${}^{L}\mathbf{G}^1 := \hat{\mathbf{G}}^1 \rtimes W_F$, where $\hat{\mathbf{G}}^1 := \hat{\mathbf{G}}^{\hat{\theta},\circ}$. If we put $\hat{\mathbf{T}}^1 := \hat{\mathbf{T}}^{\hat{\theta},\circ}$, then $\hat{\mathbf{G}}^1$ is a connected reductive group whose root system $\Phi(\hat{\mathbf{G}}^1, \hat{\mathbf{T}}^1)$ is regarded as a subset of $\Phi_{\mathrm{res}}(\hat{\mathbf{G}}, \hat{\mathbf{T}})$ (see Section 3.3). Since we fixed sets of a-data and χ-data for $\Phi_{\mathrm{res}}(\mathbf{G}, \mathbf{T}^{\diamondsuit})$, we also have sets of a-data and χ-data for $\Phi_{\mathrm{res}}(\hat{\mathbf{G}}, \hat{\mathbf{T}})$ which is equipped with a Γ-action derived from that of $\Phi_{\mathrm{res}}(\mathbf{G}, \mathbf{T}^{\diamondsuit})$. Hence, by the Langlands–Shelstad construction [LS87, Section 2.6], we obtain an L-embedding ${}^{L}j_\chi^1\colon \hat{\mathbf{T}}^1 \rtimes W_F \hookrightarrow {}^{L}\mathbf{G}^1$. Here, we emphasize that the Γ-action on $\hat{\mathbf{T}}^1$ is imported from that on $\hat{\mathbf{T}}_0$ through the isomorphism $\hat{\mathbf{T}}_0 \cong \hat{\mathbf{T}}$. Thus $\hat{\mathbf{T}}^1 \rtimes W_F$ is nothing but the L-group of the θ-coinvariant $\mathbf{T}_{0,\theta}$ of $\mathbf{T}_0$.

(2) On the other hand, as $\Phi(\mathbf{H}, \mathbf{T}^\flat)$ is regarded as a subset of $\Phi_{\mathrm{res}}(\mathbf{G}, \mathbf{T}^{\diamondsuit})$ (see Section 12.1), the fixed sets of a-data and χ-data also induce those for $\Phi(\mathbf{H}, \mathbf{T}^\flat)$. Hence, by the Langlands–Shelstad construction [LS87, Section 2.6], we obtain an L-embedding ${}^{L}j_\chi^{\mathbf{H}}\colon {}^{L}\mathbf{T}^\flat \hookrightarrow {}^{L}\mathbf{H}$.

Now we note that the homomorphism $\xi_D \circ [g_1]^{-1}\colon \mathbf{T}_0 \to \mathbf{T}^\flat$ is F-rational and induces an isomorphism $\mathbf{T}_{0,\theta} \cong \mathbf{T}^\flat$. Thus we can compare two L-embeddings ${}^{L}j_\chi^1$ and ${}^{L}j_\chi^{\mathbf{H}}$ through this identification $\mathbf{T}_{0,\theta} \cong \mathbf{T}^\flat$ and $\hat{\xi}\colon {}^{L}\mathbf{H} \hookrightarrow {}^{L}\mathbf{G}$. We define a map

$a_{\mathbf{T}_0}\colon \Gamma \to \hat{\mathbf{T}}^1 \colon \sigma \mapsto a_{\mathbf{T}_0,\sigma}$ by

$$\hat{\xi} \circ {}^L j_\chi^{\mathbf{H}}(1 \rtimes \sigma) = a_{\mathbf{T}_0,\sigma} \cdot {}^L j_\chi^1(1 \rtimes \sigma),$$

then $a_{\mathbf{T}_0}$ is regarded as a 1-cocycle $\Gamma \to \hat{\mathbf{T}}_0$ under the identification $\hat{\mathbf{T}}_0 \cong \hat{\mathbf{T}}$. We define a 1-cocycle $a_{\bar{\mathbf{T}}_0}\colon \Gamma \to \hat{\bar{\mathbf{T}}}_0$ in the same manner and put $A := (a_{\mathbf{T}_0}, a_{\bar{\mathbf{T}}_0}/a_{\mathbf{T}_0})\colon \Gamma \to \hat{\mathbf{S}}_0$. On the other hand, we take an element $s_{\mathrm{sc}} \in \hat{\mathbf{T}}_{\mathrm{sc}}$ having the same image in $\hat{\mathbf{T}}_{\mathrm{ad}}$ as $s \in \hat{\mathbf{T}}$ and write $s_{\mathbf{T}_0}$ and $s_{\bar{\mathbf{T}}_0}$ for its images in $\hat{\mathbf{T}}_{0,\mathrm{sc}}$ and $\hat{\bar{\mathbf{T}}}_{0,\mathrm{sc}}$, respectively. We put $s_{\mathbf{S}_0} := (s_{\mathbf{T}_0,\mathrm{ad}}, s_{\bar{\mathbf{T}}_0}/s_{\mathbf{T}_0}) \in \hat{\mathbf{U}}_0$. Then $(A^{-1}, s_{\mathbf{S}_0})$ forms a 1-hyper-cocycle. We let $\mathbf{A}$ be the hyper-cohomology class of $(A^{-1}, s_{\mathbf{S}_0})$.

The following proposition and its proof are inspired by [Mez13, Lemma 17].

Proposition 13.7. *Let $(\bar{\gamma}, \bar{\delta}) \in \mathcal{D}$ be another pair such that $D = (\mathbf{B}^\flat, \mathbf{T}^\flat, \mathbf{B}^\diamondsuit, \mathbf{T}^\diamondsuit) \in \mathbf{D}(\bar{\gamma}, \bar{\delta})$. Then we have*

$$\Delta_{\mathrm{III}}[a,\chi](\gamma,\delta;\bar{\gamma},\bar{\delta}) = \langle \delta/\bar{\delta}, a_{\mathbf{T}_\diamondsuit}\rangle_{\mathrm{TN}}.$$

Here, the pairing on the right-hand side is the Tate–Nakayama pairing for $\mathbf{T}^\diamondsuit$ and $a_{\mathbf{T}_\diamondsuit}$ is the 1-cocycle transported from $a_{\mathbf{T}_0}$ via the identification $[g_1]\colon \mathbf{T}_\diamondsuit \to \mathbf{T}_0$.

Proof. We examine the construction of $\Delta_{\mathrm{III}}[a,\chi](\gamma,\delta;\bar{\gamma},\bar{\delta})$ explained above by assuming that $\bar{D} = D$. We note that, under the identifications (23) and (24), we have $V_\sigma = (\bar{v}_\sigma/v_\sigma, \bar{v}_{\sigma,\mathrm{ad}})$ and $D = (\delta_0/\bar{\delta}_0, \bar{\delta}_{0,\mathrm{ad}}^{-1})$. Hence, as we have $v_\sigma = \bar{v}_\sigma$, we see that V and D are given by

$$V\colon \sigma \mapsto (1, v_{\sigma,\mathrm{ad}}) \quad \text{and} \quad D = (\delta_0/\bar{\delta}_0, \bar{\delta}_{0,\mathrm{ad}}^{-1}).$$

On the other hand, A and $s_{\mathbf{S}_0}$ are given by

$$A\colon \sigma \mapsto (a_{\mathbf{T}_0}, 1) \quad \text{and} \quad s_{\mathbf{S}_0} = (s_{\mathbf{T}_0,\mathrm{ad}}, 1).$$

We have the following commutative diagrams which are dual to each other:

$$\begin{array}{ccc}
\mathbf{T}_{0,\mathrm{sc}} & \xrightarrow{1-\theta} & \mathbf{T}_0 \\
\uparrow{\scriptstyle \mathrm{pr}_1} & & \uparrow{\scriptstyle \mathrm{pr}_1} \\
\mathbf{U}_0 \cong \mathbf{T}_{0,\mathrm{sc}} \times \mathbf{T}_{0,\mathrm{ad}} & \xrightarrow{1-\theta} & \mathbf{S}_0 \cong \mathbf{T}_0 \times \mathbf{T}_{0,\mathrm{ad}}
\end{array}$$

$$\begin{array}{ccc}
\hat{\mathbf{T}}_{0,\mathrm{ad}} & \xleftarrow{1-\hat{\theta}} & \hat{\mathbf{T}}_0 \\
\downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle i_1} \\
\hat{\mathbf{U}}_0 \cong \hat{\mathbf{T}}_{0,\mathrm{ad}} \times \hat{\mathbf{T}}_{0,\mathrm{sc}} & \xleftarrow{1-\hat{\theta}} & \hat{\mathbf{S}}_0 \cong \hat{\mathbf{T}}_0 \times \hat{\mathbf{T}}_{0,\mathrm{sc}}
\end{array}$$

Here, pr_1 and i_1 denote the first projection and the injection to the first entry, respectively. We note that $(A^{-1}, s_{\mathbf{S}_0})$ is the push-out of the 1-hyper-cocycle $(a_{\mathbf{T}_0}^{-1}, s_{\mathbf{T}_0,\mathrm{ad}})$ along the map i_1. Thus, since $(\mathrm{pr}_1(V), \mathrm{pr}_1(D)) = (1, \delta_0/\bar{\delta}_0)$, the functoriality of the Tate–Nakayama pairing (see [Wal08, Section 6.3]) implies that

$$\langle \mathrm{inv}(\gamma,\delta;\bar{\gamma},\bar{\delta}), \mathbf{A}\rangle_{\mathrm{TN}} = \langle (V,D), (A^{-1}, s_{\mathbf{S}_0})\rangle_{\mathrm{TN}} = \langle (1, \delta_0/\bar{\delta}_0), (a_{\mathbf{T}_0}^{-1}, s_{\mathbf{T}_0,\mathrm{ad}})\rangle_{\mathrm{TN}},$$

where the last pairing is the one for $(\mathbf{T}_{0,\mathrm{sc}} \xrightarrow{1-\theta}, \mathbf{T}_0, \hat{\mathbf{T}}_0 \xrightarrow{1-\hat{\theta}} \hat{\mathbf{T}}_{0,\mathrm{ad}})$.

We next note the following commutative diagrams which are dual to each other:

$$\begin{array}{ccc} \{1\} & \longrightarrow & \mathbf{T}_0 \\ \downarrow & & \downarrow \scriptstyle{\mathrm{id}} \\ \mathbf{T}_{0,\mathrm{sc}} & \xrightarrow{1-\theta} & \mathbf{T}_0 \end{array} \qquad\qquad \begin{array}{ccc} \{1\} & \longleftarrow & \hat{\mathbf{T}}_0 \\ \uparrow & & \uparrow \scriptstyle{\mathrm{id}} \\ \hat{\mathbf{T}}_{0,\mathrm{ad}} & \xleftarrow{1-\hat{\theta}} & \hat{\mathbf{T}}_0 \end{array}$$

Since $(1, \delta_0/\bar{\delta}_0)$ is obtained by the push-out from $1 \to \mathbf{T}_0$ and the image of $(a_{\mathbf{T}_0}^{-1}, s_{\mathbf{T}_0,\mathrm{ad}})$ in $\hat{\mathbf{T}}_0 \to \{1\}$ is $(a_{\mathbf{T}_0}^{-1}, 1)$, again the functoriality of the Tate–Nakayama pairing implies

$$\langle(1, \delta_0/\bar{\delta}_0), (a_{\mathbf{T}_0}^{-1}, s_{\mathbf{T}_0,\mathrm{ad}})\rangle_{\mathrm{TN}} = \langle \delta_0/\bar{\delta}_0, a_{\mathbf{T}_0}^{-1}\rangle_{\mathrm{TN}},$$

where the pairing on the right-hand side is the one for $(\{1\} \to \mathbf{T}_0, \hat{\mathbf{T}}_0 \to \{1\})$, which is nothing but the usual Tate–Nakayama pairing for $\mathbf{T}_0$ (see [KS99, A.3.13] and also [KS12, Section 4.3]). Finally, by noting that $\delta_0/\bar{\delta}_0 = [g_1](\delta/\bar{\delta})$, we get the assertion. □

Lemma 13.8. *For any positive integer $m \in \mathbb{Z}_{>0}$, we have*

$$\Delta_{\mathrm{III}}[a,\chi](\gamma_{<r} \cdot \gamma_{\geq r}^m, \delta_{<r} \cdot \delta_{\geq r}^m) = \Delta_{\mathrm{III}}[a,\chi](\gamma, \delta).$$

Proof. It suffices to show that the relative factor $\Delta_{\mathrm{III}}[a,\chi](\gamma_{<r} \cdot \gamma_{\geq r}^m, \delta_{<r} \cdot \delta_{\geq r}^m; \gamma, \delta)$ is trivial. By Proposition 13.7, this relative factor equals $\langle \delta_{\geq r}^{m-1}, a_{\mathbf{T}^\diamondsuit}\rangle_{\mathrm{TN}}$. Since we assume that χ is minimally ramified, the character $\langle -, a_{\mathbf{T}^\diamondsuit}\rangle_{\mathrm{TN}}$ of $T^\diamondsuit$ is tamely ramified. Thus we get $\langle \delta_{\geq r}^{m-1}, a_{\mathbf{T}^\diamondsuit}\rangle_{\mathrm{TN}} = 1$. □

13.4. **Fourth factor Δ_{IV}.** Recall from [KS99, Section 4.5] that the fourth factor $\Delta_{\mathrm{IV}}(\gamma, \delta)$ is defined by

$$\Delta_{\mathrm{IV}}(\gamma, \delta) := \Delta_{\mathrm{IV}}^{\tilde{\mathbf{G}}}(\delta)/\Delta_{\mathrm{IV}}^{\mathbf{H}}(\gamma),$$

where

$$\Delta_{\mathrm{IV}}^{\tilde{\mathbf{G}}}(\delta) := |\det([\delta] - 1 \mid \mathfrak{g}/\mathfrak{t}^\diamondsuit)|_F^{\frac{1}{2}} \quad \text{and} \quad \Delta_{\mathrm{IV}}^{\mathbf{H}}(\gamma) := |\det([\gamma] - 1 \mid \mathfrak{h}/\mathfrak{t}^\flat)|_F^{\frac{1}{2}}.$$

Recall that, since we are assuming that $\Phi_{\mathrm{res}}(\mathbf{G}, \mathbf{T}^\diamondsuit)$ does not contain a restricted root of type 2 or 3, we have

$$\Delta_{\mathrm{IV}}^{\tilde{\mathbf{G}}}(\delta) = \prod_{\alpha \in \Phi_{\mathrm{res}}(\mathbf{G}, \mathbf{T}^\diamondsuit)} |N(\alpha)(\nu) - 1|_F^{\frac{1}{2}}$$

(see [KS99, Section 4.5]). By noting this, we extend the definition of $\Delta_{\mathrm{IV}}^{\tilde{\mathbf{G}}}$ also for any semisimple element $\delta' \in \tilde{T}^\diamondsuit$ by

$$\Delta_{\mathrm{IV}}^{\tilde{\mathbf{G}}}(\delta') = \prod_{\substack{\alpha \in \Phi_{\mathrm{res}}(\mathbf{G}, \mathbf{T}^\diamondsuit) \\ N(\alpha)(\nu') \neq 1}} |N(\alpha)(\nu') - 1|_F^{\frac{1}{2}},$$

where $\nu' \in \mathbf{T}$ is the element such that $\tilde{\xi}_\diamondsuit(\delta') = \nu'\theta$. We define $\Delta_{\mathrm{IV}}^{\mathbf{H}}(\gamma')$ for any semisimple $\gamma' \in T^\flat$ in a similar way.

Lemma 13.9. *We have*

$$\Delta_{\mathrm{IV}}^{\tilde{\mathbf{G}}}(\delta) = \Delta_{\mathrm{IV}}^{\tilde{\mathbf{G}}}(\delta_{<r}) \cdot \Delta_{\mathrm{IV}}^{\mathbf{G}_{\delta_{<r}}}(\delta_{\geq r}) = |D_{G_{\eta_0}}^{\mathrm{red}}(\eta_+)|^{\frac{1}{2}} \cdot |D_{G_\eta}^{\mathrm{red}}(\log(\delta_{\geq r}))|^{\frac{1}{2}},$$

$$\Delta_{\mathrm{IV}}^{\mathbf{H}}(\gamma) = \Delta_{\mathrm{IV}}^{\mathbf{H}}(\gamma_{<r}) \cdot \Delta_{\mathrm{IV}}^{\mathbf{H}_{\gamma_{<r}}}(\gamma_{\geq r}).$$

Proof. We consider only $\Delta_{\mathrm{IV}}^{\tilde{\mathbf{G}}}(\delta)$ since the formula for $\Delta_{\mathrm{IV}}^{\mathbf{H}}(\gamma)$ can be showed by a simpler argument. By noting that the valuation of $N(\alpha)(\nu_{<r})-1$ is smaller than r when $N(\alpha)(\nu_{<r})\neq 1$, we have

$$|N(\alpha)(\nu)-1|_F^{\frac{1}{2}}=\begin{cases}|N(\alpha)(\nu_{<r})-1|_F^{\frac{1}{2}} & \text{if } N(\alpha)(\nu_{<r})\neq 1,\\ |N(\alpha)(\nu_{\geq r})-1|_F^{\frac{1}{2}} & \text{if } N(\alpha)(\nu_{<r})=1.\end{cases}$$

Since $\{\alpha\in\Phi_{\mathrm{res}}(\mathbf{G},\mathbf{T}^{\diamondsuit})\mid N(\alpha)(\nu_{<r})=1\}$ is identified with the set $\Phi(\mathbf{G}_{\delta_{<r}},\mathbf{T}^{\natural})$ (Section 3.3) and

$$|N(\alpha)(\nu_{\geq r})-1|_F^{\frac{1}{2}}=|\alpha(\nu_{\geq r})^{l_\alpha}-1|_F^{\frac{1}{2}}=|\alpha(\nu_{\geq r})-1|_F^{\frac{1}{2}}$$

(use that $l_\alpha=1,2$ and $p\neq 2$), we get $\Delta_{\mathrm{IV}}^{\tilde{\mathbf{G}}}(\delta)=\Delta_{\mathrm{IV}}^{\tilde{\mathbf{G}}}(\delta_{<r})\cdot\Delta_{\mathrm{IV}}^{\mathbf{G}_{\delta_{<r}}}(\delta_{\geq r})$.

By applying the same argument to $\Delta_{\mathrm{IV}}^{\tilde{\mathbf{G}}}(\delta_{<r})$, we also have a decomposition $\Delta_{\mathrm{IV}}^{\tilde{\mathbf{G}}}(\delta_{<r})=\Delta_{\mathrm{IV}}^{\tilde{\mathbf{G}}}(\delta_0)\cdot\Delta_{\mathrm{IV}}^{\mathbf{G}_{\delta_0}}(\delta_{<r}^{+})$. However, we have $|N(\alpha)(\nu_0)-1|_{\overline{F}}=1$ whenever $N(\alpha)(\nu_0)\neq 1$ since $N(\alpha)(\nu_0)$ is of prime-to-p order. Hence we get $\Delta_{\mathrm{IV}}^{\tilde{\mathbf{G}}}(\delta)=\Delta_{\mathrm{IV}}^{\mathbf{G}_{\delta_0}}(\delta_{<r}^{+})\cdot\Delta_{\mathrm{IV}}^{\mathbf{G}_{\delta_{<r}}}(\delta_{\geq r})$. This can be rewritten as $\Delta_{\mathrm{IV}}^{\tilde{\mathbf{G}}}(\delta)=|D_{G_{\eta_0}}^{\mathrm{red}}(\eta_+)|^{\frac{1}{2}}\cdot|D_{G_\eta}^{\mathrm{red}}(\delta_{\geq r})|^{\frac{1}{2}}$ by [DS18, Remark 2.12]. By also noting that $|D_{G_\eta}^{\mathrm{red}}(\delta_{\geq r})|=|D_{G_\eta}^{\mathrm{red}}(\log(\delta_{\geq r}))|$, we obtain the assertion. □

Lemma 13.10. *There exists a constant $d\in\mathbb{Z}_{\geq 0}$ determined by $\delta_{<r}$ such that, for any positive integer $m\in\mathbb{Z}_{>0}$, we have*

$$\Delta_{\mathrm{IV}}(\gamma_{<r}\cdot\gamma_{\geq r}^{p^m},\delta_{<r}\cdot\delta_{\geq r}^{p^m})=|p|_F^{md}\cdot\Delta_{\mathrm{IV}}(\gamma_{<r}\cdot\gamma_{\geq r},\delta_{<r}\cdot\delta_{\geq r}).$$

Proof. By Lemma 13.9, we have $\Delta_{\mathrm{IV}}^{\tilde{\mathbf{G}}}(\delta)=\Delta_{\mathrm{IV}}^{\tilde{\mathbf{G}}}(\delta_{<r})\cdot\Delta_{\mathrm{IV}}^{\mathbf{G}_{\delta_{<r}}}(\delta_{\geq r})$ and $\Delta_{\mathrm{IV}}^{\tilde{\mathbf{G}}}(\delta_{<r}\cdot\delta_{\geq r}^{p^m})=\Delta_{\mathrm{IV}}^{\tilde{\mathbf{G}}}(\delta_{<r})\cdot\Delta_{\mathrm{IV}}^{\mathbf{G}_{\delta_{<r}}}(\delta_{\geq r}^{p^m})$. On the other hand, by [Hal93, Lemma 3.1], we have $\Delta_{\mathrm{IV}}^{\mathbf{G}_{\delta_{<r}}}(\delta_{\geq r}^{p^m})=|p|_F^{m|\Phi(\mathbf{G}_{\delta_{<r}},\mathbf{T}^{\natural})|}\cdot\Delta_{\mathrm{IV}}^{\mathbf{G}_{\delta_{<r}}}(\delta_{\geq r})$ when $p>e_F+1$, which is assumed to hold (see the beginning of Section 11). Similarly, we have $\Delta_{\mathrm{IV}}^{\mathbf{H}}(\gamma_{<r}\cdot\gamma_{\geq r})=\Delta_{\mathrm{IV}}^{\mathbf{H}}(\gamma_{<r})\cdot\Delta_{\mathrm{IV}}^{\mathbf{H}}(\gamma_{\geq r})$, $\Delta_{\mathrm{IV}}^{\mathbf{H}}(\gamma_{<r}\cdot\gamma_{\geq r}^{p^m})=\Delta_{\mathrm{IV}}^{\mathbf{H}}(\gamma_{<r})\cdot\Delta_{\mathrm{IV}}^{\mathbf{H}}(\gamma_{\geq r}^{p^m})$, and $\Delta_{\mathrm{IV}}^{\mathbf{H}_{\gamma_{<r}}}(\gamma_{\geq r}^{p^m})=|p|_F^{m|\Phi(\mathbf{H}_{\gamma_{<r}},\mathbf{T}^{\flat})|}\cdot\Delta_{\mathrm{IV}}^{\mathbf{H}_{\gamma_{<r}}}(\gamma_{\geq r})$ when $p>e_F+1$. By putting all of these into together, we get the assertion. □

13.5. **Tail-scaling lemma on the full transfer factor.** By combining Lemmas 13.2, 13.5, 13.8, and 13.10, we get the following proposition, which is the twisted version of [Kal19b, Lemma 6.3.3].

Lemma 13.11. *For any sufficiently large positive integer $m\in\mathbb{Z}_{>0}$, we have*

$$\mathring{\Delta}(\gamma_{<r}\cdot\gamma_{\geq r}^{p^{2m}},\delta_{<r}\cdot\delta_{\geq r}^{p^{2m}})=\mathring{\Delta}(\gamma_{<r}\cdot\gamma_{\geq r},\delta_{<r}\cdot\delta_{\geq r})\quad\text{and}$$

$$\Delta(\gamma_{<r}\cdot\gamma_{\geq r}^{p^{2m}},\delta_{<r}\cdot\delta_{\geq r}^{p^{2m}})=|p|_F^{2md}\cdot\Delta(\gamma_{<r}\cdot\gamma_{\geq r},\delta_{<r}\cdot\delta_{\geq r})$$

with constant d as in Lemma 13.10.

An important consequence of this lemma is the following.

Proposition 13.12. *With the notation as in Theorem 11.3 and Corollary 11.4, for any D-norm pair $(Y,X)\in\mathfrak{h}_{y,0+}\times\mathfrak{g}_{\eta,0+}$, we have*

$$\Delta\big(y\exp(Y),\eta\exp(X)\big)=\Delta^D(\bar{Y},X_{\mathrm{sc}})\quad\text{and}$$

$$\mathring{\Delta}\big(y\exp(Y),\eta\exp(X)\big)\cdot\Delta_{\mathrm{IV}}(y,\eta)=\mathring{\Delta}^D(\bar{Y},X_{\mathrm{sc}})$$

Proof. By Lemma 13.11, we have

$$\Delta\big(y\exp(Y),\eta\exp(X)\big)=|p|_F^{-2md}\cdot\Delta(y\exp(p^{2m}Y),\eta\exp(p^{2m}X))$$

for any sufficiently large $m\in\mathbb{Z}_{>0}$. By taking m to be a sufficiently large integer so that $p^{2m}Y$ belongs to the set $\mathfrak{V}$ as in Corollary 11.4, we get

$$\Delta(y\exp(p^{2m}Y),\eta\exp(p^{2m}X))=\Delta^D(p^{2m}\bar{Y},p^{2m}X_{\mathrm{sc}})$$

by Corollary 11.4. By the homogeneity of the Lie algebra transfer factor, we have $\Delta^D(p^{2m}\bar{Y},p^{2m}X_{\mathrm{sc}})=|p|_F^{2md}\cdot\Delta^D(\bar{Y},X_{\mathrm{sc}})$ (see [Wal97, Section 2.3] and [Hal93, Section 10]). Thus we get the first equality.

By Lemma 13.9, we have $\Delta_{\mathrm{IV}}(\gamma,\delta)=\Delta_{\mathrm{IV}}(y,\eta)\cdot\Delta_{\mathrm{IV}}^{\mathbf{G}_\eta}(\delta_{\geq r})\cdot\Delta_{\mathrm{IV}}^{\mathbf{H}_y}(\gamma_{\geq r})^{-1}$. Hence, by noting $\Delta_{\mathrm{IV}}^{\mathbf{G}_\eta}(\delta_{\geq r})\cdot\Delta_{\mathrm{IV}}^{\mathbf{H}_y}(\gamma_{\geq r})^{-1}=\Delta_{\mathrm{IV}}^D(\bar{Y},X_{\mathrm{sc}})$, we get the second equality. □

14. Twisted endoscopic character relation

14.1. **Twisted endoscopic character relation.** We assume that $(\mathbf{S},\hat{\jmath},\chi,\vartheta)$ in a toral supercuspidal L-packet datum of $\mathbf{G}$ whose L-parameter ϕ factors though the L-embedding $\hat{\xi}$ for an endoscopic data $(\mathbf{H},{}^L\mathbf{H},s,\hat{\xi})$ (i.e., we are in the situation as in Section 9.2). Here, by replacing $(\mathbf{S},\hat{\jmath},\chi,\vartheta)$ with its isomorphic data if necessary, we may assume that $\chi=\chi_{\vartheta_{\hat{\jmath}}}$ (see Section 7.3.3). As in the manner of Section 9.5, we get a toral supercuspidal L-packet datum $(\mathbf{S}_{\mathbf{H}},\hat{\jmath}_{\mathbf{H}},\chi_{\mathbf{H}},\vartheta_{\mathbf{H}})$. Similarly, we may assume that $\chi_{\mathbf{H}}=\chi_{\vartheta_{\hat{\jmath}_{\mathbf{H}}}}$. Let $\Pi_\phi^{\mathbf{G}}$ (resp. $\Pi_{\phi_{\mathbf{H}}}^{\mathbf{H}}$) denote the associated toral supercuspidal L-packet of $\mathbf{G}$ (resp. $\mathbf{H}$).

The aim of this section is to establish the following in some special cases:

Expectation 14.1. *For each $\pi\in\Pi_\phi^{\mathbf{G}}$ there exists a constant $\Delta_{\phi,\pi}^{\mathrm{spec}}\in\mathbb{C}$ such that the following identity holds for any elliptic strongly regular semisimple $\delta\in\tilde{G}$:*

$$\sum_{\pi\in\Pi_\phi^{\mathbf{G}}}\Delta_{\phi,\pi}^{\mathrm{spec}}\Theta_{\tilde{\pi}}(\delta)=\sum_{\gamma\in H/\mathrm{st}}\frac{\Delta_{\mathrm{IV}}^{\mathbf{H}}(\gamma)^2}{\Delta_{\mathrm{IV}}^{\tilde{\mathbf{G}}}(\delta)^2}\Delta(\gamma,\delta)\sum_{\pi_{\mathbf{H}}\in\Pi_{\phi_{\mathbf{H}}}^{\mathbf{H}}}\Theta_{\pi_{\mathbf{H}}}(\gamma),$$

or equivalently,

$$\sum_{\pi\in\Pi_\phi^{\mathbf{G}}}\Delta_{\phi,\pi}^{\mathrm{spec}}\Phi_{\tilde{\pi}}(\delta)=\sum_{\gamma\in H/\mathrm{st}}\mathring{\Delta}(\gamma,\delta)\sum_{\pi_{\mathbf{H}}\in\Pi_{\phi_{\mathbf{H}}}^{\mathbf{H}}}\Phi_{\pi_{\mathbf{H}}}(\gamma),\tag{25}$$

where the first sum on the right-hand sides is over the stable conjugacy classes of strongly $\mathbf{G}$-regular semisimple elements of H and we put $\Phi_{\tilde{\pi}}(\delta):=\Delta_{\mathrm{IV}}^{\tilde{\mathbf{G}}}(\delta)\cdot\Theta_{\tilde{\pi}}(\delta)$ and $\Phi_{\pi_{\mathbf{H}}}(\gamma):=\Delta_{\mathrm{IV}}^{\mathbf{H}}(\gamma)\cdot\Theta_{\pi_{\mathbf{H}}}(\gamma)$.

14.2. **Several preliminary considerations.**

14.2.1. *Initial observation on the index sets.* In the following, we fix an elliptic strongly regular semisimple element $\delta\in\tilde{G}$ and also fix a normal r-approximation $\delta=\delta_0\delta_{<r}^{+}\delta_{\geq r}$ in the sense of Definition 3.15 (recall that we can always find a normal r-approximation by Proposition 3.17). We let η denote $\delta_{<r}\in\tilde{G}_{\mathrm{ss}}$. We take a set $\mathfrak{H}_\eta\subset H_{\mathrm{ss}}$ as in Section 10.3, i.e., $\mathfrak{H}_\eta$ is a set of representatives for the stable conjugacy classes of semisimple elements of H such that y corresponds to η and $\mathbf{H}_y$ is quasi-split for any $y\in\mathfrak{H}_\eta$.

Recall that the θ-stable members of $\Pi_\phi^{\mathbf{G}}$ are parametrized by

$$\tilde{\mathcal{J}}_G^{\mathbf{G}}:=\{j\colon\tilde{\mathbf{S}}\hookrightarrow\tilde{\mathbf{G}}\mid j\text{ is }F\text{-rational and }j\sim_{\mathbf{G}}\jmath^{-1}\}/{\sim_G}.$$

More precisely, for each $j \in \tilde{\mathcal{J}}_G^{\mathbf{G}}$, the corresponding member is given to be the toral supercuspidal representation (let us write π_j) arising from the tame elliptic regular pair $(\mathbf{S}_j, \vartheta'_j)$ of $\mathbf{G}$ (see Section 7.3.3). According to the twisted character formula (Theorem 6.10), the twisted character $\Phi_{\tilde{\pi}_j}(\delta)$ is expressed by a sum over the set $\{g \in S_j \backslash G/G_\eta \mid {}^g\eta \in \tilde{S}_j\}$. For each $j \in \tilde{\mathcal{J}}_G^{\mathbf{G}}$, we put

$$\tilde{\mathcal{J}}_{G_\eta}^G(j) := \{k \colon \tilde{\mathbf{S}} \hookrightarrow \tilde{\mathbf{G}} \mid k \text{ is } F\text{-rational, } k \sim_G j, \text{ and } \eta \in \tilde{S}_k\}/{\sim_{G_\eta}}.$$

Lemma 14.2. *We have a bijection*

$$\{g \in S_j \backslash G/G_\eta \mid {}^g\eta \in \tilde{S}_j\} \xleftrightarrow{1:1} \tilde{\mathcal{J}}_{G_\eta}^G(j) \colon g \mapsto [g^{-1}] \circ j.$$

Proof. It suffices to check that the surjective map from $\{g \in S_j \backslash G \mid {}^g\eta \in \tilde{S}_j\}$ to $\{k \colon \tilde{\mathbf{S}} \hookrightarrow \tilde{\mathbf{G}} \mid k \text{ is } F\text{-rational, } k \sim_G j, \text{ and } \eta \in \tilde{S}_k\}$ given by $g \mapsto [g]^{-1} \circ j$ is in fact injective. (Then we can get the assertion by taking the quotient by G_η.) Let us suppose that two elements g and g' of G map to the same element, i.e., we have $[g]^{-1} \circ j = [g']^{-1} \circ j$. Then $g'g^{-1}$ belongs to S_j, hence g and g' belong to the same double coset. Hence the map in the assertion is injective. $\square$

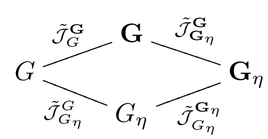
By this lemma, the $\mathbf{G}$-side $\sum_{\pi \in \Pi_\phi^{\mathbf{G}}} \Delta^{\mathrm{spec}}(\phi, \pi)\Theta_{\tilde{\pi}}(\delta)$ of the twisted endoscopic character relation (25) can be written as a double sum over the sets $\tilde{\mathcal{J}}_G^{\mathbf{G}}$ and $\tilde{\mathcal{J}}_{G_\eta}^G(j)$ (for $j \in \tilde{\mathcal{J}}_G^{\mathbf{G}}$). We rearrange this double as follows. We first combine $\tilde{\mathcal{J}}_G^{\mathbf{G}}$ and $\tilde{\mathcal{J}}_{G_\eta}^G(j)$ (for $j \in \tilde{\mathcal{J}}_G^{\mathbf{G}}$) into the following single set:

$$\tilde{\mathcal{J}}_{G_\eta}^{\mathbf{G}} := \{k \colon \tilde{\mathbf{S}} \hookrightarrow \tilde{\mathbf{G}} \mid k \text{ is } F\text{-rational, } k \sim_{\mathbf{G}} \jmath^{-1}, \text{ and } \eta \in \tilde{S}_k\}/{\sim_{G_\eta}}.$$

Then we again divide $\mathcal{J}_{G_\eta}^{\mathbf{G}}$ into the sets $\tilde{\mathcal{J}}_{\mathbf{G}_\eta}^{\mathbf{G}}$ and $\tilde{\mathcal{J}}_{G_\eta}^{\mathbf{G}_\eta}(j)$ (for $j \in \tilde{\mathcal{J}}_{\mathbf{G}_\eta}^{\mathbf{G}}$), where

- $\tilde{\mathcal{J}}_{\mathbf{G}_\eta}^{\mathbf{G}} := \{j \colon \tilde{\mathbf{S}} \hookrightarrow \tilde{\mathbf{G}} \mid j \text{ is } F\text{-rational, } j \sim_{\mathbf{G}} \jmath^{-1}, \text{ and } \eta \in \tilde{S}_j\}/{\sim_{\mathbf{G}_\eta}}$,
- $\tilde{\mathcal{J}}_{G_\eta}^{\mathbf{G}_\eta}(j) := \{k \colon \tilde{\mathbf{S}} \hookrightarrow \tilde{\mathbf{G}} \mid k \text{ is } F\text{-rational, } k \sim_{\mathbf{G}_\eta} j, \text{ and } \eta \in \tilde{S}_k\}/{\sim_{G_\eta}}$.

In the following arguments, we fix representatives of these sets and loosely identify these sets with the fixed sets of representatives.



Keeping this observation in mind, let us first consider the case where

$$\mathbf{D}(y, \eta) = \varnothing \text{ for any } y \in \mathfrak{H}_\eta.$$

In this case, by Proposition 10.11 (and Remark 10.12), $\tilde{\mathcal{J}}_{\mathbf{G}_\eta}^{\mathbf{G}}$ is necessarily empty. This implies that the $\mathbf{G}$-side of the twisted endoscopic character relation contains a sum over the empty set, hence equals 0. On the other hand, we see that also the $\mathbf{H}$-side equals 0 by the following lemma:

Lemma 14.3. *If $\mathbf{D}(y, \eta) = \varnothing$ for any $y \in \mathfrak{H}_\eta$, then there is no norm of δ in H.*

Proof. For the sake of contradiction, let us suppose that there exists a norm $\gamma \in H$ of δ. Then we have a diagram $D \in \mathbf{D}(\gamma, \delta)$ associated to (γ, δ) by Lemma 10.7. If we put $\epsilon := \tilde{\xi}_D(\eta)$, D is also a diagram associated to (ϵ, η). Then, by the same argument as in the proof of the surjectivity part of Proposition 10.11, we can construct a unique $y \in \mathfrak{H}_\eta$ and a diagram D' associated to (y, η) by modifying D via $\mathbf{H}$-conjugacy. Thus we get a contradiction. $\square$

Therefore, in the following, we focus on an(y) elliptic strongly regular semisimple element $\delta \in \mathbf{G}$ such that

$$\mathbf{D}(y, \eta) \neq \varnothing \text{ for some } y \in \mathfrak{H}_\eta.$$

Remark 14.4. This argument shows that if there does not exist an elliptic strongly regular semisimple element $\delta \in \mathbf{G}$ such that $\mathbf{D}(y, \eta) \neq \varnothing$ for some $y \in \mathfrak{H}_\eta$, then there is nothing to prove anymore. In this case, we simply put $\Delta^{\mathrm{spec}}(\phi, \pi) := 0$ for any $\pi \in \Pi_\phi^{\mathbf{G}}$.

14.2.2. *Head-tail stratification of the endoscopic index set.* When $\gamma \in H$ is a norm of $\delta \in \tilde{G}$, we take a diagram $D \in \mathbf{D}(\gamma, \delta)$ and consider the associated map $\tilde{\xi}_D$. According to Lemma 10.7, such a diagram always exists uniquely up to equivalence and the map $\tilde{\xi}_D$ is independent of the choice of D. By noting this, we put $\gamma_i := \tilde{\xi}_D(\delta_i)$ for $i \in \mathbb{R}_{\geq 0}$. Then we get a normal r-approximation $\gamma = \gamma_{<r} \cdot \gamma_{\geq r}$ (see Lemma 13.1). Note that the r-approximation to γ induced from that to δ in this way is consistent with respect to the stable $\mathbf{H}$-conjugacy. More precisely, for any norms $\gamma \in \mathbf{H}$ and $\bar{\gamma} \in \mathbf{H}$ of δ which are stably conjugate by $h \in \mathbf{H}$ (say $\bar{\gamma} = {}^h\gamma$), we have $\bar{\gamma}_i = {}^h\gamma_i$.

For each $y \in \mathfrak{H}_\eta$, we put

$$H_y[\delta]_r := \left\{ z \in H_{y,\mathrm{srs}} \;\middle|\; \begin{array}{l} yz \in H \text{ is a norm of } \delta, \\ y \cdot z \text{ is the fixed normal } r\text{-approximation to } yz \end{array} \right\}.$$

Lemma 14.5 ([Kal15, Lemma 6.4])**.** *The map*

$$\bigsqcup_{y \in \mathfrak{H}_\eta} H_y[\delta]_r/{\sim_{\mathbf{H}_y}} \to \{\gamma \in H_{\mathrm{srs}} \mid \gamma \text{ is a norm of } \delta\}/{\sim_{\mathbf{H}}} \colon z \mapsto yz$$

is a $\pi_0(\mathbf{H}^y)(F)$-torsor on each disjoint summand $H_y[\delta]_r/{\sim_{\mathbf{H}_y}}$ (onto its image). Furthermore, the induced map

$$\bigsqcup_{y \in \mathfrak{H}_\eta} (H_y[\delta]_r/{\sim_{\mathbf{H}_y}})/\pi_0(\mathbf{H}^y)(F) \to \{\gamma \in H_{\mathrm{srs}} \mid \gamma \text{ is a norm of } \delta\}/{\sim_{\mathbf{H}}} \colon z \mapsto yz$$

is bijective. Here, $\sim_{\mathbf{H}_y}$ on the left-hand side (resp. $\sim_{\mathbf{H}}$ on the right-hand side) denotes the stable conjugacy in $\mathbf{H}_y$ (resp. $\mathbf{H}$).

Proof. The well-definedness of the map is obvious.

We first show the surjectivity of the map. Let $\gamma \in H_{\mathrm{srs}}$ be a norm of δ. Then, according to Lemma 10.7, there exists a diagram $D \in \mathbf{D}(\gamma, \delta)$ unique up to equivalence. We put $\epsilon := \tilde{\xi}_D(\eta)$. Then, by the definition of $\mathfrak{H}_\eta$, there uniquely exists a $y \in \mathfrak{H}_\eta$ which is stably $\mathbf{H}$-conjugate to ϵ. Let us take an element $h \in \mathbf{H}$ giving this stable conjugacy, that is, $[h](\epsilon) = y$ and $\sigma(h)^{-1}h \in \mathbf{H}_\epsilon$ for any $\sigma \in \Gamma$. Then the map $[h]$ gives an inner twist between $\mathbf{H}_\epsilon$ and the quasi-split connected reductive group $\mathbf{H}_y$. Since any maximal torus defined over F transfers to the quasi-split inner form, we may suppose that the map $[h] \colon \mathbf{H}_\epsilon \to \mathbf{H}_y$ induces an F-rational isomorphism from $\mathbf{H}_\gamma$ (this is a maximal torus of $\mathbf{H}_\epsilon$) to a maximal torus of $\mathbf{H}_y$. Then $z := [h](\xi_D(\delta_{\geq r}))$ is an element of $\mathbf{H}_y$ such that $yz = [h](\gamma)$. This means that the map in the assertion is surjective.

We next investigate the fibers of the map. Suppose that we have $z \in H_y[\delta]_r$ and $\bar{z} \in H_{\bar{y}}[\delta]_r$ for $y, \bar{y} \in \mathfrak{H}_\eta$ such that yz and $\bar{y}\bar{z}$ are stably $\mathbf{H}$-conjugate. Let

$h \in \mathbf{H}$ be an element giving the stable conjugacy, i.e., $[h](yz) = \bar{y}\bar{z}$. As we took the normal r-approximations to be consistent with the stable conjugacy, this implies that $[h](y) = \bar{y}$. Then, by the definition of $\mathfrak{H}_\eta$, we get $y = \bar{y}$, hence $h \in \mathbf{H}^y$. Since we also have $[h](z) = \bar{z}$, we know that z and $\bar{z}$ are conjugate under the action of $\pi_0(\mathbf{H}^y)(F)$ by [Kal15, Lemma 6.3]. $\square$

Lemma 14.6. *We put $X := \log(\delta_{\geq r}) \in \mathfrak{g}_\eta$. Let $z \in H_y[\delta]_r$ and we put $Y := \log(z) \in \mathfrak{h}_y$. There exists a diagram $D \in \mathbf{D}(y, \eta)$ uniquely up to equivalence such that (Y, X) is a D-norm pair.*

Proof. By the definition of the set $H_y[\delta]_r$, yz is a norm of δ and $y \cdot z$ is the fixed r-approximation to yz. Hence, according to our choice of normal r-approximations, there exists a diagram $D = (\mathbf{B}^\flat, \mathbf{T}^\flat, \mathbf{B}^\diamondsuit, \mathbf{T}^\diamondsuit) \in \mathbf{D}(y, \eta)$ satisfying $\tilde{\xi}_D(\eta) = y$ and $\xi_D(\delta_{\geq r}) = z$. This implies that (Y, X) is a D-norm pair.

To check the uniqueness of D, let us suppose that (Y, X) is a $\bar{D}$-norm pair for another diagram $\bar{D} = (\bar{\mathbf{B}}^\flat, \bar{\mathbf{T}}^\flat, \bar{\mathbf{B}}^\diamondsuit, \bar{\mathbf{T}}^\diamondsuit) \in \mathbf{D}(y, \eta)$. Then, by replacing $\bar{D}$ with its equivalent diagram appropriately, we may assume that $\bar{\mathbf{T}}^\flat = \mathbf{T}^\flat$ and $\bar{\mathbf{T}}^\diamondsuit = \mathbf{T}^\diamondsuit$ and that $\xi_D(\exp(X)) = \exp(Y) = \xi_{\bar{D}}(\exp(X))$ (cf. the argument in the proof of Lemma 10.8). This implies that both D and $\bar{D}$ belong to $\mathbf{D}(y\exp(Y), \eta\exp(X))$. Thus, by Lemma 10.7, D and $\bar{D}$ are equivalent in $\mathbf{D}(y\exp(Y), \eta\exp(X))$, hence also in $\mathbf{D}(y, \eta)$. $\square$

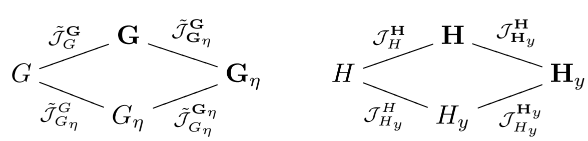
By the invariance of the logarithm map, Lemma 14.6 implies the following.

Lemma 14.7. *The association $z \mapsto \log(z)$ induces a bijection*

$$H_y[\delta]_r/{\sim_{\mathbf{H}_y}} \xrightarrow{1:1} \bigsqcup_{D \in \mathbb{D}(y,\eta)} \{Y \overset{D}{\longleftrightarrow} X\}/{\sim_{\mathbf{H}_y}},$$

where the left-hand side denotes the set of $\mathbf{H}_y$-conjugacy classes of elements of $H_y[\delta]_r$ and the right-hand side denotes the set of $\mathbf{H}_y$-conjugacy classes of elements $Y \in \mathfrak{h}_y$ which constitute a D-norm pair with $X := \log(\delta_{\geq r})$ (over $D \in \mathbb{D}(y, \eta)$).

14.2.3. *Lie algebra transfer: revisited.* We introduce the sets $\mathcal{J}_H^{\mathbf{H}}$, $\mathcal{J}_{H_y}^{H}$, $\mathcal{J}_{\mathbf{H}_y}^{\mathbf{H}}$ and $\mathcal{J}_{H_y}^{\mathbf{H}_y}$ to rearrange the index sets on the $\mathbf{H}$-side of (25) in a similar manner to Section 14.2.1.



Recall that, by Proposition 10.11, we have a bijective map

$$\mathfrak{tran}\colon \bigsqcup_{y \in \mathfrak{H}_\eta} \left(\mathbb{D}(y, \eta) \times \mathcal{J}_{\mathbf{H}_y}^{\mathbf{H}}\right)/\pi_0(\mathbf{H}^y)(F) \to \tilde{\mathcal{J}}_{\mathbf{G}_\eta}^{\mathbf{G}}.$$

Suppose that $D \in \mathbf{D}(y, \eta)$, $j_{\mathbf{H}} \in \mathcal{J}_{\mathbf{H}_y}^{\mathbf{H}}$, and $j \in \tilde{\mathcal{J}}_{\mathbf{G}_\eta}^{\mathbf{G}}$ satisfy $\mathfrak{tran}(D, j_{\mathbf{H}}) = j$.

We also recall that, in Section 9, we fixed $X^* \in \mathfrak{s}^*$ and $Y^* \in \mathfrak{s}_{\mathbf{H}}^*$, which are the elements realizing the toral characters ϑ and $\vartheta_{\mathbf{H}}$, respectively (see Corollary 9.21). In the following, for $k \in \mathcal{J}^{\mathbf{G}}$, we put

$$X_k^* := (dk^*)^{-1}(X^*) \in \mathfrak{s}_k^* \hookrightarrow \mathfrak{g}^*$$

(note that $dk\colon \mathfrak{s} \cong \mathfrak{s}_k$, hence $dk^*\colon \mathfrak{s}_k^* \cong \mathfrak{s}^*$). This can be also thought of as an element of $\mathfrak{s}_k^*$ representing the character $\vartheta_k|_{S_{k,r}} = \vartheta_k'|_{S_{k,r}}$. Note that when k

belongs to $\tilde{\mathcal{J}}^{\mathbf{G}}_{G_\eta}$, we also have an element $X^*_k \in \mathfrak{s}^{\natural *}_k \hookrightarrow \mathfrak{g}^*_\eta$, which can be thought of as the image of the above X^*_k under the natural map $\mathfrak{s}^*_k \twoheadrightarrow \mathfrak{s}^{\natural *}_k$. Similarly, for $k_{\mathbf{H}} \in \mathcal{J}^{\mathbf{H}}$, we put

$$Y^*_{k_{\mathbf{H}}} := (dk^*_{\mathbf{H}})^{-1}(Y^*) \in \mathfrak{s}^*_{k_{\mathbf{H}}} \hookrightarrow \mathfrak{h}^*.$$

By the construction of the map $\mathfrak{tran}$, the maps $\xi_{\mathbf{S}}$ and ξ_D coincide under the embeddings j and $j_{\mathbf{H}}$, i.e., $\xi_D \circ j = j_{\mathbf{H}} \circ \xi_{\mathbf{S}}$. This implies that $d\xi^*_D(Y^*_{j_{\mathbf{H}}}) = X^*_j \in (\mathfrak{t}^{\diamondsuit *})^{\theta_\diamondsuit}$.

Lemma 14.8. (1) *The set $\{Y^*_{k_{\mathbf{H}}} \mid k_{\mathbf{H}} \in \mathcal{J}^{\mathbf{H}_y}_{H_y}(j_{\mathbf{H}})\}$ represents the H_y-conjugacy classes within a stable $\mathbf{H}_y$-conjugacy class.*

(2) *The set $\{X^*_k \mid k \in \tilde{\mathcal{J}}^{\mathbf{G}_\eta}_{G_\eta}(j)\}$ represents the G_η-conjugacy classes of elements of $\mathfrak{g}_{\eta,0+}$ constituting a D-norm pair with $Y^*_{k_{\mathbf{H}}}$ for a(ny) $k_{\mathbf{H}} \in \mathcal{J}^{\mathbf{H}_y}_{H_y}(j_{\mathbf{H}})$.*

Proof. The assertion (1) is obvious by the definitions of $Y^*_{k_{\mathbf{H}}}$ and $\mathcal{J}^{\mathbf{H}_y}_{H_y}(j_{\mathbf{H}})$. Since we have $d\xi^*_D(Y^*_{j_{\mathbf{H}}}) = X^*_j$, $(Y^*_{j_{\mathbf{H}}}, X^*_j)$ is a D-norm pair. Noting that all elements of $\mathfrak{g}_{\eta,0+}$ constituting a norm pair with $Y^*_{k_{\mathbf{H}}}$ are $\mathbf{G}_\eta$-conjugate, the assertion (2) follows. □

Lemma 14.8 enables us to rewrite Proposition 11.8 as follows:

Proposition 14.9. *We have*

$$\sum_{Y \overset{D}{\leftrightarrow} X/\sim_{\mathbf{H}_y}} \mathring{\Delta}^D(\bar{Y}, X_{\mathrm{sc}}) \sum_{k_{\mathbf{H}} \in \mathcal{J}^{\mathbf{H}_y}_{H_y}(j_{\mathbf{H}})} D^{\mathbf{H}_y}_{Y^*_{k_{\mathbf{H}}}}(Y) = \sum_{k \in \tilde{\mathcal{J}}^{\mathbf{G}_\eta}_{G_\eta}(j)} \mathring{\Delta}^D(\bar{Y}^*_{j_{\mathbf{H}}}, X^*_{k,\mathrm{sc}}) D^{\mathbf{G}_\eta}_{X^*_k}(X).$$

14.2.4. *a-data and χ-data for restricted roots.* In our computation of the transfer factor carried out later, we need to fix sets of a-data and χ-data for the restricted roots. We explain our choice in the following.

We first discuss the $\mathbf{G}$-side. Suppose that $j \in \tilde{\mathcal{J}}^{\mathbf{G}}_{\mathbf{G}_\eta}$. For any $k \in \tilde{\mathcal{J}}^{\mathbf{G}_\eta}_{G_\eta}(j)$, we get an η-stable (hence also η_0-stable) tame elliptic toral pair $(\mathbf{S}_k, \vartheta'_k)$ of $\mathbf{G}$. Then we have the set $\Phi_{\mathrm{res}}(\mathbf{G}, \mathbf{S}_k)$ of restricted roots. We define a set $a^{\mathrm{res}}_k = \{a^{\mathrm{res}}_{k,\alpha_{\mathrm{res}}}\}_{\alpha_{\mathrm{res}} \in \Phi_{\mathrm{res}}(\mathbf{G},\mathbf{S}_k)}$ of a-data for $\Phi_{\mathrm{res}}(\mathbf{G}, \mathbf{S}_k)$ by

$$a^{\mathrm{res}}_{k,\alpha_{\mathrm{res}}} = \langle H_{\alpha_{\mathrm{res}}}, X^*_k \rangle,$$

where

- $H_{\alpha_{\mathrm{res}}} := d\alpha^\vee_{\mathrm{res}}(1) \in \mathfrak{s}^\natural_k(F_\alpha)$, and
- $X^*_k \in \mathfrak{s}^{\natural *}_{k,-r}$ is an element associated to ϑ'_k as in Section 14.2.3.

We define a set $\chi^{\mathrm{res}}_k = \{\chi^{\mathrm{res}}_{k,\alpha_{\mathrm{res}}}\}_{\alpha_{\mathrm{res}} \in \Phi_{\mathrm{res}}(\mathbf{G},\mathbf{S}_k)}$ of χ-data for $\Phi_{\mathrm{res}}(\mathbf{G}, \mathbf{S}_k)$ as follows:

- For $\alpha_{\mathrm{res}} \in \Phi_{\mathrm{res}}(\mathbf{G}, \mathbf{S}_k)_{\mathrm{asym}}$, let $\chi^{\mathrm{res}}_{k,\alpha_{\mathrm{res}}}$ be the trivial character of $F^\times_{\alpha_{\mathrm{res}}}$.
- For $\alpha_{\mathrm{res}} \in \Phi_{\mathrm{res}}(\mathbf{G}, \mathbf{S}_k)_{\mathrm{ur}}$, let $\chi^{\mathrm{res}}_{k,\alpha_{\mathrm{res}}}$ be the unique unramified nontrivial quadratic character of $F^\times_{\alpha_{\mathrm{res}}}$.
- For $\alpha_{\mathrm{res}} \in \Phi_{\mathrm{res}}(\mathbf{G}, \mathbf{S}_k)_{\mathrm{ram}}$, let $\chi^{\mathrm{res}}_{k,\alpha_{\mathrm{res}}}$ be the unique tamely ramified character of $F^\times_{\alpha_{\mathrm{res}}}$ characterized by the following properties:

$$\chi^{\mathrm{res}}_{k,\alpha_{\mathrm{res}}}|_{F^\times_{\pm\alpha_{\mathrm{res}}}} = \kappa_{\alpha_{\mathrm{res}}} \quad \text{and} \quad \chi^{\mathrm{res}}_{j,\alpha_{\mathrm{res}}}(2a^{\mathrm{res}}_{k,\alpha_{\mathrm{res}}}) = \lambda_{\alpha_{\mathrm{res}}}.$$

Remark 14.10. We can check that the above conditions uniquely specify the tamely ramified quadratic character $\chi^{\mathrm{res}}_{k,\alpha_{\mathrm{res}}}$ for $\alpha_{\mathrm{res}} \in \Phi_{\mathrm{res}}(\mathbf{G}, \mathbf{S}_k)_{\mathrm{ram}}$ in the same manner as in [Kal19b, Section 4.7]. Indeed, if we let $\sigma_{\alpha_{\mathrm{res}}} \in \mathrm{Gal}(F_{\alpha_{\mathrm{res}}}/F_{\pm\alpha_{\mathrm{res}}})$ be the unique nontrivial element, then we have $\sigma_{\alpha_{\mathrm{res}}}(H_{\alpha_{\mathrm{res}}}) = \sigma_{\alpha_{\mathrm{res}}}(d\alpha_{\mathrm{res}}^{\vee}(1)) = d\sigma_{\alpha_{\mathrm{res}}}(\alpha_{\mathrm{res}}^{\vee})(1) = -d\alpha_{\mathrm{res}}^{\vee}(1) = -H_{\alpha_{\mathrm{res}}}$ and $\sigma_{\alpha_{\mathrm{res}}}(X_k^*) = X_k^*$. Hence $\sigma_{\alpha_{\mathrm{res}}}(a_{k,\alpha_{\mathrm{res}}}^{\mathrm{res}}) = -a_{k,\alpha_{\mathrm{res}}}^{\mathrm{res}}$. This implies that the valuation (normalized with respect to $F_{\alpha_{\mathrm{res}}}$) of $a_{k,\alpha_{\mathrm{res}}}^{\mathrm{res}}$ is odd.

We note that, by restriction, $(\mathbf{S}_k, \vartheta'_k)$ induces a tame elliptic toral pair $(\mathbf{S}_k^{\natural}, \vartheta_k'^{\natural})$ of $\mathbf{G}_{\eta_0}$, where $\vartheta_k'^{\natural} = \vartheta'_k|_{S_k^{\natural}}$ (Lemma 5.4). Then, by the construction of Kaletha (see Section 7.1), we have a set $a_{\vartheta_k'^{\natural}} = \{a_{\vartheta_k'^{\natural},\alpha}\}_{\alpha\in\Phi(\mathbf{G}_{\eta_0},\mathbf{S}_k^{\natural})}$ of a-data and a set $\chi_{\vartheta_k'^{\natural}} = \{\chi_{\vartheta_k'^{\natural},\alpha}\}_{\alpha\in\Phi(\mathbf{G}_{\eta_0},\mathbf{S}_k^{\natural})}$ of χ-data for $\Phi(\mathbf{G}_{\eta_0}, \mathbf{S}_k^{\natural})$. We shortly write $(a_k^{\natural}, \chi_k^{\natural})$ for $(a_{\vartheta_k'^{\natural}}, \chi_{\vartheta_k'^{\natural}})$. As explained in Section 12.1, the set $\Phi(\mathbf{G}_{\eta_0}, \mathbf{S}_k^{\natural})$ can be regarded as a subset (root subsystem) of $\Phi_{\mathrm{res}}(\mathbf{G}, \mathbf{S}_k)$. By construction, we have the following:

Lemma 14.11. *The sets of a-data and χ-data $(a_k^{\natural}, \chi_k^{\natural})$ are restrictions of $(a_k^{\mathrm{res}}, \chi_k^{\mathrm{res}})$.*

We next discuss the $\mathbf{H}$-side. Suppose that $D \in \mathbf{D}(y, \eta)$, $j_{\mathbf{H}} \in \mathcal{J}_{\mathbf{H}_y}^{\mathbf{H}}$, and $j \in \tilde{\mathcal{J}}_{\mathbf{G}_\eta}^{\mathbf{G}}$ satisfy $\mathfrak{tran}(D, j_{\mathbf{H}}) = j$. For any $k_{\mathbf{H}} \in \mathcal{J}_{H_y}^{\mathbf{H}_y}$, we get a tame elliptic toral pair $(\mathbf{S}_{k_{\mathbf{H}}}, \vartheta'_{k_{\mathbf{H}}}) := (\mathbf{S}_{\mathbf{H},k_{\mathbf{H}}}, \vartheta'_{\mathbf{H},k_{\mathbf{H}}})$ of $\mathbf{H}$. By applying Kaletha's construction (Section 7.1) to $(\mathbf{S}_{k_{\mathbf{H}}}, \vartheta'_{k_{\mathbf{H}}})$, we get the sets $a_{k_{\mathbf{H}}} := a_{\vartheta'_{k_{\mathbf{H}}}}$ of a-data and $\chi_{k_{\mathbf{H}}} := \chi_{\vartheta'_{k_{\mathbf{H}}}}$ of χ-data with respect to $\Phi(\mathbf{H}, \mathbf{S}_{k_{\mathbf{H}}})$. Suppose that $y \in S_{k_{\mathbf{H}}}$. Then, as explained in Section 12.1, the set $\Phi(\mathbf{H}_y, \mathbf{S}_{k_{\mathbf{H}}})$ can be regarded as a subset (root subsystem) of $\Phi(\mathbf{G}_\eta, \mathbf{S}_k^{\natural})$, which is a subset of $\Phi(\mathbf{G}_{\eta_0}, \mathbf{S}_k^{\natural})$.

Lemma 14.12. *Suppose that $\alpha_y \in \Phi(\mathbf{H}_y, \mathbf{S}_{k_{\mathbf{H}}})$ is identified with $\alpha_\eta \in \Phi(\mathbf{G}_\eta, \mathbf{S}_k^{\natural})$. Then, we have $l_\alpha \cdot a_{k_{\mathbf{H}},\alpha_y} = a_{k,\alpha_\eta}^{\mathrm{res}}$.*

Proof. By definition, we have $a_{k_{\mathbf{H}},\alpha_y} = \langle H_{\alpha_y}, Y_{k_{\mathbf{H}}}^* \rangle$ and $a_{k,\alpha_\eta}^{\mathrm{res}} = \langle H_{\alpha_\eta}, X_k^* \rangle$. Since $(Y_{k_{\mathbf{H}}}^*, X_k^*)$ is a D-norm pair (Lemma 14.8) and we have $H_{\alpha_y} = d\alpha_y^{\vee}(1)$, $H_{\alpha_\eta} = d\alpha_\eta^{\vee}(1)$, we get the equality $l_\alpha \cdot a_{k_{\mathbf{H}},\alpha_y} = a_{k,\alpha_\eta}^{\mathrm{res}}$ by Lemma 12.1. ☐

14.2.5. *Twisted character formula of a normalized form.* In the following, for each $j \in \tilde{\mathcal{J}}_G^{\mathbf{G}}$, we fix a set of elements $\{g_k \in G \mid k \in \tilde{\mathcal{J}}_{G_\eta}^G(j)\}$ such that $\{[g_k]^{-1} \circ j\}$ is a (fixed) set of representatives of $\tilde{\mathcal{J}}_{G_\eta}^G(j)$. (Note that this set also represents $\{g \in S_j\backslash G/G_\eta \mid {}^g\eta \in \tilde{S}_j\}$ by Lemma 14.2.) Moreover, we fix a base point $\underline{\eta}_j$ of the twisted space $\tilde{S}_j$ For each $k \in \tilde{\mathcal{J}}_{G_\eta}^G(j)$, we fix a base point $\underline{\eta}_k$ of the twisted space $\tilde{S}_k$ by $\underline{\eta}_k := [g_k]^{-1}(\underline{\eta}_j)$.

For any $k \in \tilde{\mathcal{J}}_{G_\eta}^{\mathbf{G}}$, we define a character $\epsilon_{\vartheta_k}^{\star}$ of S_k by

$$\epsilon_{\vartheta_k}^{\star}(s) := \prod_{\substack{\alpha\in\ddot{\Xi}(\mathbf{G},\mathbf{S}_k)\\ \alpha_{\mathrm{res}}:\,\mathrm{ram}}} \epsilon_\alpha(s).$$

Proposition 14.13. *Let $j \in \tilde{\mathcal{J}}_G^{\mathbf{G}}$. For each $k \in \tilde{\mathcal{J}}_{G_\eta}^G(j)$, we write $\eta = s_k \cdot \underline{\eta}_k \in \tilde{S}_k$. Then we have*

$$\Phi_{\tilde{\pi}_j}(\delta) = C_{\underline{\eta}_j} \cdot (-1)^{|\ddot{\Xi}_{\eta_0,\mathrm{ur}}|} \cdot e(\mathbf{G}_{\eta_0}) \cdot e(\mathbf{G}_\eta) \cdot \varepsilon(\mathbf{T}_{\mathbf{G}_{\eta_0}^*}) \cdot \varepsilon(\mathbf{T}_{\mathbf{G}_\eta^*})^{-1} \cdot \sum_{k\in\tilde{\mathcal{J}}_{G_\eta}^G(j)} \vartheta_k(s_k) \cdot \epsilon_{\mathbf{S}_k,\mathrm{ram}}(s_k) \cdot \epsilon_{\vartheta_k}^{\star}(s_k) \cdot \Delta_{\mathrm{II}}^{\mathbf{G}_{\eta_0}}[a_k^{\mathrm{res}}, \chi_k^{\mathrm{res}}](\eta_+) \cdot \hat{\iota}_{X_k^*}^{\mathbf{G}_\eta}\big(\log(\delta_{\geq r})\big),$$

where a_k^{res} and χ_k^{res} are the sets of a-data and χ-data for $\Phi(\mathbf{G}_{\eta_0}, \mathbf{S}_k^{\natural})$ as in Section 14.2.4.

Proof. By Proposition 6.11, we have

$$\Phi_{\tilde{\pi}_j}(\delta) = C_{\underline{\eta}_j} \cdot (-1)^{|\ddot{\Xi}_{\eta_0,\mathrm{ur}}|} \cdot e(\mathbf{G}_{\eta_0}) \cdot e(\mathbf{G}_\eta) \cdot \varepsilon(\mathbf{T}_{\mathbf{G}^*_{\eta_0}}) \cdot \varepsilon(\mathbf{T}_{\mathbf{G}^*_\eta})^{-1} \sum_{\substack{g \in S_j \backslash G/G_\eta \\ {}^g\eta \in \tilde{S}_j}} \vartheta'_j(s_{j,g}) \cdot \tilde{\epsilon}_\Xi(s_{j,g}) \cdot \Delta_{\mathrm{II}}^{\mathbf{G}_{{}^g\eta_0}}[a^\natural_{\vartheta'_j}, \chi^\natural_{\vartheta'_j}]({}^g\eta_+) \cdot \hat{\iota}^{\mathbf{G}_{{}^g\eta}}_{X^*_j}(\log({}^g\delta_{\geq r})),$$

where $s_{j,g} \in S_j$ is the element satisfying ${}^g\eta = s_{j,g}\underline{\eta}_j$. Recall that $\vartheta'_j(s_{j,g}) = \epsilon_{\vartheta_j} \cdot \vartheta_j$ (Section 7.3.3). Here we caution that we took the initial regular supercuspidal packet datum such that its χ-data is equal to χ_{ϑ_j}, hence the zeta character contained in ϑ'_j is trivial. As we have $\epsilon_{\vartheta_j} = \epsilon_{\vartheta_j,\mathrm{asym}} \cdot \epsilon_{\vartheta_j,\mathrm{ur}} \cdot \epsilon_{\mathbf{S}_j,\mathrm{ram}}$ and $\tilde{\epsilon}_\Xi$ is the product of ϵ_α's for $\alpha \in \ddot{\Xi}$ such that α_{res} is asymmetric or unramified (see Section 6.7), we get

$$\vartheta'_j(s_{j,g}) \cdot \tilde{\epsilon}_\Xi(s_{j,g}) = \vartheta_j(s_{j,g}) \cdot \epsilon_{\mathbf{S}_j,\mathrm{ram}}(s_{j,g}) \cdot \epsilon^\star_{\vartheta_j}(s_{j,g}).$$

By our choice of base points, when $g = g_k$, we have $\underline{\eta}_j = {}^g\underline{\eta}_k$, hence $s_{j,g} = {}^g s_k$. Thus we get $\vartheta_j(s_{j,g}) = \vartheta(j^{-1}(s_{j,g})) = \vartheta(j^{-1} \circ [g](s_k)) = \vartheta(k^{-1}(s_k)) = \vartheta_k(s_k)$. Similarly, we have $\epsilon_{\mathbf{S}_j,\mathrm{ram}}(s_{j,g}) = \epsilon_{\mathbf{S}_j,\mathrm{ram}}({}^g s_k) = \epsilon_{\mathbf{S}_k,\mathrm{ram}}(s_k)$, $\epsilon^\star_{\vartheta_j}(s_{j,g}) = \epsilon^\star_{\vartheta_j}({}^g s_k) = \epsilon^\star_{\vartheta_k}(s_k)$, and $\hat{\iota}^{\mathbf{G}_{{}^g\eta}}_{X^*_j}(\log({}^g\delta_{\geq r})) = \hat{\iota}^{\mathbf{G}_\eta}_{{}^{g^{-1}}X^*_j}(\log(\delta_{\geq r})) = \hat{\iota}^{\mathbf{G}_\eta}_{X^*_k}(\log(\delta_{\geq r}))$. Moreover, by noting that $\vartheta'^\natural_j$ and $\vartheta^\natural_j$ give rise to the same a-data and χ-data (Section 7.1) and using Lemma 14.11, we have $\Delta_{\mathrm{II}}^{\mathbf{G}_{{}^g\eta_0}}[a^\natural_{\vartheta'_j}, \chi^\natural_{\vartheta'_j}]({}^g\eta_+) = \Delta_{\mathrm{II}}^{\mathbf{G}_{\eta_0}}[a^{\mathrm{res}}_k, \chi^{\mathrm{res}}_k](\eta_+)$. Thus we arrive at the claimed formula. □

14.2.6. *Third factor Δ_{III}: revisited.* We next rewrite Proposition 13.7 in a form suitable for our purpose. Suppose that $D = (\mathbf{B}^\flat, \mathbf{T}^\flat, \mathbf{B}^\diamondsuit, \mathbf{T}^\diamondsuit) \in \mathbf{D}(y, \eta)$, $j_{\mathbf{H}} \in \mathcal{J}^{\mathbf{H}}_{\mathbf{H}_y}$, and $j \in \tilde{\mathcal{J}}^{\mathbf{G}}_{\mathbf{G}_\eta}$ satisfy $\mathfrak{tran}(D, j_{\mathbf{H}}) = j$. Hence, we may and do assume that $\mathbf{T}^\flat = \mathbf{S}_{j_{\mathbf{H}}}$ and $\mathbf{T}^\diamondsuit = \mathbf{S}_j$. Let $k \in \tilde{\mathcal{J}}^{\mathbf{G}_\eta}_{G_\eta}(j)$.

We introduce the following character according to [Kal19a, Proposition 5.25]:

Definition 14.14. Let $\zeta_{\mathrm{desc}} \colon S_k \to \mathbb{C}^\times$ be a character given by

$$\zeta_{\mathrm{desc}}(s) := \prod_{\substack{\alpha \in \ddot{\Phi}_{\mathrm{asym}}(\mathbf{G}, \mathbf{S}_k) \\ \alpha_{\mathrm{res}}\colon \text{ ramified}}} \epsilon_\alpha(s) \prod_{\substack{\alpha \in \dot{\Phi}_{\mathrm{ur}}(\mathbf{G}, \mathbf{S}_k) \\ \alpha_{\mathrm{res}}\colon \text{ ramified}}} \epsilon_\alpha(s).$$

Our aim here is to show the following:

Proposition 14.15. *Suppose that $(\bar{\gamma}, \bar{\delta}), (\bar{\gamma},' \bar{\delta}') \in \mathcal{D}$ are such that $D \in \mathbf{D}(\bar{\gamma}, \bar{\delta})$ and $D \in \mathbf{D}(\bar{\gamma}', \bar{\delta}')$. Then we have*

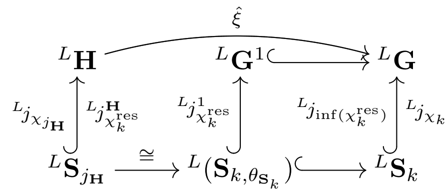
$$\Delta_{\mathrm{III}}[a^{\mathrm{res}}_k, \chi^{\mathrm{res}}_k](\bar{\gamma}, \bar{\delta}; \bar{\gamma}', \bar{\delta}') = \frac{\vartheta_k(\bar{\delta}/\bar{\delta}')}{\vartheta_{j_{\mathbf{H}}}(\bar{\gamma}/\bar{\gamma}')} \cdot \zeta_{\mathrm{desc}}(\bar{\delta}/\bar{\delta}') \cdot \zeta_{\chi^{\mathrm{res}}_j/\chi_{j_{\mathbf{H}}}, S_{j_{\mathbf{H}}}}(\bar{\gamma}/\bar{\gamma}').$$

Recall that we introduced a 1-cocycle $a_{\mathbf{S}_k}$ which measures the difference between ${}^L j^1_{\chi^{\mathrm{res}}_k}$ and $\hat{\xi} \circ {}^L j^{\mathbf{H}}_{\chi^{\mathrm{res}}_k}$ in Section 13.3. Let us write $a[{}^L j^{\mathbf{H}}_{\chi^{\mathrm{res}}_k}/{}^L j^1_{\chi^{\mathrm{res}}_k}]$ for $a_{\mathbf{S}_k}$.

On the other hand, we also have L-embeddings ${}^L j_{\chi_k} \colon {}^L\mathbf{S}_k \hookrightarrow {}^L\mathbf{G}$ and ${}^L j_{\chi_{\mathbf{H}}} \colon {}^L\mathbf{S}_{j_{\mathbf{H}}} \hookrightarrow {}^L\mathbf{H}$ obtained by applying the Langlands–Shelstad construction to the χ-data χ_k and $\chi_{j_{\mathbf{H}}}$ as in Section 14.2.4. We define a 1-cocycle $a[{}^L j_{\chi_{j_{\mathbf{H}}}}/{}^L j_{\chi_k}]$ by $a[{}^L j_{\chi_{j_{\mathbf{H}}}}/{}^L j_{\chi_k}] \cdot {}^L j_{\chi_k} = \hat{\xi} \circ {}^L j_{\chi_{j_{\mathbf{H}}}}$

We furthermore introduce one more L-embedding ${}^L j_{\inf(\chi_k^{\mathrm{res}})}\colon {}^L\mathbf{S}_k \hookrightarrow {}^L\mathbf{G}$. For this, we first define a set of χ-data $\inf(\chi_k^{\mathrm{res}})$ of $\Phi(\mathbf{G},\mathbf{S}_k)$ by inflating the set of χ-data χ_k^{res} of $\Phi_{\mathrm{res}}(\mathbf{G},\mathbf{S}_k)$ along the natural restriction map $\Phi(\mathbf{G},\mathbf{S}_k) \twoheadrightarrow \Phi_{\mathrm{res}}(\mathbf{G},\mathbf{S}_k)$ (see [Kal19a, Definition 5.14]) and then apply the Langlands–Shelstad construction.

We define 1-cocycles $a[{}^L j_{\inf(\chi_k^{\mathrm{res}})}/{}^L j_{\chi_k}]$, $a[{}^L j^1_{\chi_k^{\mathrm{res}}}/{}^L j_{\inf(\chi_k^{\mathrm{res}})}]$, and $a[{}^L j_{\chi_{j_{\mathbf{H}}}}/{}^L j^{\mathbf{H}}_{\chi_k^{\mathrm{res}}}]$ in a similar way to $a[{}^L j_{\chi_{j_{\mathbf{H}}}}/{}^L j_{\chi_k}]$.



Lemma 14.16. *For any $s \in S_k$, we have*

(1) $\langle s, a[{}^L j_{\chi_{j_{\mathbf{H}}}}/{}^L j_{\chi_k}]\rangle_{\mathrm{TN}} = \vartheta_k/\vartheta_{j_{\mathbf{H}}}(s)$,
(2) $\langle s, a[{}^L j_{\inf(\chi_k^{\mathrm{res}})}/{}^L j_{\chi_k}]\rangle_{\mathrm{TN}} = \zeta_{\mathrm{desc}}(s)^{-1}$,
(3) $\langle s, a[{}^L j^1_{\chi_k^{\mathrm{res}}}/{}^L j_{\inf(\chi_k^{\mathrm{res}})}]\rangle_{\mathrm{TN}} = 1$.

Proof. We first consider (1). Recall that we have ${}^L j_{\chi} \circ \phi_{\vartheta} = \hat{\xi} \circ {}^L j_{\chi_{\mathbf{H}}} \circ \phi_{\vartheta_{\mathbf{H}}}$ (see Section 9.5, especially, (14)). Since we assumed that $\chi = \chi_{\vartheta_{\hat{\jmath}}}$ and $\chi_{\mathbf{H}} = \chi_{\vartheta_{\hat{\jmath}_{\mathbf{H}}}}$ (see the beginning of Section 14.2.1), this identity can be rewritten as ${}^L j_{\chi_k} \circ \phi_{\vartheta_k} = \hat{\xi} \circ {}^L j_{\chi_{j_{\mathbf{H}}}} \circ \phi_{\vartheta_{j_{\mathbf{H}}}}$. This implies that $\phi_{\vartheta_k} = a[{}^L j_{\chi_{j_{\mathbf{H}}}}/{}^L j_{\chi_k}] \cdot \phi_{\vartheta_{j_{\mathbf{H}}}}$. Hence we get the identity (1).

We next consider (2). We note that the set χ_k of χ-data is minimally ramified, which is obtained by the "minimalization" of $\inf(\chi_k^{\mathrm{res}})$ ([Kal19a, Definition 5.24]). Therefore the claimed identity is a direct consequence of [Kal19a, Proposition 5.25]. (Just note that the integers "$e(\alpha/\alpha_{\mathrm{res}})$" in [Kal19a, Proposition 5.25] are all equal to 1, which can be checked by looking at the proofs of Lemma 12.2 and 12.3).

It is a routine work to check the assertion (3) by going back to the Langlands–Shelstad construction. □

Proof of Proposition 14.15. By Proposition 13.7, we have

$$\Delta_{\mathrm{III}}[a_k^{\mathrm{res}}, \chi_k^{\mathrm{res}}](\bar{\gamma}, \bar{\delta}; \bar{\gamma}', \bar{\delta}') = \langle \bar{\delta}/\bar{\delta}', a[{}^L j^{\mathbf{H}}_{\chi_k^{\mathrm{res}}}/{}^L j^1_{\chi_k^{\mathrm{res}}}]\rangle_{\mathrm{TN}}.$$

We note that $a[{}^L j_{\chi_{j_{\mathbf{H}}}}/{}^L j_{\chi_k}]$ is equal to

$$a[{}^L j_{\chi_{j_{\mathbf{H}}}}/{}^L j^{\mathbf{H}}_{\chi_k^{\mathrm{res}}}] \cdot a[{}^L j^{\mathbf{H}}_{\chi_k^{\mathrm{res}}}/{}^L j^1_{\chi_k^{\mathrm{res}}}] \cdot a[{}^L j^1_{\chi_k^{\mathrm{res}}}/{}^L j_{\inf(\chi_k^{\mathrm{res}})}] \cdot a[{}^L j_{\inf(\chi_k^{\mathrm{res}})}/{}^L j_{\chi_k}].$$

Hence, Lemma 14.16 implies that

$$\vartheta_k/\vartheta_{j_{\mathbf{H}}}(\bar{\delta}/\bar{\delta}') = \zeta_{\chi_{j_{\mathbf{H}}}/\chi_k^{\mathrm{res}}, S_{j_{\mathbf{H}}}}(\bar{\gamma}/\bar{\gamma}') \cdot \Delta_{\mathrm{III}}[a_k^{\mathrm{res}}, \chi_k^{\mathrm{res}}](\bar{\gamma}, \bar{\delta}; \bar{\gamma}', \bar{\delta}') \cdot \zeta_{\mathrm{desc}}(\bar{\delta}/\bar{\delta}')^{-1}$$

Thus we get the assertion. □

14.3. **Appearance of the spectral transfer factor.** We start with rewriting the endoscopic side of (25). We put $\Phi_{\phi}^{\mathbf{H},\mathrm{st}} := \sum_{\pi_{\mathbf{H}} \in \Pi^{\mathbf{H}}_{\phi_{\mathbf{H}}}} \Phi_{\pi_{\mathbf{H}}}$. By Lemma 14.5, we have

$$(26)\quad \sum_{\gamma \in H_{\mathrm{srs}}/\sim_{\mathbf{H}}} \mathring{\Delta}(\gamma, \delta)\Phi_{\phi}^{\mathbf{H},\mathrm{st}}(\gamma) = \sum_{y \in \mathfrak{H}_{\eta}} \frac{1}{|\pi_0(\mathbf{H}^y)(F)|} \sum_{z \in H_y[\delta]_r/\sim_{\mathbf{H}_y}} \mathring{\Delta}(yz, \delta)\Phi_{\phi}^{\mathbf{H},\mathrm{st}}(yz).$$

In the following, we put $X := \log(\delta_{\geq r}) \in \mathfrak{g}_\eta$. Let $z \in H_y[\delta]_r$ and we put $Y := \log(z) \in \mathfrak{h}_y$. By Lemma 14.6, there exists a unique $D \in \mathbb{D}(y,\eta)$ such that (Y, X) is a D-norm pair. Therefore, by Proposition 13.12, we have

$$\mathring{\Delta}^D(\bar{Y}, X_{\mathrm{sc}}) = \begin{cases} \mathring{\Delta}(yz,\delta)\cdot\Delta_{\mathrm{IV}}(y,\eta) & \text{for a unique } D \in \mathbb{D}(y,\eta), \\ 0 & \text{otherwise.} \end{cases}$$

Thus, by also using Lemma 14.7, we see that the right-hand side of (26) equals

$$\sum_{y\in\mathfrak{H}_\eta} \frac{1}{|\pi_0(\mathbf{H}^y)(F)|} \sum_{D\in\mathbb{D}(y,\eta)} \sum_{Y \overset{D}{\leftrightarrow} X/\sim_{\mathbf{H}_y}} \mathring{\Delta}^D(\bar{Y}, X_{\mathrm{sc}})\Delta_{\mathrm{IV}}(y,\eta)^{-1}\Phi_\phi^{\mathbf{H},\mathrm{st}}(yz). \tag{27}$$

Now we utilize the character formula ([Kal19b, Lemma 6.3.1]):

$$\Phi_\phi^{\mathbf{H},\mathrm{st}}(yz) = \varepsilon(\mathbf{T_H})\varepsilon(\mathbf{T}_{\mathbf{H}_y})^{-1} \sum_{j_\mathbf{H}\in\mathcal{J}_{\mathbf{H}_y}^{\mathbf{H}}} \Delta_{\mathrm{II}}^{\mathbf{H}}[a_{j_\mathbf{H}}, \chi_{j_\mathbf{H}}](y)\vartheta_{j_\mathbf{H}}(y) \sum_{k_\mathbf{H}\in\mathcal{J}_{H_y}^{\mathbf{H}_y}(j_\mathbf{H})} \hat{\iota}^{\mathbf{H}_y}_{Y^*_{k_\mathbf{H}}}(Y),$$

where $a_{j_\mathbf{H}} := a_{\vartheta_{j_\mathbf{H}}}$, $\chi_{j_\mathbf{H}} := \chi_{\vartheta_{j_\mathbf{H}}}$ (see Section 14.2.4). Note that the above formula is simplified compared to [Kal19b, Lemma 6.3.1] because now $\mathbf{H}_y$ is quasi-split and $\langle \mathrm{inv}(j_{\mathbf{H},\mathfrak{w}_\mathbf{H}}, k_\mathbf{H}), 1\rangle = 1$ since the groups $S^+_{\phi_\mathbf{H}}$ is abelian. Thus (27) equals

$$\sum_{y\in\mathfrak{H}_\eta} \frac{\varepsilon(\mathbf{T_H})\varepsilon(\mathbf{T}_{\mathbf{H}_y})^{-1}}{|\pi_0(\mathbf{H}^y)(F)|} \sum_{D\in\mathbb{D}(y,\eta)} \sum_{j_\mathbf{H}\in\mathcal{J}_{\mathbf{H}_y}^{\mathbf{H}}} \Delta_{\mathrm{IV}}(y,\eta)^{-1} \Delta_{\mathrm{II}}^{\mathbf{H}}[a_{j_\mathbf{H}}, \chi_{j_\mathbf{H}}](y)\vartheta_{j_\mathbf{H}}(y) \sum_{Y \overset{D}{\leftrightarrow} X/\sim_{\mathbf{H}_y}} \mathring{\Delta}^D(\bar{Y}, X_{\mathrm{sc}}) \sum_{k_\mathbf{H}\in\mathcal{J}_{H_y}^{\mathbf{H}_y}(j_\mathbf{H})} \hat{\iota}^{\mathbf{H}_y}_{Y^*_{k_\mathbf{H}}}(Y). \tag{28}$$

According to Proposition 10.11, the first three index sets with $1/|\pi_0(\mathbf{H}^y)(F)|$ are combined into one index set $\tilde{\mathcal{J}}^{\mathbf{G}}_{\mathbf{G}_\eta}$. Hence we can rewrite the above sum as

$$\sum_{j\in\tilde{\mathcal{J}}^{\mathbf{G}}_{\mathbf{G}_\eta}} \varepsilon(\mathbf{T_H})\varepsilon(\mathbf{T}_{\mathbf{H}_y})^{-1}\Delta_{\mathrm{IV}}(y,\eta)^{-1}\Delta_{\mathrm{II}}^{\mathbf{H}}[a_{j_\mathbf{H}}, \chi_{j_\mathbf{H}}](y)\vartheta_{j_\mathbf{H}}(y) \sum_{Y \overset{D}{\leftrightarrow} X/\sim_{\mathbf{H}_y}} \mathring{\Delta}^D(\bar{Y}, X_{\mathrm{sc}}) \sum_{k_\mathbf{H}\in\mathcal{J}_{H_y}^{\mathbf{H}_y}(j_\mathbf{H})} \hat{\iota}^{\mathbf{H}_y}_{Y^*_{k_\mathbf{H}}}(Y). \tag{29}$$

Here, for each $j \in \tilde{\mathcal{J}}^{\mathbf{G}}_{\mathbf{G}_\eta}$, we let $y \in \mathfrak{H}_\eta$ and $j_\mathbf{H} \in \mathcal{J}^{\mathbf{H}}_{\mathbf{H}_y}$ denote the unique (up to $\pi_0(\mathbf{H}^y)(F)$-action) elements determined by Proposition 10.11. By applying the Lie algebra transfer for twisted endoscopy (Proposition 14.9):

$$\sum_{Y \overset{D}{\leftrightarrow} X/\sim_{\mathbf{H}_y}} \mathring{\Delta}^D(\bar{Y}, X_{\mathrm{sc}}) \sum_{k_\mathbf{H}\in\mathcal{J}_{H_y}^{\mathbf{H}_y}(j_\mathbf{H})} D^{\mathbf{H}_y}_{Y^*_{k_\mathbf{H}}}(Y) = \sum_{k\in\mathcal{J}_{G_\eta}^{\mathbf{G}_\eta}(j)} \mathring{\Delta}^D(\bar{Y}^*_{j_\mathbf{H}}, X^*_{k,\mathrm{sc}}) D^{\mathbf{G}_\eta}_{X^*_k}(X)$$

to the last double sum of (29), we see that (29) is equal to

$$\sum_{j\in\tilde{\mathcal{J}}^{\mathbf{G}}_{\mathbf{G}_\eta}} \varepsilon(\mathbf{T_H})\varepsilon(\mathbf{T}_{\mathbf{H}_y})^{-1}\Delta_{\mathrm{IV}}(y,\eta)^{-1}\Delta_{\mathrm{II}}^{\mathbf{H}}[a_{j_\mathbf{H}}, \chi_{j_\mathbf{H}}](y)\vartheta_{j_\mathbf{H}}(y) \cdot \gamma(\mathfrak{g}_\eta)\gamma(\mathfrak{h}_y)^{-1} \sum_{k\in\mathcal{J}_{G_\eta}^{\mathbf{G}_\eta}(j)} \mathring{\Delta}^D(\bar{Y}^*_{j_\mathbf{H}}, X^*_{k,\mathrm{sc}}) \hat{\iota}^{\mathbf{G}_\eta}_{X^*_k}(X) \tag{30}$$

(recall that $D^{\mathbf{H}_y}_{Y^*_{k_{\mathbf{H}}}}(Y)=\gamma(\mathfrak{h}_y)\hat{\iota}^{\mathbf{H}_y}_{Y^*_{k_{\mathbf{H}}}}(Y)$ and $D^{\mathbf{G}_\eta}_{X^*_k}(X)=\gamma(\mathfrak{g}_\eta)\hat{\iota}^{\mathbf{G}_\eta}_{X^*_k}(X)$). Again using Proposition 13.12, this equals

$$(31)\quad \sum_{j\in\tilde{\mathcal{J}}^{\mathbf{G}}_{\mathbf{G}_\eta}} \varepsilon(\mathbf{T}_{\mathbf{H}})\varepsilon(\mathbf{T}_{\mathbf{H}_y})^{-1}\Delta^{\mathbf{H}}_{\mathrm{II}}[a_{j_{\mathbf{H}}},\chi_{j_{\mathbf{H}}}](y)\vartheta_{j_{\mathbf{H}}}(y) \cdot\gamma(\mathfrak{g}_\eta)\gamma(\mathfrak{h}_y)^{-1}\sum_{k\in\mathcal{J}^{\mathbf{G}_\eta}_{G_\eta}(j)}\mathring{\Delta}\big(y\exp(Y^*_{j_{\mathbf{H}}}),\eta\exp(X^*_k)\big)\hat{\iota}^{\mathbf{G}_\eta}_{X^*_k}(X).$$

By definition, $\mathring{\Delta}$ is given by the product of $\varepsilon(\mathbf{T}_{\mathbf{G}_\theta})\varepsilon(\mathbf{T}_{\mathbf{H}})^{-1}$, Δ_{I}, Δ_{II}, and Δ_{III}. In the following, we choose the a-data a_k^{res} and χ-data χ_k^{res} as in Section 14.2.4 to compute these factors. Then, in summary, the **H**-side (31) equals

$$(32)\quad \sum_{j\in\tilde{\mathcal{J}}^{\mathbf{G}}_{\mathbf{G}_\eta}} \varepsilon(\mathbf{T}_{\mathbf{G}_\theta})\varepsilon(\mathbf{T}_{\mathbf{H}_y})^{-1}\cdot\Delta^{\mathbf{H}}_{\mathrm{II}}[a_{j_{\mathbf{H}}},\chi_{j_{\mathbf{H}}}](y)\vartheta_{j_{\mathbf{H}}}(y)\gamma(\mathfrak{g}_\eta)\gamma(\mathfrak{h}_y)^{-1} \cdot\sum_{k\in\mathcal{J}^{\mathbf{G}_\eta}_{G_\eta}(j)}\Delta_{\mathrm{I,II,III}}[a_k^{\mathrm{res}},\chi_k^{\mathrm{res}}]\big(y\exp(Y^*_{j_{\mathbf{H}}}),\eta\exp(X^*_k)\big)\hat{\iota}^{\mathbf{G}_\eta}_{X^*_k}(X).$$

Now we are reduced to comparing the above to the **G**-side of (25). Let $\{\Delta^{\mathrm{spec}}_{\phi,j}\}_{j\in\mathcal{J}^{\mathbf{G}}_G}$ be any family of constants such that $\Delta^{\mathrm{spec}}_{\phi,j}=0$ for any $j\in\mathcal{J}^{\mathbf{G}}_G\smallsetminus\tilde{\mathcal{J}}^{\mathbf{G}}_G$. By Proposition 14.13, $\sum_{j\in\tilde{\mathcal{J}}^{\mathbf{G}}_G}\Delta^{\mathrm{spec}}_{\phi,j}\Phi_{\tilde{\pi}_j}(\delta)$ equals

$$(33)\quad \sum_{j\in\tilde{\mathcal{J}}^{\mathbf{G}}_G}\Delta^{\mathrm{spec}}_{\phi,j}C_{\underline{\eta}_j}(-1)^{|\ddot{\Xi}_{\eta_0,\mathrm{ur}}|}e(\mathbf{G}_{\eta_0})e(\mathbf{G}_\eta)\varepsilon(\mathbf{T}_{\mathbf{G}^*_{\eta_0}})\varepsilon(\mathbf{T}_{\mathbf{G}^*_\eta})^{-1} \cdot\sum_{k\in\tilde{\mathcal{J}}^G_{G_\eta}(j)}\vartheta_k(s_k)\cdot\epsilon_{\mathbf{S}_k,\mathrm{ram}}(s_k)\cdot\epsilon^\star_{\vartheta_k}(s_k)\cdot\Delta^{\mathbf{G}_{\eta_0}}_{\mathrm{II}}[a_k^{\mathrm{res}},\chi_k^{\mathrm{res}}](\eta_+)\cdot\hat{\iota}^{\mathbf{G}_\eta}_{X^*_k}(X).$$

By putting $\bar{\Delta}^{\mathrm{spec}}_{\phi,k}:=\Delta^{\mathrm{spec}}_{\phi,j}C_{\underline{\eta}_j}$ for any $k\in\tilde{\mathcal{J}}^G_{G_\eta}(j)$, (33) equals

$$(34)\quad \sum_{j\in\tilde{\mathcal{J}}^{\mathbf{G}}_G}\sum_{k\in\tilde{\mathcal{J}}^G_{G_\eta}(j)}\bar{\Delta}^{\mathrm{spec}}_{\phi,k}\cdot(-1)^{|\ddot{\Xi}_{\eta_0,\mathrm{ur}}|}\cdot e(\mathbf{G}_{\eta_0})e(\mathbf{G}_\eta)\cdot\varepsilon(\mathbf{T}_{\mathbf{G}^*_{\eta_0}})\varepsilon(\mathbf{T}_{\mathbf{G}^*_\eta})^{-1} \cdot\vartheta_k(s_k)\cdot\epsilon_{\mathbf{S}_k,\mathrm{ram}}(s_k)\cdot\epsilon^\star_{\vartheta_k}(s_k)\cdot\Delta^{\mathbf{G}_{\eta_0}}_{\mathrm{II}}[a_k^{\mathrm{res}},\chi_k^{\mathrm{res}}](\eta_+)\cdot\hat{\iota}^{\mathbf{G}_\eta}_{X^*_k}(X).$$

Therefore it suffices to prove that, for every $j\in\tilde{\mathcal{J}}^{\mathbf{G}}_{\mathbf{G}_\eta}$, the contribution of each $k\in\tilde{\mathcal{J}}^{\mathbf{G}_\eta}_{G_\eta}(j)$ to the **H**-side (32)

$$(35)\quad \varepsilon(\mathbf{T}_{\mathbf{G}_\theta})\varepsilon(\mathbf{T}_{\mathbf{H}_y})^{-1}\cdot\Delta^{\mathbf{H}}_{\mathrm{II}}[a_{j_{\mathbf{H}}},\chi_{j_{\mathbf{H}}}](y)\vartheta_{j_{\mathbf{H}}}(y)\cdot\gamma(\mathfrak{g}_\eta)\gamma(\mathfrak{h}_y)^{-1} \cdot\Delta_{\mathrm{I,II,III}}[a_k^{\mathrm{res}},\chi_k^{\mathrm{res}}]\big(y\exp(Y^*_{j_{\mathbf{H}}}),\eta\exp(X^*_k)\big)$$

(other than the Fourier transform of the orbital integral $\hat{\iota}^{\mathbf{G}_\eta}_{X^*_k}(X)$) is equal to that to the **G**-side (34)

$$(36)\quad \bar{\Delta}^{\mathrm{spec}}_{\phi,k}\cdot(-1)^{|\dot{\Xi}_{\eta_0,\mathrm{ur}}|}\cdot e(\mathbf{G}_{\eta_0})e(\mathbf{G}_\eta)\cdot\varepsilon(\mathbf{T}_{\mathbf{G}^*_{\eta_0}})\varepsilon(\mathbf{T}_{\mathbf{G}^*_\eta})^{-1} \cdot\vartheta_k(s_k)\cdot\epsilon_{\mathbf{S}_k,\mathrm{ram}}(s_k)\cdot\epsilon^\star_{\vartheta_k}(s_k)\cdot\Delta^{\mathbf{G}_{\eta_0}}_{\mathrm{II}}[a_k^{\mathrm{res}},\chi_k^{\mathrm{res}}](\eta_+)$$

under an appropriate choice of $\Delta^{\mathrm{spec}}_{\phi,k}$ such that $\Delta^{\mathrm{spec}}_{\phi,k}$ is constant on $k \in \tilde{\mathcal{J}}^{G}_{G_\eta}(j)$. Hence let us just define $\Delta^{\mathrm{spec}}_{\phi,k}$, or equivalently $\bar{\Delta}^{\mathrm{spec}}_{\phi,k}$, so that (35) equals (36):

$$(37)\quad \bar{\Delta}^{\mathrm{spec}}_{\phi,k} := \frac{\varepsilon(\mathbf{T}_{\mathbf{G}^*_\eta}) \cdot \varepsilon(\mathbf{T}_{\mathbf{H}_y})^{-1} \cdot \gamma(\mathfrak{g}_\eta)\gamma(\mathfrak{h}_y)^{-1}}{e(\mathbf{G}_\eta)} \cdot \frac{\Delta^{\mathbf{H}}_{\mathrm{II}}[a_{j_\mathbf{H}}, \chi_{j_\mathbf{H}}](y)}{\Delta^{\mathbf{G}_{\eta_0}}_{\mathrm{II}}[a^{\mathrm{res}}_k, \chi^{\mathrm{res}}_k](\eta_+)} \cdot \frac{\varepsilon(\mathbf{T}_{\mathbf{G}_\theta}) \cdot \varepsilon(\mathbf{T}_{\mathbf{G}^*_{\eta_0}})^{-1}}{e(\mathbf{G}_{\eta_0}) \cdot (-1)^{|\dot{\Xi}_{\eta_0,\mathrm{ur}}|} \cdot \epsilon_{\mathbf{S}_k,\mathrm{ram}}(s_k) \cdot \epsilon^\star_{\vartheta_k}(s_k)} \cdot \frac{\vartheta_{j_\mathbf{H}}(y)}{\vartheta_k(s_k)} \cdot \Delta_{\mathrm{I,II,III}}[a^{\mathrm{res}}_k, \chi^{\mathrm{res}}_k]\big(y\exp(Y^*_{j_\mathbf{H}}), \eta\exp(X^*_k)\big).$$

Then the problem is that this quantity heavily depends on η and y. What we have to do now is to check the well-definedness of $\Delta^{\mathrm{spec}}_{\phi,k}$; in other words,

(1) $\bar{\Delta}^{\mathrm{spec}}_{\phi,k}$ is constant for $k \in \tilde{\mathcal{J}}^{G}_{G_\eta}(j)$, and
(2) $\bar{\Delta}^{\mathrm{spec}}_{\phi,k}$ is independent of η.

We first recall the following formula of Kaletha–Kottwitz:

Proposition 14.17 ([Kal15, Lemma 4.8, Theorem 4.10])**.** *Let $\mathbf{J}$ be a connected reductive group over F and $\mathbf{S_J}$ an F-rational maximal torus of $\mathbf{J}$. We fix a J-invariant symmetric non-degenerate bilinear form $B_\mathfrak{j}$ on $\mathfrak{j}$. Then we have*

$$\varepsilon(\mathbf{S_J})\varepsilon(\mathbf{T_{J^*}})^{-1} = e(\mathbf{J})\gamma(\mathfrak{j}) \prod_{\alpha\in\dot{\Phi}(\mathbf{J},\mathbf{S_J})_{\mathrm{sym}}} \kappa_\alpha(B_{\mathfrak{j},\alpha})^{-1},$$

where $\mathbf{T_{J^*}}$ denotes a minimal Levi subgroup of the quasi-split inner form of $\mathbf{J}$ and $B_{\mathfrak{j},\alpha} := B_\mathfrak{j}(X_\alpha, Y_\alpha) \in F^\times_{\pm\alpha}$ for any elements $X_\alpha \in \mathbf{j}_\alpha(F_\alpha)$ and $Y_\alpha \in \mathbf{j}_{-\alpha}(F_\alpha)$ satisfying $[X_\alpha, Y_\alpha] = H_\alpha (:= d\alpha^\vee(1))$.

Lemma 14.18. *We have*

$$\frac{\varepsilon(\mathbf{T}_{\mathbf{G}^*_\eta}) \cdot \varepsilon(\mathbf{T}_{\mathbf{H}_y})^{-1} \cdot \gamma(\mathfrak{g}_\eta)\gamma(\mathfrak{h}_y)^{-1}}{e(\mathbf{G}_\eta)} = \frac{\Delta^{\mathbf{H}_y}_{\mathrm{II}}[a_{j_\mathbf{H}}, \chi_{j_\mathbf{H}}](\exp(Y^*_{j_\mathbf{H}}))}{\Delta^{\mathbf{G}_\eta}_{\mathrm{II}}[a^{\mathrm{res}}_k, \chi^{\mathrm{res}}_k](\exp(X^*_k))}.$$

Proof. By Proposition 14.17, we have

$$\varepsilon(\mathbf{S}^\natural_k) \cdot \varepsilon(\mathbf{T}_{\mathbf{G}^*_\eta})^{-1} = e(\mathbf{G}_\eta)\gamma(\mathfrak{g}_\eta) \prod_{\alpha_\eta\in\dot{\Phi}(\mathbf{G}_\eta,\mathbf{S}^\natural_k)_{\mathrm{sym}}} \kappa_{\alpha_\eta}(B_{\mathfrak{g}_\eta,\alpha_\eta})^{-1},$$
$$\varepsilon(\mathbf{S}_{j_\mathbf{H}}) \cdot \varepsilon(\mathbf{T}_{\mathbf{H}_y})^{-1} = \gamma(\mathfrak{h}_y) \prod_{\alpha_y\in\dot{\Phi}(\mathbf{H}_y,\mathbf{S}_{j_\mathbf{H}})_{\mathrm{sym}}} \kappa_{\alpha_y}(B_{\mathfrak{h}_y,\alpha_y})^{-1}.$$

(note that $e(\mathbf{H}_y) = 1$ since $\mathbf{H}_y$ is quasi-split). Hence, by noting that $X^*(\mathbf{S}_{j_\mathbf{H}})_\mathbb{C} \cong X^*(\mathbf{S}^\natural_k)_\mathbb{C}$, the left-hand side of the assertion is equal to

$$(38)\qquad \prod_{\alpha_\eta\in\dot{\Phi}(\mathbf{G}_\eta,\mathbf{S}^\natural_k)_{\mathrm{sym}}} \kappa_{\alpha_\eta}(B_{\mathfrak{g}_\eta,\alpha_\eta}) \prod_{\alpha_y\in\dot{\Phi}(\mathbf{H}_y,\mathbf{S}_{j_\mathbf{H}})_{\mathrm{sym}}} \kappa_{\alpha_y}(B_{\mathfrak{h}_y,\alpha_y})^{-1}.$$

This can be computed by the same argument as in the final paragraph of the proof of [Kal19b, Theorem 6.3.4] as follows. For any $\alpha_\eta \in \Phi(\mathbf{G}_\eta, \mathbf{S}^\natural_k)_{\mathrm{sym}}$, we have

$$\lim_{m\to\infty} \frac{\alpha_\eta(\exp(p^{2m}X^*_k)) - 1}{p^{2m}} = d\alpha_\eta(X^*_k),$$

where $X_k^* \in \mathfrak{g}_\eta^*$ is regarded as an element of $\mathfrak{g}_\eta$ via non-degenerate bilinear form $B_{\mathfrak{g}_\eta}$ on $\mathfrak{g}_\eta$. By noting that $B_{\mathfrak{g}_\eta}$ is an invariant bilinear form, we have

$$\langle H_{\alpha_\eta}, X_k^*\rangle = B_{\mathfrak{g}_\eta}(X_k^*, H_{\alpha_\eta}) = B_{\mathfrak{g}_\eta}(X_k^*, [X_{\alpha_\eta}, Y_{\alpha_\eta}]) = B_{\mathfrak{g}_\eta}([X_k^*, X_{\alpha_\eta}], Y_{\alpha_\eta}).$$

Since we have $[X_k^*, X_{\alpha_\eta}] = d\alpha_\eta(X_k^*)X_{\alpha_\eta}$, we get $\langle H_{\alpha_\eta}, X_k^*\rangle = d\alpha_\eta(X_k^*) \cdot B_{\mathfrak{g}_\eta,\alpha_\eta}$. Hence, as we have $a^{\mathrm{res}}_{k,\alpha_\eta} = \langle H_{\alpha_\eta}, X_k^*\rangle$, we get

$$\chi^{\mathrm{res}}_{k,\alpha_\eta}\biggl(\frac{\alpha_\eta(\exp(X_k^*)) - 1}{a^{\mathrm{res}}_{k,\alpha_\eta}}\biggr) = \chi^{\mathrm{res}}_{k,\alpha_\eta}\biggl(\frac{\alpha_\eta(\exp(p^{2m}X_k^*)) - 1}{a^{\mathrm{res}}_{k,\alpha_\eta}}\biggr) = \kappa_{\alpha_\eta}(B_{\mathfrak{g}_\eta,\alpha_\eta})^{-1},$$

where we used that χ_k^{res} is minimally ramified in the first equality.

Similarly, for any $\alpha_y \in \Phi(\mathbf{S_H}, \mathbf{H}_y)_{\mathrm{sym}}$, we have

$$\lim_{m\to\infty} \frac{\alpha_y(\exp(p^{2m}Y^*_{j_\mathbf{H}})) - 1}{p^{2m}} = d\alpha_y(Y^*_{j_\mathbf{H}}) = \langle H_{\alpha_y}, Y^*_{j_\mathbf{H}}\rangle \cdot B^{-1}_{\mathfrak{h}_y,\alpha_y}.$$

Since we have $a_{j_\mathbf{H},\alpha_y} = \langle H_{\alpha_y}, Y^*_{j_\mathbf{H}}\rangle$, we get

$$\chi_{j_\mathbf{H},\alpha_y}\biggl(\frac{\alpha_y(\exp(Y^*_{j_\mathbf{H}})) - 1}{a_{j_\mathbf{H},\alpha_y}}\biggr) = \chi_{j_\mathbf{H},\alpha_y}\biggl(\frac{\alpha_y(\exp(p^{2m}Y^*_{j_\mathbf{H}})) - 1}{a_{j_\mathbf{H},\alpha_y}}\biggr) = \kappa_{\alpha_y}(B_{\mathfrak{h}_y,\alpha_y})^{-1}.$$

Therefore we see that (38) is given by the ratio of $\Delta_{\mathrm{II}}^{\mathbf{H}_y}[a_{j_\mathbf{H}}, \chi_{j_\mathbf{H}}](\exp(Y^*_{j_\mathbf{H}}))$ to $\Delta_{\mathrm{II}}^{\mathbf{G}_\eta}[a_k^{\mathrm{res}}, \chi_k^{\mathrm{res}}](\exp(X_k^*))$ as in the right-hand side of the assertion. □

From Lemma 14.18 and the descent properties of the second transfer factors (both in the twisted and untwisted cases, Lemma 13.3 and [Kal19b, Lemma 4.6.7]), we see that (37) equals

$$\begin{aligned}(39)\quad \Delta_{\mathrm{II}}^{\tilde{\mathbf{G}}}[a_k^{\mathrm{res}}, \chi_k^{\mathrm{res}}](\eta_0) \cdot \chi_k^{\mathrm{res}}(\eta_0) \cdot \frac{\Delta_{\mathrm{II}}^{\mathbf{H}}[a_{j_\mathbf{H}}, \chi_{j_\mathbf{H}}](y\exp(Y^*_{j_\mathbf{H}}))}{\Delta_{\mathrm{II}}^{\mathbf{H}}[a_k^{\mathrm{res}}, \chi_k^{\mathrm{res}}](y\exp(Y^*_{j_\mathbf{H}}))}\\ \cdot \frac{\varepsilon(\mathbf{T}_{\mathbf{G}_\theta}) \cdot \varepsilon(\mathbf{T}_{\mathbf{G}^*_{\eta_0}})^{-1}}{e(\mathbf{G}_{\eta_0}) \cdot (-1)^{|\dot{\Xi}_{\eta_0,\mathrm{ur}}|} \cdot \epsilon_{\mathbf{S}_k,\mathrm{ram}}(s_k) \cdot \epsilon^\star_{\vartheta_k}(s_k)}\\ \cdot \frac{\vartheta_{j_\mathbf{H}}(y)}{\vartheta_k(s_k)} \cdot \Delta_{\mathrm{I,III}}[a_k^{\mathrm{res}}, \chi_k^{\mathrm{res}}]\bigl(y\exp(Y^*_{j_\mathbf{H}}), \eta\exp(X_k^*)\bigr),\end{aligned}$$

where $\chi_k^{\mathrm{res}}(\eta_0)$ is as in Lemma 13.3:

$$\chi_k^{\mathrm{res}}(\eta_0) := \prod_{\alpha_{\mathrm{res}}\in\dot{\Phi}(\mathbf{G}_{\eta_0},\mathbf{S}_k^\natural)} \chi^{\mathrm{res}}_{k,\alpha_{\mathrm{res}}}(l_\alpha).$$

Since $\chi_k^{\mathrm{res}}(\eta_0)$ is independent of η_0 by Lemma 13.4, let us write $\chi_k^{\mathrm{res}}(\mathbf{S}_k^\natural)$ for $\chi_k^{\mathrm{res}}(\eta_0)$ in the following.

Lemma 14.19. *The quantity* $\Delta_{\mathrm{II}}^{\tilde{\mathbf{G}}}[a_k^{\mathrm{res}}, \chi_k^{\mathrm{res}}](\eta_0)$ *is given by*

$$\lambda^{\mathrm{res}}_{k,\mathrm{ur}} \cdot (-1)^{r^{\mathrm{res}}_{k,\mathrm{ur}} + |\dot{\Xi}_{\eta_0,\mathrm{ur}}|} \cdot \prod_{\alpha_{\mathrm{res}}\in\dot{\Phi}(\mathbf{G}_{\eta_0},\mathbf{S}_k^\natural)_{\mathrm{ur}}} f_{(\mathbf{G}_{\eta_0},\mathbf{S}_k^\natural)}(\alpha_{\mathrm{res}}) \cdot \prod_{\substack{\alpha_{\mathrm{res}}\in\dot{\Phi}_{\mathrm{res}}(\mathbf{G},\mathbf{S}_k)_{\mathrm{sym}}\\ N(\alpha)(\nu_0)\neq 1}} \lambda_{\alpha_{\mathrm{res}}},$$

where we put $r^{\mathrm{res}}_{k,\mathrm{ur}} := \sum_{\alpha_{\mathrm{res}}\in\dot{\Phi}_{\mathrm{res}}(\mathbf{G},\mathbf{S}_k)_{\mathrm{ur}}} e_{\alpha_{\mathrm{res}}} r$ *and* $\lambda^{\mathrm{res}}_{k,\mathrm{ur}} = \prod_{\alpha_{\mathrm{res}}\in\dot{\Phi}_{\mathrm{res}}(\mathbf{G},\mathbf{S}_k)_{\mathrm{ur}}} \lambda_{\alpha_{\mathrm{res}}}$.

Proof. By definition, we have

$$\Delta_{\mathrm{II}}^{\tilde{\mathbf{G}}}[a_k^{\mathrm{res}}, \chi_k^{\mathrm{res}}](\eta_0) = \prod_{\substack{\alpha_{\mathrm{res}} \in \dot{\Phi}_{\mathrm{res}}(\mathbf{G}, \mathbf{S}_k) \\ N(\alpha)(\nu_0) \neq 1}} \chi_{k,\alpha_{\mathrm{res}}}^{\mathrm{res}}\biggl(\frac{N(\alpha)(\nu_0) - 1}{a_{k,\alpha_{\mathrm{res}}}^{\mathrm{res}}}\biggr).$$

Since $\nu_0\theta$ is topologically semisimple, the valuation of $N(\alpha)(\nu_0) - 1$ is zero whenever $N(\alpha)(\nu_0) \neq 1$. Hence, for any $\alpha_{\mathrm{res}} \in \dot{\Phi}_{\mathrm{res}}(\mathbf{G}, \mathbf{S}_k)$ such that $N(\alpha)(\nu_0) \neq 1$, we can compute each factor as follows (cf. [Kal19b, Lemma 4.7.1]):

The case where α_{res} is asymmetric: Since $\chi_{k,\alpha_{\mathrm{res}}}^{\mathrm{res}}$ is the trivial character of $F_{\alpha_{\mathrm{res}}}^{\times}$ in this case, we have

$$\chi_{k,\alpha_{\mathrm{res}}}^{\mathrm{res}}\biggl(\frac{N(\alpha)(\nu_0) - 1}{a_{k,\alpha_{\mathrm{res}}}^{\mathrm{res}}}\biggr) = 1.$$

The case where α_{res} is symmetric unramified: Since $\chi_{k,\alpha_{\mathrm{res}}}^{\mathrm{res}}$ is the unique nontrivial quadratic unramified character of $F_{\alpha_{\mathrm{res}}}^{\times}$ and

$$\mathrm{val}_F(a_{k,\alpha_{\mathrm{res}}}^{\mathrm{res}}) = \mathrm{val}_F(\langle H_{\alpha_{\mathrm{res}}}, X_k^* \rangle) = r \in \mathrm{val}_F(F_{\alpha_{\mathrm{res}}}^{\times}),$$

we have

$$\chi_{k,\alpha_{\mathrm{res}}}^{\mathrm{res}}\biggl(\frac{N(\alpha)(\nu_0) - 1}{a_{k,\alpha_{\mathrm{res}}}^{\mathrm{res}}}\biggr) = (-1)^{e_{\alpha_{\mathrm{res}}} r}.$$

The case where α_{res} is symmetric ramified: Since $\nu_0\theta$ is topologically semisimple and $N(\alpha)(\nu_0)$ belongs to the kernel of the norm map $\mathrm{Nr}\colon F_{\alpha_{\mathrm{res}}}^{\times} \to F_{\pm\alpha_{\mathrm{res}}}^{\times}$, we have $N(\alpha)(\nu_0) \equiv -1 \pmod{\mathfrak{p}_{F_{\alpha_{\mathrm{res}}}}}$ whenever $N(\alpha)(\nu_0) \neq 1$. By noting that $\chi_{k,\alpha_{\mathrm{res}}}^{\mathrm{res}}$ is tamely ramified, we get

$$\chi_{k,\alpha_{\mathrm{res}}}^{\mathrm{res}}\biggl(\frac{N(\alpha)(\nu_0) - 1}{a_{k,\alpha_{\mathrm{res}}}^{\mathrm{res}}}\biggr) = \chi_{k,\alpha_{\mathrm{res}}}^{\mathrm{res}}(-2a_{k,\alpha_{\mathrm{res}}}^{\mathrm{res},-1}).$$

As we have $\mathrm{Tr}_{F_{\alpha_{\mathrm{res}}}/F_{\pm\alpha_{\mathrm{res}}}}(a_{k,\alpha_{\mathrm{res}}}^{\mathrm{res}}) = 0$, we have $\mathrm{Nr}_{F_{\alpha_{\mathrm{res}}}/F_{\pm\alpha_{\mathrm{res}}}}(a_{k,\alpha_{\mathrm{res}}}^{\mathrm{res}}) = -a_{k,\alpha_{\mathrm{res}}}^{\mathrm{res},2}$. Hence $\chi_{k,\alpha_{\mathrm{res}}}^{\mathrm{res}}(-2a_{k,\alpha_{\mathrm{res}}}^{\mathrm{res},-1}) = \chi_{k,\alpha_{\mathrm{res}}}^{\mathrm{res}}(2a_{k,\alpha_{\mathrm{res}}}^{\mathrm{res}}) = \lambda_{\alpha_{\mathrm{res}}}$.

Therefore, we get

$$\begin{aligned}
\Delta_{\mathrm{II}}^{\tilde{\mathbf{G}}}[a_k^{\mathrm{res}}, \chi_k^{\mathrm{res}}](\eta_0) &= \prod_{\substack{\alpha_{\mathrm{res}} \in \dot{\Phi}_{\mathrm{res}}(\mathbf{G}, \mathbf{S}_k)_{\mathrm{ur}} \\ N(\alpha)(\nu_0) \neq 1}} (-1)^{e_{\alpha_{\mathrm{res}}} r} \prod_{\substack{\alpha_{\mathrm{res}} \in \dot{\Phi}_{\mathrm{res}}(\mathbf{G}, \mathbf{S}_k)_{\mathrm{ram}} \\ N(\alpha)(\nu_0) \neq 1}} \lambda_{\alpha_{\mathrm{res}}} \\
&= (-1)^{r_{k,\mathrm{ur}}^{\mathrm{res}}} \prod_{\substack{\alpha_{\mathrm{res}} \in \dot{\Phi}_{\mathrm{res}}(\mathbf{G}, \mathbf{S}_k)_{\mathrm{ur}} \\ N(\alpha)(\nu_0) = 1}} (-1)^{e_{\alpha_{\mathrm{res}}} r} \prod_{\substack{\alpha_{\mathrm{res}} \in \dot{\Phi}_{\mathrm{res}}(\mathbf{G}, \mathbf{S}_k)_{\mathrm{ram}} \\ N(\alpha)(\nu_0) \neq 1}} \lambda_{\alpha_{\mathrm{res}}}.
\end{aligned}$$

We compute $(-1)^{e_{\alpha_{\mathrm{res}}} r}$ for $\alpha_{\mathrm{res}} \in \dot{\Phi}_{\mathrm{res}}(\mathbf{G}, \mathbf{S}_k)_{\mathrm{ur}}$ satisfying $N(\alpha)(\nu_0) = 1$ by noting whether $\alpha_{\mathrm{res}} \in \Xi_{\eta_0}$ (i.e., α_{res} appears in the Heisenberg quotient of $\mathbf{G}_{\eta_0}$ with respect to $(\mathbf{x}, \frac{r}{2})$) or not. When $\alpha_{\mathrm{res}} \in \Xi_{\eta_0}$, we have

$$\begin{cases} \frac{r}{2} \in e_{\alpha_{\mathrm{res}}}^{-1}\mathbb{Z} & \text{if } f_{(\mathbf{G}_{\eta_0}, \mathbf{S}_k^{\natural})}(\alpha_{\mathrm{res}}) = +1, \\ \frac{r}{2} \in e_{\alpha_{\mathrm{res}}}^{-1}(\mathbb{Z} + \frac{1}{2}) & \text{if } f_{(\mathbf{G}_{\eta_0}, \mathbf{S}_k^{\natural})}(\alpha_{\mathrm{res}}) = -1, \end{cases}$$

by [Kal19b, Proposition 4.5.1]. This is equivalent to that

$$\begin{cases} e_{\alpha_{\mathrm{res}}} r \equiv 0 \pmod 2 & \text{if } f_{(\mathbf{G}_{\eta_0}, \mathbf{S}_k^{\natural})}(\alpha_{\mathrm{res}}) = +1, \\ e_{\alpha_{\mathrm{res}}} r \equiv 1 \pmod 2 & \text{if } f_{(\mathbf{G}_{\eta_0}, \mathbf{S}_k^{\natural})}(\alpha_{\mathrm{res}}) = -1. \end{cases}$$

By noting that these conditions are simply swapped when $\alpha_{\mathrm{res}} \notin \Xi_{\eta_0}$, we see that

$$(-1)^{e_{\alpha_{\mathrm{res}}} r} = \begin{cases} f_{(\mathbf{G}_{\eta_0}, \mathbf{S}_k^\natural)}(\alpha_{\mathrm{res}}) \\ -f_{(\mathbf{G}_{\eta_0}, \mathbf{S}_k^\natural)}(\alpha_{\mathrm{res}}) \end{cases} = \begin{cases} -f_{(\mathbf{G}_{\eta_0}, \mathbf{S}_k^\natural)}(\alpha_{\mathrm{res}}) \cdot \lambda_{\alpha_{\mathrm{res}}} & \text{if } \alpha_{\mathrm{res}} \in \Xi_{\eta_0}, \\ f_{(\mathbf{G}_{\eta_0}, \mathbf{S}_k^\natural)}(\alpha_{\mathrm{res}}) \cdot \lambda_{\alpha_{\mathrm{res}}} & \text{if } \alpha_{\mathrm{res}} \notin \Xi_{\eta_0} \end{cases}$$

(recall that $\lambda_{\alpha_{\mathrm{res}}} = -1$ since $F_{\alpha_{\mathrm{res}}}/F_{\pm\alpha_{\mathrm{res}}}$ is unramified). Thus we get

$$\prod_{\substack{\alpha_{\mathrm{res}} \in \dot{\Phi}_{\mathrm{res}}(\mathbf{G}, \mathbf{S}_k)_{\mathrm{ur}} \\ N(\alpha)(\nu_0)=1}} (-1)^{e_{\alpha_{\mathrm{res}}} r} = (-1)^{|\dot{\Xi}_{\eta_0,\mathrm{ur}}|} \prod_{\substack{\alpha_{\mathrm{res}} \in \dot{\Phi}_{\mathrm{res}}(\mathbf{G}, \mathbf{S}_k)_{\mathrm{ur}} \\ N(\alpha)(\nu_0)=1}} f_{(\mathbf{G}_{\eta_0}, \mathbf{S}_k^\natural)}(\alpha_{\mathrm{res}}) \cdot \lambda_{\alpha_{\mathrm{res}}}.$$

Again noting that $\lambda_{\alpha_{\mathrm{res}}} = -1$, we have

$$\prod_{\substack{\alpha_{\mathrm{res}} \in \dot{\Phi}_{\mathrm{res}}(\mathbf{G}, \mathbf{S}_k)_{\mathrm{ur}} \\ N(\alpha)(\nu_0)=1}} \lambda_{\alpha_{\mathrm{res}}} = \lambda_{k,\mathrm{ur}}^{\mathrm{res}} \cdot \prod_{\substack{\alpha_{\mathrm{res}} \in \dot{\Phi}_{\mathrm{res}}(\mathbf{G}, \mathbf{S}_k)_{\mathrm{ur}} \\ N(\alpha)(\nu_0)\neq 1}} \lambda_{\alpha_{\mathrm{res}}}.$$

Recalling that $\Phi(\mathbf{G}_{\eta_0}, \mathbf{S}_k^\natural) = \{\alpha_{\mathrm{res}} \in \Phi_{\mathrm{res}}(\mathbf{G}, \mathbf{S}_k) \mid N(\alpha)(\nu_0) = 1\}$, we get the assertion. □

Lemma 14.20. *We have*

$$\varepsilon(\mathbf{S}_k^\natural) \cdot \varepsilon(\mathbf{T}_{\mathbf{G}_{\eta_0}^*})^{-1} = e(\mathbf{G}_{\eta_0}) \prod_{\alpha_{\mathrm{res}} \in \dot{\Phi}(\mathbf{G}_{\eta_0}, \mathbf{S}_k^\natural)_{\mathrm{sym}}} f_{(\mathbf{G}_{\eta_0}, \mathbf{S}_k^\natural)}(\alpha_{\mathrm{res}}) \cdot \lambda_{\alpha_{\mathrm{res}}}$$

Proof. This is a variant of the formula of Kaletha–Kottwitz (Proposition 14.17), which is stated in [Kal15, Corollary 4.11]. □

By noting that $\{\alpha_{\mathrm{res}} \in \dot{\Phi}_{\mathrm{res}}(\mathbf{G}, \mathbf{S}_k) \mid N(\alpha)(\nu_0) \neq 1\} = \dot{\Phi}_{\mathrm{res}}(\mathbf{G}, \mathbf{S}_k) \smallsetminus \dot{\Phi}(\mathbf{G}_{\eta_0}, \mathbf{S}_k^\natural)$, Lemmas 14.19 and 14.20 imply that (39) equals

$$\begin{aligned}(40) \quad & \chi_k^{\mathrm{res}}(\mathbf{S}_k^\natural) \cdot (-1)^{r_{k,\mathrm{ur}}^{\mathrm{res}}} \cdot \lambda_{k,\mathrm{ur}}^{\mathrm{res}} \cdot \prod_{\alpha_{\mathrm{res}} \in \dot{\Phi}_{\mathrm{res}}(\mathbf{G}, \mathbf{S}_k)_{\mathrm{sym}}} \lambda_{\alpha_{\mathrm{res}}} \cdot \prod_{\alpha_{\mathrm{res}} \in \dot{\Phi}(\mathbf{G}_{\eta_0}, \mathbf{S}_k^\natural)_{\mathrm{ram}}} f_{(\mathbf{G}_{\eta_0}, \mathbf{S}_k^\natural)}(\alpha_{\mathrm{res}}) \\ & \cdot \frac{\Delta_{\mathrm{II}}^{\mathbf{H}}[a_{j_{\mathbf{H}}}, \chi_{j_{\mathbf{H}}}](y \exp(Y_{j_{\mathbf{H}}}^*))}{\Delta_{\mathrm{II}}^{\mathbf{H}}[a_k^{\mathrm{res}}, \chi_k^{\mathrm{res}}](y \exp(Y_{j_{\mathbf{H}}}^*))} \cdot \frac{\varepsilon(\mathbf{T}_{\mathbf{G}_\theta}) \cdot \varepsilon(\mathbf{S}_k^\natural)^{-1}}{\epsilon_{\mathbf{S}_k,\mathrm{ram}}(s_k) \cdot \epsilon_{\vartheta_k}^\star(s_k)} \\ & \cdot \frac{\vartheta_{j_{\mathbf{H}}}(y)}{\vartheta_k(s_k)} \cdot \Delta_{\mathrm{I,III}}[a_k^{\mathrm{res}}, \chi_k^{\mathrm{res}}]\big(y \exp(Y_{j_{\mathbf{H}}}^*), \eta \exp(X_k^*)\big).\end{aligned}$$

We put

$$\chi_j^{\mathrm{res}}(\mathbf{S}_{j_{\mathbf{H}}}) := \prod_{\alpha_{\mathrm{res}} \in \dot{\Phi}(\mathbf{H}, \mathbf{S}_{j_{\mathbf{H}}})} \chi_j^{\mathrm{res}}(l_\alpha).$$

Lemma 14.21. *We have*

$$\frac{\Delta_{\mathrm{II}}^{\mathbf{H}}[a_{j_{\mathbf{H}}}, \chi_{j_{\mathbf{H}}}](y \exp(Y_{j_{\mathbf{H}}}^*))}{\Delta_{\mathrm{II}}^{\mathbf{H}}[a_k^{\mathrm{res}}, \chi_k^{\mathrm{res}}](y \exp(Y_{j_{\mathbf{H}}}^*))} = \zeta_{\chi_{j_{\mathbf{H}}}/\chi_j^{\mathrm{res}}, S_{j_{\mathbf{H}}}}(y \exp(Y_{j_{\mathbf{H}}}^*)) \cdot \chi_j^{\mathrm{res}}(\mathbf{S}_{j_{\mathbf{H}}}).$$

Proof. Since $y \exp(Y_{j_{\mathbf{H}}}^*)$ is regular semisimple in $\mathbf{H}$, we have

$$\Delta_{\mathrm{II}}^{\mathbf{H}}[a_{j_{\mathbf{H}}}, \chi_{j_{\mathbf{H}}}](y \exp(Y_{j_{\mathbf{H}}}^*)) = \prod_{\alpha_{\mathrm{res}} \in \dot{\Phi}(\mathbf{H}, \mathbf{S}_{j_{\mathbf{H}}})} \chi_{j_{\mathbf{H}}}\Big(\frac{\alpha_{\mathrm{res}}(y \exp(Y_{j_{\mathbf{H}}}^*)) - 1}{a_{j_{\mathbf{H}}, \alpha_{\mathrm{res}}}}\Big),$$

$$\Delta_{\mathrm{II}}^{\mathbf{H}}[a_k^{\mathrm{res}}, \chi_k^{\mathrm{res}}](y \exp(p^{2m} Y_{j_{\mathbf{H}}}^*)) = \prod_{\alpha_{\mathrm{res}} \in \dot{\Phi}(\mathbf{H}, \mathbf{S}_{j_{\mathbf{H}}})} \chi_k^{\mathrm{res}}\Big(\frac{\alpha_{\mathrm{res}}(y \exp(p^{2m} Y_{j_{\mathbf{H}}}^*)) - 1}{a_{k,\alpha_{\mathrm{res}}}^{\mathrm{res}}}\Big).$$

By Lemma 14.12, we have $a^{\mathrm{res}}_{k,\alpha_{\mathrm{res}}} = l_\alpha \cdot a_{j_{\mathbf{H}},\alpha_{\mathrm{res}}}$. Thus we get

$$\frac{\Delta^{\mathbf{H}}_{\mathrm{II}}[a_{j_{\mathbf{H}}}, \chi_{j_{\mathbf{H}}}](y\exp(Y^*_{j_{\mathbf{H}}}))}{\Delta^{\mathbf{H}}_{\mathrm{II}}[a^{\mathrm{res}}_k, \chi^{\mathrm{res}}_k](y\exp(Y^*_{j_{\mathbf{H}}}))} = \frac{\Delta^{\mathbf{H}}_{\mathrm{II}}[a_{j_{\mathbf{H}}}, \chi_{j_{\mathbf{H}}}](y\exp(Y^*_{j_{\mathbf{H}}}))}{\Delta^{\mathbf{H}}_{\mathrm{II}}[a_{j_{\mathbf{H}}}, \chi^{\mathrm{res}}_k](y\exp(Y^*_{j_{\mathbf{H}}}))} \cdot \prod_{\alpha_{\mathrm{res}} \in \dot{\Phi}(\mathbf{H},\mathbf{S}_{j_{\mathbf{H}}})} \chi^{\mathrm{res}}_k(l_\alpha).$$

Here, on the right-hand side, the ratio of two second transfer factors is given by $\zeta_{\chi_{j_{\mathbf{H}}}/\chi^{\mathrm{res}}_k, S_{j_{\mathbf{H}}}}(y\exp(Y^*_{j_{\mathbf{H}}}))$ by[Kal19b, Lemma 4.6.6]. Thus, by noting that both χ^{res}_k and χ^{res}_j induce the same set of χ-data on $\Phi(\mathbf{H}, \mathbf{S}_{j_{\mathbf{H}}})$, we get the assertion. □

Now recall that Proposition 10.11 associates to $j \in \tilde{\mathcal{J}}^{\mathbf{G}}_{\mathbf{G}_\eta}$ a unique element $y \in \mathfrak{H}_\eta$ and $(D, j_{\mathbf{H}}) \in \mathbf{D}(y, \eta) \times \mathcal{J}^{\mathbf{H}}_{\mathbf{H}_y}$. Also recall that we have fixed an element $\underline{\eta}_j \in \tilde{S}_j$. We put $\underline{y}_j := \tilde{\xi}_D(\underline{\eta}_j) \in S_{j_{\mathbf{H}}}$. Then, by Lemma 13.2 and Proposition 14.15, we have

$$(41) \quad \frac{\Delta_{\mathrm{I,III}}[a^{\mathrm{res}}_k, \chi^{\mathrm{res}}_k](y\exp(Y^*_{j_{\mathbf{H}}}), \eta\exp(X^*_k))}{\Delta_{\mathrm{I,III}}[a^{\mathrm{res}}_k, \chi^{\mathrm{res}}_k](\underline{y}_j\exp(Y^*_{j_{\mathbf{H}}}), \underline{\eta}_k\exp(X^*_k))} = \frac{\vartheta_k(s_k)}{\vartheta_{j_{\mathbf{H}}}(y/\underline{y}_j)} \cdot \zeta_{\mathrm{desc}}(s_k) \cdot \zeta_{\chi^{\mathrm{res}}_j/\chi_{j_{\mathbf{H}}}, S_{j_{\mathbf{H}}}}(y/\underline{y}_j).$$

(Recall that $\eta = s_k\underline{\eta}_k$.) Therefore, by using Lemma 14.21, we see that (40) equals

$$(42) \quad \chi^{\mathrm{res}}_k(\mathbf{S}^\natural_k) \cdot \chi^{\mathrm{res}}_j(\mathbf{S}_{j_{\mathbf{H}}}) \cdot (-1)^{r^{\mathrm{res}}_{k,\mathrm{ur}}} \cdot \lambda^{\mathrm{res}}_{k,\mathrm{ram}} \cdot \varepsilon(\mathbf{T}_{\mathbf{G}_\theta}) \cdot \varepsilon(\mathbf{S}^\natural_k)^{-1}$$
$$\cdot \vartheta_{j_{\mathbf{H}}}(\underline{y}_j) \cdot \zeta_{\chi_{j_{\mathbf{H}}}/\chi^{\mathrm{res}}_j, S_{j_{\mathbf{H}}}}(\underline{y}_j\exp(Y^*_{j_{\mathbf{H}}})) \cdot \Delta_{\mathrm{I,III}}[a^{\mathrm{res}}_k, \chi^{\mathrm{res}}_k](\underline{y}_j\exp(Y^*_{j_{\mathbf{H}}}), \underline{\eta}_k\exp(X^*_k))$$
$$\cdot \epsilon_{\mathbf{S}_k,\mathrm{ram}}(s_k) \cdot \epsilon^\star_{\vartheta_k}(s_k) \cdot \zeta_{\mathrm{desc}}(s_k) \cdot \prod_{\alpha_{\mathrm{res}} \in \dot{\Phi}(\mathbf{G}_{\eta_0}, \mathbf{S}^\natural_k)_{\mathrm{ram}}} f_{(\mathbf{G}_{\eta_0}, \mathbf{S}^\natural_k)}(\alpha_{\mathrm{res}}),$$

where we put $\lambda^{\mathrm{res}}_{k,\mathrm{ram}} = \prod_{\alpha_{\mathrm{res}} \in \dot{\Phi}_{\mathrm{res}}(\mathbf{G},\mathbf{S}_k)_{\mathrm{ram}}} \lambda_{\alpha_{\mathrm{res}}}$.

Now let us examine the factors contained in (42). The factor $\varepsilon(\mathbf{T}_{\mathbf{G}_\theta})$ obviously independent of j. Let $j' \in \tilde{\mathcal{J}}^{\mathbf{G}}_{\mathbf{G}_\eta}$ and $k' \in \tilde{\mathcal{J}}^{\mathbf{G}_\eta}_{G_\eta}(j')$ such that k and k' are G-conjugate. Suppose that $g \in G$ be an element such that $k' = [g] \circ k$ and $\underline{\eta}_{k'} = {}^g\underline{\eta}_k$. Since the F-rational isomorphism $[g] \colon \mathbf{S}_k \to \mathbf{S}_{k'}$ gives a Γ-equivariant isomorphism $\Phi(\mathbf{G}, \mathbf{S}_k) \to \Phi(\mathbf{G}, \mathbf{S}_{k'})$ compatible with twists, we get $r^{\mathrm{res}}_{k,\mathrm{ur}} = r^{\mathrm{res}}_{k',\mathrm{ur}}$, $\lambda^{\mathrm{res}}_{k,\mathrm{ram}} = \lambda^{\mathrm{res}}_{k',\mathrm{ram}}$, and $\varepsilon(\mathbf{S}^\natural_k) = \varepsilon(\mathbf{S}^\natural_{k'})$. Lemma 13.4 implies that $\chi^{\mathrm{res}}_{k'}(\mathbf{S}^\natural_{k'}) = \chi^{\mathrm{res}}_{k'}({}^g\eta_0) = \chi^{\mathrm{res}}_k(\eta_0) = \chi^{\mathrm{res}}_k(\mathbf{S}_k)$. It is a routine work to check that the factors $\chi^{\mathrm{res}}_j(\mathbf{S}_{j_{\mathbf{H}}})$, $\vartheta_{j_{\mathbf{H}}}(\underline{y}_j)$, $\zeta_{\chi_{j_{\mathbf{H}}}/\chi^{\mathrm{res}}_j, S_{j_{\mathbf{H}}}}(\underline{y}_j\exp(Y^*_{j_{\mathbf{H}}}))$, and $\Delta_{\mathrm{I,III}}[a^{\mathrm{res}}_k, \chi^{\mathrm{res}}_k](\underline{y}_j\exp(Y^*_{j_{\mathbf{H}}}), \underline{\eta}_k\exp(X^*_k))$ do not change even if we replace (j, k) with (j', k').

We summarize our discussion so far. We obtained

$$(43) \quad \bar{\Delta}^{\mathrm{spec}}_{\phi,k} = \chi^{\mathrm{res}}_k(\mathbf{S}^\natural_k) \cdot \chi^{\mathrm{res}}_j(\mathbf{S}_{j_{\mathbf{H}}}) \cdot (-1)^{r^{\mathrm{res}}_{k,\mathrm{ur}}} \cdot \lambda^{\mathrm{res}}_{k,\mathrm{ram}} \cdot \varepsilon(\mathbf{T}_{\mathbf{G}_\theta}) \cdot \varepsilon(\mathbf{S}^\natural_k)^{-1} \cdot \vartheta_{j_{\mathbf{H}}}(\underline{y}_j)$$
$$\cdot \zeta_{\chi_{j_{\mathbf{H}}}/\chi^{\mathrm{res}}_j, S_{j_{\mathbf{H}}}}(\underline{y}_j\exp(Y^*_{j_{\mathbf{H}}})) \cdot \Delta_{\mathrm{I,III}}[a^{\mathrm{res}}_k, \chi^{\mathrm{res}}_k](\underline{y}_j\exp(Y^*_{j_{\mathbf{H}}}), \underline{\eta}_k\exp(X^*_k))$$
$$\cdot \epsilon_{\mathbf{S}_k,\mathrm{ram}}(s_k) \cdot \epsilon^\star_{\vartheta_k}(s_k) \cdot \zeta_{\mathrm{desc}}(s_k) \cdot \prod_{\alpha_{\mathrm{res}} \in \dot{\Phi}(\mathbf{G}_{\eta_0}, \mathbf{S}^\natural_k)_{\mathrm{ram}}} f_{(\mathbf{G}_{\eta_0}, \mathbf{S}^\natural_k)}(\alpha_{\mathrm{res}}).$$

Moreover, we checked that all factors contained in the first and second lines of the right-hand side of (43) depend only on the G-conjugacy class of k and, of course, are independent of η. In other words, our remaining task is to check that

$$\epsilon_{\mathbf{S}_k,\mathrm{ram}}(s_k)\cdot\epsilon^{\star}_{\vartheta_k}(s_k)\cdot\zeta_{\mathrm{desc}}(s_k)\cdot\prod_{\alpha_{\mathrm{res}}\in\dot{\Phi}(\mathbf{G}_{\eta_0},\mathbf{S}_k^{\natural})_{\mathrm{ram}}} f_{(\mathbf{G}_{\eta_0},\mathbf{S}_k^{\natural})}(\alpha_{\mathrm{res}}). \tag{44}$$

depends only on the G-conjugacy class of k and is independent of η.

We note that all the factors in (44) are products over sets of ramified (restricted) roots. Thus, there is nothing to prove if $\Phi(\mathbf{G},\mathbf{S}_k)$ and $\Phi_{\mathrm{res}}(\mathbf{G},\mathbf{S}_k)$ do not contain a ramified symmetric element. For example, a sufficient condition for this is that $\mathbf{S}$ splits over a finite extension E of F whose ramification index $e(E/F)$ is odd. Indeed, we have the following diagram:

$$\begin{array}{ccccccc} & & F_{\alpha_{\mathrm{res}}} & \subset & F_\alpha & \subset & E \\ & & \cup & & \cup & & \\ F & \subset & F_{\pm\alpha_{\mathrm{res}}} & \subset & F_{\pm\alpha} & & \end{array}$$

Hence, if $e(E/F)$ is odd, then the extension $F_\alpha/F_{\pm\alpha}$ and $F_{\alpha_{\mathrm{res}}}/F_{\pm\alpha_{\mathrm{res}}}$ cannot be quadratic ramified. Let us record this observation here.

Theorem 14.22. *The spectral transfer factor $\Delta^{\mathrm{spec}}_{\phi,k}$ is well-defined if $\mathbf{S}$ splits over a finite extension E of F whose ramification index $e(E/F)$ is odd. In particular, the twisted endoscopic character relation* (25) *is satisfied.*

What we will do in the rest of paper is to show that (44) indeed depends only on the G-conjugacy class of k and is independent of η in the case where $\mathbf{G}=\mathrm{GL}_n$.

Remark 14.23. Recall that the members of $\Pi^{\mathbf{G}}_\phi$ are parametrized by the set $\mathcal{J}^{\mathbf{G}}_G$. In the case of standard endoscopy, in [Kal19b, Section 5.3], Kaletha introduced the paring

$$\langle -,-\rangle_{\mathfrak{w}}\colon \mathcal{J}^{\mathbf{G}}_G\times\pi_0(S^+_\phi)\to\mathbb{C}^\times;\quad \langle j,s\rangle_{\mathfrak{w}}\mapsto\langle \mathrm{inv}(j_{\mathfrak{w}},j),s\rangle$$

(see [Kal19b, 1155 page] for the details). This is nothing but the spectral transfer factor in the sense of this paper in the untwisted case. In other words, we have

$$\Delta^{\mathrm{spec}}_{\phi,j}=\langle \mathrm{inv}(j,j_{\mathfrak{w}}),s\rangle$$

when θ is trivial. We may understand that Kaletha's proof of the standard endoscopic character relation ([Kal19b, Theorem 6.3.4]) contains this explicit determination of the spectral transfer factor in the standard case.

15. GL_n consideration

In the following, let $\mathbf{G}:=\mathrm{GL}_n$ and $\theta:=J_n{}^t(-)^{-1}J_n^{-1}$, where n is even. (Recall that this assumption is harmless for our purpose; see Remark 6.1).

15.1. Twisted elliptic maximal tori of GL_n.

Let us assume that $(\tilde{\mathbf{S}},\mathbf{S})$ is an F-rational twisted maximal torus of $\mathbf{G}$ whose $\mathbf{S}$ is elliptic. Then $(\tilde{\mathbf{S}},\mathbf{S})$ is elliptic by Remark 3.5. It is well-known that there exists a finite extension E of F of degree n such that $\mathbf{S}$ is isomorphic to $\mathrm{Res}_{E/F}\,\mathbb{G}_{\mathrm{m}}$.

Lemma 15.1. *There exists an element $\tau_\theta\in\mathrm{Aut}_F(E)$ of order 2 such that $\theta_{\mathbf{S}}(s)=\tau_\theta(s)^{-1}$ for any $s\in S\cong E^\times$.*

Proof. Note that $\theta_{\mathbf{S}}$ is of the form $x^t(-)^{-1}x^{-1}$ for some $x \in \mathrm{GL}_n(F)$. Since $\theta_{\mathbf{S}}$ preserves $S \subset \mathrm{GL}_n(F)$, the map $X \mapsto x^t X x^{-1}$ preserves $\mathfrak{s} \subset \mathfrak{gl}_n(F)$. As the map $X \mapsto x^t X x^{-1}$ is an involutive F-algebra homomorphism on $\mathfrak{s} \cong E$, it is given by an element τ_θ of $\mathrm{Aut}_F(E)$ whose order is either 1 or 2. In other words, we have $\theta_{\mathbf{S}}(s) = \tau_\theta(s)^{-1}$ for any $s \in S \cong E^\times$.

Let us show that τ_θ is not trivial. For the sake of contradiction, we suppose that τ_θ is trivial. Then the automorphism $\theta_{\mathbf{S}}$ of $\mathbf{S}$ is given by $s \mapsto s^{-1}$ on $S \cong E^\times$. Hence we have $\mathbf{S}^\natural(F) \subset \mathbf{S}^{\theta_{\mathbf{S}}}(F) = S^{\theta_{\mathbf{S}}} = \{\pm 1\}$. On the other hand, since $(\tilde{\mathbf{S}}, \mathbf{S})$ is a twisted maximal torus, there exists an element $g \in \mathbf{G}$ such that ${}^g\mathbf{S} = \mathbf{T}$ and $\theta_{\mathbf{S}}$ is mapped to $\theta|_{\mathbf{T}}$. This implies that the torus $\mathbf{S}^\natural$ is isomorphic to $\mathbf{T}^{\theta,\circ}$ over $\overline{F}$. In particular, the rank of $\mathbf{S}^\natural$ is given by $n/2$. However, there is no F-rational torus whose rank is nonzero such that the set of F-valued points is of order at most 2. Hence we get a contradiction. □

In the following, we let $\tau_\theta \in \mathrm{Aut}_F(E)$ be the element as in Lemma 15.1. Let $E_\pm$ be the fixed field of τ_θ in E.

15.2. **Roots of elliptic maximal tori of** GL_n**.** We next recall a description of the set of roots of $\mathbf{S}$ in GL_n following[Tam16, Sections 3.1 and 3.2] (see also [OT21, Sections 3.2 and 5.1]). First we fix a set $\{g_1, \ldots, g_n\}$ of representatives of the quotient Γ/Γ_E such that $g_1 = \mathrm{id}$. Then we get an isomorphism $\mathbf{S}(\overline{F}) \cong \prod_{i=1}^n \overline{F}^\times$ which maps $x \in E^\times \cong \mathbf{S}(F)$ to $(g_1(x), \ldots, g_n(x))$. Then the projections

$$\delta_i \colon \mathbf{S}(\overline{F}) \xrightarrow{\cong} \prod_{i=1}^n \overline{F}^\times \to \overline{F}^\times; \quad (x_1, \ldots, x_n) \mapsto x_i$$

form a $\mathbb{Z}$-basis of $X^*(\mathbf{S})$. The set $\Phi(\mathbf{G}, \mathbf{S})$ of roots of $\mathbf{S}$ in $\mathbf{G}$ is given by

$$\left\{ \begin{bmatrix} g_i \\ g_j \end{bmatrix} := \delta_i - \delta_j \,\middle|\, 1 \leq i \neq j \leq n \right\}$$

and the set $\dot{\Phi}(\mathbf{G}, \mathbf{S})$ is described as follows:

$$(\Gamma_E \backslash \Gamma / \Gamma_E)' \xrightarrow{1:1} \dot{\Phi}(\mathbf{G}, \mathbf{S}); \quad \Gamma_E g_i \Gamma_E \mapsto \Gamma \cdot \begin{bmatrix} 1 \\ g_i \end{bmatrix},$$

where $(\Gamma_E \backslash \Gamma / \Gamma_E)'$ is the set of nontrivial double-Γ_E-cosets in Γ.

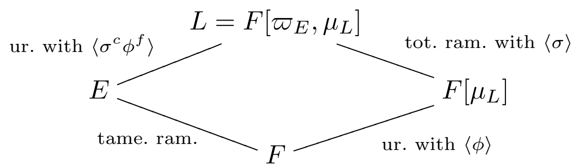
Suppose that E/F is tamely ramified in the following. We simply write e (resp. f) for the ramification index $e(E/F)$ (resp. residue degree $f(E/F)$). We first recall an explicit choice of a set of representatives of Γ/Γ_E, following [Tam16, Section 3.2]. Let μ_E denote the set of roots of unity in E. We take uniformizers ϖ_E and ϖ_F of E and F, respectively, so that $\varpi_E^e = \zeta_{E/F}\varpi_F$ for some $\zeta_{E/F} \in \mu_E$. We fix a primitive e-th root ζ_e of unity and an e-th root $\zeta_{E/F,e}$ of $\zeta_{E/F}$, and put $\zeta_\phi := \zeta_{E/F,e}^{q-1}$. Then $L := E[\zeta_e, \zeta_{E/F,e}]$ is a tamely ramified extension of F which contains the Galois closure of E/F and is unramified over E. The Galois group $\mathrm{Gal}(L/F)$ of the extension L/F is given by the semi-direct product $\langle\sigma\rangle \rtimes \langle\phi\rangle$, where

$$\sigma \colon \zeta \mapsto \zeta \quad (\zeta \in \mu_L), \quad \varpi_E \mapsto \zeta_e \varpi_E$$
$$\phi \colon \zeta \mapsto \zeta^q \quad (\zeta \in \mu_L), \quad \varpi_E \mapsto \zeta_\phi \varpi_E$$

and $\phi\sigma\phi^{-1} = \sigma^q$. Moreover, as explained in [Tam16, Proposition 3.3 (i)], we can take a set of representatives of Γ/Γ_E to be

$$\{\Gamma_F/\Gamma_E\} := \{\sigma^k \phi^i \mid 0 \leq k \leq e-1,\, 0 \leq i \leq f-1\}.$$

Here we implicitly regard each $\sigma^k\phi^i \in \Gamma_{L/F}$ as an element of Γ_F by taking its extension to $\overline{F}$ from L. We note that, as L/E is unramified, there exists an integer c such that $\mathrm{Gal}(L/E) = \langle \sigma^c\phi^f \rangle$.



We recall a fact about symmetric ramified roots of $\mathbf{S}$ in $\mathbf{G}$.

Proposition 15.2 ([OT21, Proposition 5.3]). *Let $\alpha \in \Phi(\mathbf{G}, \mathbf{S})$ be a root of the form $\left[\begin{smallmatrix} 1 \\ g \end{smallmatrix}\right]$ for some $g = \sigma^k\phi^i$. The root α is symmetric ramified if and only if $g = \sigma^{\frac{e}{2}}$ (hence e must be even in this case).*

Lemma 15.3. *If e is even, then $E/E_{\pm}$ must be ramified so that a $\theta_{\mathbf{S}}$-stable toral character of S exists.*

Proof. Let ϑ be a $\theta_{\mathbf{S}}$-stable toral character of S. If we let $r \in \mathbb{R}_{>0}$ be the depth of ϑ, then we can take a $\theta_{\mathbf{S}}$-stable element $X^* \in \mathfrak{s}^*_{-r}$ representing $\vartheta|_{S_r}$ (Lemma 5.3). By the torality of ϑ, X^* must satisfy Yu's condition **GE2** (see [Yu01, Section 8] or Section 4.2), which means that $\mathrm{val}_F(\langle X^*, H_\alpha \rangle) = -r$ for any $\alpha \in \Phi(\mathbf{G}, \mathbf{S})$.

We identify $\mathfrak{s}^* \cong E^* = \mathrm{Hom}_F(E, F)$ with E via the F-linear isomorphism $[Y \mapsto \mathrm{Tr}_{E/F}(XY)] \leftrightarrow X$. Write X for the element of E corresponding to $X^* \in E^*$ under this identification. If we write $\alpha = \left[\begin{smallmatrix} g_i \\ g_j \end{smallmatrix}\right]$ as in the above notation, then we have $\langle X^*, H_\alpha \rangle = g_i(X) - g_j(X)$. Thus, for any α belonging to the Γ-orbit of $\alpha = \left[\begin{smallmatrix} 1 \\ g \end{smallmatrix}\right]$ with $g = \sigma^k\phi^i$, we have $\mathrm{val}_F(\langle X^*, H_\alpha \rangle) = \mathrm{val}_F(X - g(X))$.

Now, for the sake of contradiction, let us suppose that $E/E_{\pm}$ is unramified. Since X^* is $\theta_{\mathbf{S}}$-invariant and the above identification between E^* and E is Galois-equivariant, X must satisfy $-\tau_\theta(X) = X$. We write $X = \varpi_E^t u$ with $t \in \mathbb{Z}$ and $u \in \mathcal{O}_E^\times$. Then we have $\sigma^{\frac{e}{2}}(X) \equiv (-1)^t X \pmod{\mathfrak{p}_E^{t+1}}$. On the other hand, we have $\tau_\theta\sigma^{\frac{e}{2}}(X) \equiv (-1)^{t+1} X \pmod{\mathfrak{p}_E^{t+1}}$. Since $E/E_{\pm}$ is unramified, $\tau_\theta\sigma^{\frac{e}{2}} \neq \mathrm{id}$. Thus, by considering the condition **GE2** for $g = \sigma^{\frac{e}{2}}$ and $g = \tau_\theta\sigma^{\frac{e}{2}}$, we get

$$\mathrm{val}_F(X - \sigma^{\frac{e}{2}}(X)) = r = \mathrm{val}_F(X - \tau_\theta\sigma^{\frac{e}{2}}(X)).$$

However, this is impossible because we have

$$X - \sigma^{\frac{e}{2}}(X) \equiv X - (-1)^t X \pmod{\mathfrak{p}_E^{t+1}} \quad \text{and}$$
$$X - \tau_\theta\sigma^{\frac{e}{2}}(X) \equiv X - (-1)^{t+1} X \pmod{\mathfrak{p}_E^{t+1}}$$

and exactly one of these is nonzero. □

15.3. **Computation of spectral transfer factors.** Now we go back to the situation as in Section 14. Thus the explanation given in the previous subsections are applied to the F-rational elliptic twisted maximal torus $(\tilde{\mathbf{S}}_k, \mathbf{S}_k)$ of $(\tilde{\mathbf{G}}, \mathbf{G})$. Recall that we want to show that the quantity (44), which is given by

$$\epsilon_{\mathbf{S}_k,\mathrm{ram}}(s_k) \cdot \epsilon^\star_{\vartheta_k}(s_k) \cdot \zeta_{\mathrm{desc}}(s_k)^{-1} \cdot \prod_{\alpha_{\mathrm{res}} \in \dot{\Phi}(\mathbf{G}_{\eta_0}, \mathbf{S}_k^\natural)_{\mathrm{ram}}} f_{(\mathbf{G}_{\eta_0}, \mathbf{S}_k^\natural)}(\alpha_{\mathrm{res}}),$$

depends only on the G-conjugacy class of k and is independent of η.

Suppose that $\mathbf{S}_k$ is isomorphic to $\mathrm{Res}_{E/F}\,\mathbb{G}_{\mathrm{m}}$, where E/F is a tamely ramified extension of degree n. If the ramification index e of E/F is odd, then so is that of the Galois closure of E/F, which implies that $\mathbf{S}_k$ splits over a finite extension of F with odd ramification index. Since such a case is already treated in Theorem 14.22, we assume that e is even in the following. In particular, by Lemma 15.3, $E/E_{\pm}$ is a ramified quadratic extension with Galois group generated by $\tau_\theta = \sigma^{\frac{e}{2}}$.

Note that the toral invariant is always trivial when $\mathbf{G} = \mathrm{GL}_n$ (see [OT21, Proposition 4.4]), hence the character $\epsilon_{\mathbf{S}_k,\mathrm{ram}}$ is trivial. Moreover, we have

Lemma 15.4. *If e is even, then the character $\epsilon^{\star}_{\vartheta_k}$ is trivial.*

Proof. Recall that, for any $s \in S_k$, $\epsilon^{\star}_{\vartheta_k}(s)$ is defined to be the product of $\epsilon_\alpha(s)$ over $\alpha \in \ddot{\Xi}(\mathbf{G}, \mathbf{S}_k)$ whose restricted root α_{res} is ramified. If the ramification index e of E/F is even, then there exists a ramified symmetric root of $\mathbf{S}$ in $\mathbf{G}$ by Proposition 15.2. Then, as discussed in [Kal19b, Section 4.7] (see also [OT21, Section 6.4]), the depth r of the toral character ϑ_k of S_k is given by $\frac{2s+1}{e}$ for some integer s. However, this implies that the set $\Xi(\mathbf{G}, \mathbf{S}_k)$ of roots appearing in the Heisenberg space is empty (see [OT21, Remark 5.10]). Thus we get the assertion. $\square$

Hence we are reduced to investigate the following product:

$$\prod_{\substack{\alpha \in \ddot{\Phi}(\mathbf{G},\mathbf{S}_k)_{\mathrm{asym}} \\ \alpha_{\mathrm{res}}:\mathrm{ram}}} \epsilon_\alpha(s_k) \prod_{\substack{\alpha \in \dot{\Phi}(\mathbf{G},\mathbf{S}_k)_{\mathrm{ur}} \\ \alpha_{\mathrm{res}}:\mathrm{ram}}} \epsilon_\alpha(s_k) \prod_{\alpha_{\mathrm{res}} \in \dot{\Phi}(\mathbf{G}_{\eta_0},\mathbf{S}_k^{\natural})_{\mathrm{ram}}} f_{(\mathbf{G}_{\eta_0},\mathbf{S}_k^{\natural})}(\alpha_{\mathrm{res}}). \tag{45}$$

Lemma 15.5. *The third product in* (45) *equals*

$$\prod_{\alpha_{\mathrm{res}} \in \dot{\Phi}(\mathbf{G}_{\eta_0},\mathbf{S}_k^{\natural})_{\mathrm{ram}}^{(\mathrm{asym})}} f_{(\mathbf{G}_{\eta_0},\mathbf{S}_k^{\natural})}(\alpha_{\mathrm{res}}) \cdot \prod_{\alpha_{\mathrm{res}} \in \dot{\Phi}(\mathbf{G}_{\eta_0},\mathbf{S}_k^{\natural})_{\mathrm{ram}}^{(\mathrm{ur})}} f_{(\mathbf{G}_{\eta_0},\mathbf{S}_k^{\natural})}(\alpha_{\mathrm{res}}).$$

Proof. Let $\alpha \in \Phi(\mathbf{G}, \mathbf{S}_k)_{\mathrm{ram}}$ be an element satisfying $\alpha_{\mathrm{res}} \in \dot{\Phi}(\mathbf{G}_{\eta_0}, \mathbf{S}_k^{\natural})_{\mathrm{ram}}^{(\mathrm{ram})}$. Then α is fixed by $\theta_{\mathbf{S}}$. Indeed, we may suppose that α is of the form $\begin{bmatrix} 1 \\ \sigma^{\frac{e}{2}} \end{bmatrix}$. Since $\tau_\theta = \sigma^{\frac{e}{2}}$,

$$\theta_{\mathbf{S}}(\alpha) = \tau_\theta \cdot \begin{bmatrix} 1 \\ \sigma^{\frac{e}{2}} \end{bmatrix}^{-1} = \sigma^{\frac{e}{2}} \begin{bmatrix} \sigma^{\frac{e}{2}} \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \sigma^{\frac{e}{2}} \end{bmatrix} = \alpha.$$

Hence, we have $f_{(\mathbf{G}_{\eta_0},\mathbf{S}_k^{\natural})}(\alpha_{\mathrm{res}}) = f_{(\mathbf{G},\mathbf{S}_k)}(\alpha)$ as noted in the proof of Proposition 12.7. Again by using that $f_{(\mathbf{G},\mathbf{S}_k)}(\alpha) = 1$ ([OT21, Proposition 4.4]), we get $f_{(\mathbf{G}_{\eta_0},\mathbf{S}_k^{\natural})}(\alpha_{\mathrm{res}}) = 1$. $\square$

Lemma 15.6. *There exists an element of $\tilde{S}_k$ of order* 2.

Proof. We utilize a realization of $\tilde{G}$ as the space of bilinear forms as in [Wal10, Section 1.2] (see also [Li13, Section 3.6]).

Let V be an n-dimensional F-vector space equipped with basis $\{e_i\}_{i=1,\dots,n}$. We let $\tilde{\theta}$ be a symplectic form on V such that the representation matrix of $\tilde{\theta}$ with respect to $\{e_i\}_{i=1,\dots,n}$ is J_{2n}, i.e., $\tilde{\theta}(e_k, e_l) = (-1)^{k-1}\delta_{k,2n+1-l}$. Let $\mathrm{Hom}_F^{\mathrm{nondeg}}(V \otimes_F V, F)$ denote the space of non-degenerate F-bilinear forms on V. Note that $\mathrm{Hom}_F^{\mathrm{nondeg}}(V \otimes_F V, F)$ has a bi-$\mathrm{GL}_F(V)$-torsor structure by

$$(g \cdot q \cdot g')(v, v') := q(g^{-1}v, g'v')$$

for any $q \in \mathrm{Hom}_F^{\mathrm{nondeg}}(V \otimes_F V, F)$ and $g, g' \in \mathrm{GL}_F(V)$. (Thus we may regard $\tilde{\theta}$ as a "base point" of $\mathrm{Hom}_F^{\mathrm{nondeg}}(V \otimes_F V, F)$.) Then $\mathrm{Hom}_F^{\mathrm{nondeg}}(V \otimes_F V, F)$ is identified with $\tilde{G} = \mathrm{GL}_n(F) \rtimes \theta$ bi-$\mathrm{GL}_n(F)$-equivariantly by the following association:

$$\mathrm{Hom}_F^{\mathrm{nondeg}}(V \otimes_F V, F) \leftrightarrow \mathrm{GL}_n(F) \rtimes \theta \colon g \cdot \tilde{\theta} \mapsto g \rtimes \theta.$$

Let us examine how the condition that $g \rtimes \theta$ is of order 2 can be rephrased on the space $\mathrm{Hom}_F^{\mathrm{nondeg}}(V \otimes_F V, F)$. The order of $g \rtimes \theta \in \tilde{G}$ is 2 if and only if we have $g \cdot \tilde{\theta} = \tilde{\theta} \cdot g^{-1}$. Let ι be the involution on the space $\mathrm{Hom}_F^{\mathrm{nondeg}}(V \otimes_F V, F)$ given by swapping two entries of $V \otimes_F V$, i.e.,

$$\iota(q)(v, v') = q(v', v)$$

for $q \in \mathrm{Hom}_F^{\mathrm{nondeg}}(V \otimes_F V, F)$ and $v, v' \in V$. Then we have $\iota(g \cdot \tilde{\theta}) = -\tilde{\theta} \cdot g^{-1}$. Indeed, we have

$$\iota(g \cdot \tilde{\theta})(v, v') = (g \cdot \tilde{\theta})(v', v) = \tilde{\theta}(g^{-1}v', v) = -\tilde{\theta}(v, g^{-1}v') = -(\tilde{\theta} \cdot g^{-1})(v, v')$$

for $v, v' \in V$ (we used that $\tilde{\theta}$ is symplectic in the third equality). Hence, $g \rtimes \theta$ is of order 2 if and only if $\iota(g \cdot \tilde{\theta}) = -g \cdot \tilde{\theta}$, in other words, $g \cdot \tilde{\theta}$ is symplectic.

Now we note that elements of $\tilde{S}_k$ can be realized in $\mathrm{Hom}_F^{\mathrm{nondeg}}(V \otimes_F V, F)$ in the following way ([Wal10, Section 1.3]). Recall that $S_k \cong E^\times$ and we have a degree 2 subextension $E/E_\pm$ with Galois group $\langle \tau_\theta \rangle$. For any $x \in E^\times$, we define an F-bilinear form $\tilde{x}$ on E by

$$\tilde{x}(v, v') := \mathrm{Tr}_{E/F}(v\tau_\theta(v')x).$$

Then, by choosing an F-basis of E, we can embed $\{\tilde{x} \mid x \in E^\times\}$ in $\mathrm{Hom}_F^{\mathrm{nondeg}}(V \otimes_F V, F)$. This subset realizes $\tilde{S}_k$.

Therefore, in order to show the claim, it suffices to find an element $x \in E^\times$ such that $\tilde{x}$ is symplectic. If we let $x \in E^\times$ be any element satisfying $\mathrm{Tr}_{E/E_\pm}(x) = 0$, then $\tilde{x}$ is symplectic. □

Proposition 15.7. *If e is even, then we have natural identifications*

$$\{\alpha \in \ddot{\Phi}_{\mathrm{asym}}(\mathbf{G}, \mathbf{S}_k) \mid \alpha_{\mathrm{res}} : \mathrm{ram}\} \xrightarrow{1:1} \dot{\Phi}(\mathbf{G}_{\eta_0}, \mathbf{S}_k^\natural)_{\mathrm{ram}}^{(\mathrm{asym})} \colon \alpha \mapsto \alpha_{\mathrm{res}},$$

$$\{\alpha \in \dot{\Phi}_{\mathrm{ur}}(\mathbf{G}, \mathbf{S}_k) \mid \alpha_{\mathrm{res}} : \mathrm{ram}\} \xrightarrow{1:1} \dot{\Phi}(\mathbf{G}_{\eta_0}, \mathbf{S}_k^\natural)_{\mathrm{ram}}^{(\mathrm{ur})} \colon \alpha \mapsto \alpha_{\mathrm{res}}.$$

Proof. We consider only the case of asymmetric roots with ramified restriction since the case of symmetric unramified roots with ramified restriction can be treated in the same manner. To show that the association $\alpha \mapsto \alpha_{\mathrm{res}}$ gives the asserted identification, we must check the following:

(1) α and $\theta(\alpha)$ belong to the same class in $\ddot{\Phi}_{\mathrm{asym}}(\mathbf{G}, \mathbf{S}_k)$;
(2) any $\alpha \in \Phi_{\mathrm{asym}}(\mathbf{G}, \mathbf{S}_k)$ whose α_{res} is ramified descends to $\mathbf{G}_{\eta_0}$.

As investigated in the proof of Lemma 12.2, we must have $\theta(\alpha) \neq \alpha$. Moreover, if we let τ_α be the nontrivial element of $\mathrm{Gal}(F_{\alpha_{\mathrm{res}}}/F_{\pm\alpha_{\mathrm{res}}})$, then we have $\tau_\alpha(\alpha) = -\theta(\alpha)$. This implies the condition (1). For the condition (2), we note that α descends to $\mathbf{G}_{\eta_0}$ if and only if α descends to $\mathbf{G}_{\eta'_0}$ for any topologically semisimple $\eta'_0 \in \tilde{S}_k$, which is equivalent to $\alpha(\eta_0'^2) = 1$ (see Lemma 12.2 and its proof). Any element η'_0 of $\tilde{S}$ of order 2, which exists by Lemma 15.6, satisfies the latter condition. □

By combining Lemma 15.5 with Proposition 15.7, we see that (45) equals the following product:

$$\prod_{\alpha_{\mathrm{res}}\in\dot{\Phi}(\mathbf{G}_{\eta_0},\mathbf{S}_k^{\natural})^{(\mathrm{asym})}_{\mathrm{ram}}\sqcup\dot{\Phi}(\mathbf{G}_{\eta_0},\mathbf{S}_k^{\natural})^{(\mathrm{ur})}_{\mathrm{ram}}} \epsilon_\alpha(s_k)\cdot f_{(\mathbf{G}_{\eta_0},\mathbf{S}_k^{\natural})}(\alpha_{\mathrm{res}}). \tag{46}$$

Therefore, by recalling that $\eta = s_k\underline{\eta}_k$, we see that (46) equals

$$\prod_{\alpha_{\mathrm{res}}\in\dot{\Phi}(\mathbf{G}_{\underline{\eta}_k},\mathbf{S}_k^{\natural})^{(\mathrm{asym})}_{\mathrm{ram}}\sqcup\dot{\Phi}(\mathbf{G}_{\underline{\eta}_k},\mathbf{S}_k^{\natural})^{(\mathrm{ur})}_{\mathrm{ram}}} f_{(\mathbf{G}_{\underline{\eta}_k},\mathbf{S}_k^{\natural})}(\alpha_{\mathrm{res}}) \tag{47}$$

by Propositions 12.5 and 12.6. Now we can prove the following.

Proposition 15.8. *If e is even, then* (46) *depends only on the G-conjugacy class of k and is independent of η.*

Proof. It is obvious that (46) is independent of η. Let $j' \in \tilde{\mathcal{J}}^{\mathbf{G}}_{\mathbf{G}_\eta}$ and $k' \in \tilde{\mathcal{J}}^{\mathbf{G}_\eta}_{G_\eta}(j)$ such that k and k' are G-conjugate. Suppose that $g \in G$ be an element such that $k' = [g]\circ k$ and $\underline{\eta}_{k'} = {}^g\underline{\eta}_k$. Then the g-conjugation induces F-rational isomorphisms $[g]\colon \mathbf{G}_{\underline{\eta}_k} \to \mathbf{G}_{\underline{\eta}_{k'}}$ and $\mathbf{S}_k^{\natural} \to \mathbf{S}_{k'}^{\natural}$. Hence we get the assertion. □

We summarize what we obtained.

Theorem 15.9. *The spectral transfer factor $\Delta^{\mathrm{spec}}_{\phi,k}$ is well-defined for GL_n with even n. In particular, the twisted endoscopic character relation* (25) *is satisfied.*

15.4. **A consequence.**

Lemma 15.10. *Let $\mathbf{H}$ be either a quasi-split special orthogonal or symplectic group over F which is an endoscopic group of $(\mathbf{G},\theta)$. Let $\phi_{\mathbf{H}}$ be a toral supercuspidal L-parameter of depth $r\in\mathbb{R}_{>0}$ in the sense of Kaletha (Definition 7.19). Suppose that $\mathcal{S}_{\phi_{\mathbf{H}}} := \pi_0(\mathbf{Z}_{\hat{\mathbf{G}}}(\mathrm{Im}(\phi_{\mathbf{H}}))/\mathbf{Z}_{\hat{\mathbf{G}}})$ is trivial. Then $\hat{\xi}\circ\phi_{\mathbf{H}}$ is toral supercuspidal as an L-parameter of $\mathbf{G}$ of depth $r\in\mathbb{R}_{>0}$.*

Proof. We put $\phi := \hat{\xi}\circ\phi_{\mathbf{H}}$. Let us check that the three conditions (0), (1), (2) of Definition 7.19 are satisfied by ϕ. By the assumption that $\mathcal{S}_{\phi_{\mathbf{H}}}$ is trivial, we see that ϕ is irreducible as an n-dimensional representation of W_F, which implies (0). Since $\phi_{\mathbf{H}}$ is toral supercuspidal of depth $r\in\mathbb{R}_{>0}$, $\mathbf{Z}_{\hat{\mathbf{H}}}(\phi_{\mathbf{H}}(I_F^r))$ is a maximal torus of $\hat{\mathbf{H}}$ containing $\phi_{\mathbf{H}}(P_F)$. We note that $\mathbf{Z}_{\hat{\mathbf{G}}}(\phi(I_F^r))$ is a Levi subgroup of $\hat{\mathbf{G}}$ by (the proof of) [Kal19b, Lemma 5.2.2 (1)]. In other words, $\mathbf{Z}_{\hat{\mathbf{G}}}(\phi(I_F^r))$ is a $\hat{\theta}$-stable Levi subgroup of $\hat{\mathbf{G}}$ whose $\mathbf{Z}_{\hat{\mathbf{H}}}(\phi_{\mathbf{H}}(I_F^r)) = \mathbf{Z}_{\hat{\mathbf{G}}}(\phi(I_F^r))^{\hat{\theta},\circ}$ is a maximal torus of $\hat{\mathbf{H}}$. This implies that the Levi subgroup $\mathbf{Z}_{\hat{\mathbf{G}}}(\phi(I_F^r))$ is necessarily a ($\hat{\theta}$-stable) maximal torus of $\mathbf{G}$. Thus we get (1). The condition (3) is obviously satisfied. □

Now we arrive at the following consequence.

Theorem 15.11. *Let $\mathbf{H}$ be either a split odd special orthogonal or symplectic group over F. Let $\Pi^{\mathbf{H}}_{\phi_{\mathbf{H}}}$ be a toral supercuspidal L-packet with L-parameter $\phi_{\mathbf{H}}$ in the sense of Kaletha (see Section 7). Let $\Pi^{\mathbf{H}}_{\phi_{\mathbf{H}},\mathrm{Art}}$ be the L-packet of $\mathbf{H}$ corresponding to $\phi_{\mathbf{H}}$ in the sense of Arthur (*[Art13, Theorem 2.2.1]*). Then we have $\Pi^{\mathbf{H}}_{\phi_{\mathbf{H}}} = \Pi^{\mathbf{H}}_{\phi_{\mathbf{H}},\mathrm{Art}}$.*

Proof. Recall that both $\Pi^{\mathbf{H}}_{\phi_{\mathbf{H}}}$ and $\Pi^{\mathbf{H}}_{\phi_{\mathbf{H}},\mathrm{Art}}$ are bijective to the set of irreducible characters of the "S-group" $\mathcal{S}_{\phi_{\mathbf{H}}}$ ([Kal19b, Section 5.3] and [Art13, Theorem 2.2.1]). We first note that we may assume $|\Pi^{\mathbf{H}}_{\phi_{\mathbf{H}}}|=|\Pi^{\mathbf{H}}_{\phi_{\mathbf{H}},\mathrm{Art}}|=1$ by a standard argument based on the theory of standard endoscopy. Indeed, suppose that $|\Pi^{\mathbf{H}}_{\phi_{\mathbf{H}}}|=|\Pi^{\mathbf{H}}_{\phi_{\mathbf{H}},\mathrm{Art}}|>1$. Then the S-group contains a nontrivial element, which means that the L-parameter $\phi_{\mathbf{H}}$ factors through the L-group of a nontrivial standard endoscopic group $\mathbf{H}'$ of $\mathbf{H}$. Let $\phi_{\mathbf{H}'}$ be an L-parameter of $\mathbf{H}'$ such that its lift to $\mathbf{H}$ is $\phi_{\mathbf{H}}$. By [Kal19b, Theorem 6.3.4] and [Art13, Theorem 2.2.1], both $\Pi^{\mathbf{H}}_{\phi_{\mathbf{H}}}$ and $\Pi^{\mathbf{H}}_{\phi_{\mathbf{H}},\mathrm{Art}}$ satisfy the standard endoscopic character relation with $\Pi^{\mathbf{H}'}_{\phi_{\mathbf{H}'}}$ and $\Pi^{\mathbf{H}'}_{\phi_{\mathbf{H}'},\mathrm{Art}}$ for any elliptic strongly regular semisimple element of H, respectively. Therefore, if we can show that $\Pi^{\mathbf{H}'}_{\phi_{\mathbf{H}'}}=\Pi^{\mathbf{H}'}_{\phi_{\mathbf{H}'},\mathrm{Art}}$, then we see that the signed sum of the characters of members of $\Pi^{\mathbf{H}}_{\phi_{\mathbf{H}}}$ coincides with that of $\Pi^{\mathbf{H}}_{\phi_{\mathbf{H}},\mathrm{Art}}$ for any elliptic strongly regular semisimple element of H. Since the strongly regular semisimple locus of $\mathbf{H}$ is Zariski dense in the regular semisimple locus of $\mathbf{H}$, we see that the signed sum of $\Pi^{\mathbf{H}}_{\phi_{\mathbf{H}}}$ and that of $\Pi^{\mathbf{H}}_{\phi_{\mathbf{H}},\mathrm{Art}}$ coincide for any elliptic regular semisimple elements of H. Hence, by the orthogonality relation of the elliptic inner product ([Clo91, Theorem 3]), we get $\Pi^{\mathbf{H}}_{\phi_{\mathbf{H}}}=\Pi^{\mathbf{H}}_{\phi_{\mathbf{H}},\mathrm{Art}}$. Since the order of $\Pi^{\mathbf{H}'}_{\phi_{\mathbf{H}'}}$ (or $\Pi^{\mathbf{H}'}_{\phi_{\mathbf{H}'},\mathrm{Art}}$) is smaller than that of $\Pi^{\mathbf{H}}_{\phi_{\mathbf{H}}}$, by repeating this argument inductively, we may assume that $|\Pi^{\mathbf{H}}_{\phi_{\mathbf{H}}}|=|\Pi^{\mathbf{H}}_{\phi_{\mathbf{H}},\mathrm{Art}}|=1$.

Let us put $\phi:=\hat{\xi}\circ\phi_{\mathbf{H}}$. When $|\Pi^{\mathbf{H}}_{\phi_{\mathbf{H}}}|=|\Pi^{\mathbf{H}}_{\phi_{\mathbf{H}},\mathrm{Art}}|=1$, or equivalently, $\mathcal{S}_{\phi_{\mathbf{H}}}$ is trivial, ϕ is a toral supercuspidal L-parameter of GL_n by Lemma 15.10. Thus we can apply Theorem 15.9; $\Pi^{\mathbf{G}}_{\phi}$ and $\Pi^{\mathbf{H}}_{\phi_{\mathbf{H}}}$ satisfy the twisted endoscopic character relation, i.e., we have

$$\Delta^{\mathrm{spec}}_{\phi,\pi}\Phi_{\tilde{\pi}}(\delta)=\sum_{\gamma\in H/\mathrm{st}}\mathring{\Delta}(\gamma,\delta)\Phi_{\pi_{\mathbf{H}}}(\gamma)$$

for any elliptic strongly regular semisimple element $\delta\in\tilde{G}$, where π and $\pi_{\mathbf{H}}$ are the unique members of $\Pi^{\mathbf{G}}_{\phi}$ and $\Pi^{\mathbf{H}}_{\phi_{\mathbf{H}}}$, respectively. Similarly, we also have

$$\Phi_{\tilde{\pi}_{\mathrm{Art}}}(\delta)=\sum_{\gamma\in H/\mathrm{st}}\mathring{\Delta}(\gamma,\delta)\Phi_{\pi_{\mathbf{H},\mathrm{Art}}}(\gamma)$$

for any elliptic strongly regular semisimple element $\delta\in\tilde{G}$, where π_{Art} and $\pi_{\mathbf{H},\mathrm{Art}}$ be the unique members of $\Pi^{\mathbf{G}}_{\phi,\mathrm{Art}}$ and $\Pi^{\mathbf{H}}_{\phi_{\mathbf{H}},\mathrm{Art}}$, respectively. We note that, for any elliptic strongly regular semisimple element $\delta\in\tilde{G}$, there exists an elliptic strongly $\mathbf{G}$-regular semisimple element $\gamma\in H$ satisfying $(\gamma,\delta)\in\mathcal{D}$ at most uniquely up to stable conjugacy. In other words, the index set of the above sums can be thought of as a singleton at most. Moreover, for any elliptic strongly $\mathbf{G}$-regular semisimple element $\gamma\in H$, there exists an elliptic strongly regular semisimple element $\delta\in\tilde{G}$. (These facts follow from, e.g., an explicit parametrization of semisimple conjugacy classes of these groups; see [Wal10, Sections 1.3 and 1.9].) Since we have $\pi=\pi_{\mathrm{Art}}$ by [OT21], we get $\Phi_{\pi_{\mathbf{H}}}(\gamma)=\Delta^{\mathrm{spec}}_{\phi,\pi}\Phi_{\pi_{\mathbf{H},\mathrm{Art}}}(\gamma)$ for any elliptic strongly $\mathbf{G}$-regular semisimple element $\gamma\in H$ (recall that $\mathring{\Delta}(\gamma,\delta)\neq 0$ whenever $(\gamma,\delta)\in\mathcal{D}$). As the strongly $\mathbf{G}$-regular semisimple locus of $\mathbf{H}$ is Zariski dense in the regular semisimple locus of $\mathbf{H}$, we see that the identity $\Phi_{\pi_{\mathbf{H}}}(\gamma)=\Delta^{\mathrm{spec}}_{\phi,\pi}\Phi_{\pi_{\mathbf{H},\mathrm{Art}}}(\gamma)$ holds for any elliptic regular semisimple element $\gamma\in H$. Therefore, again by the orthogonality relation of the elliptic inner product, we conclude that $\pi_{\mathbf{H}}=\pi_{\mathbf{H},\mathrm{Art}}$ (and also $\Delta^{\mathrm{spec}}_{\phi,\pi}=1$). □

We note that Arthur's local Langlands correspondence is established only up to the action of outer automorphisms for quasi-split even special orthogonal groups. By exactly the same argument as above, we can show the following (note that, in that case, for any elliptic strongly regular semisimple element $\delta \in \tilde{G}$, there exists an elliptic strongly $\mathbf{G}$-regular semisimple element $\gamma \in H$ satisfying $(\gamma, \delta) \in \mathcal{D}$ at most uniquely up to stable conjugacy and the action of the outer automorphisms).

Theorem 15.12. *Let $\mathbf{H}$ be a quasi-split even special orthogonal group over F. Let $\Pi^{\mathbf{H}}_{\phi_{\mathbf{H}}}$ be a toral supercuspidal L-packet with L-parameter $\phi_{\mathbf{H}}$ in the sense of Kaletha (see Section 7). Let $\Pi^{\mathbf{H}}_{\phi_{\mathbf{H}},\mathrm{Art}}$ be the L-packet of $\mathbf{H}$ corresponding to $\phi_{\mathbf{H}}$ in the sense of Arthur (*[Art13, Theorem 2.2.1]*). Then we have $\Pi^{\mathbf{H}}_{\phi_{\mathbf{H}}} = \Pi^{\mathbf{H}}_{\phi_{\mathbf{H}},\mathrm{Art}}$ up to the action of outer automorphisms.*

Appendix A. Some facts on Heisenberg–Weil representations

A.1. **Decomposition formula of twisted characters.** Let us consider an abstract situation where the following data are given:

- a finite-dimensional symplectic space V over $\mathbb{F}_p$, where $p \neq 2$,
- mutually orthogonal finite-dimensional symplectic subspaces V^i_j where $i = 0, \ldots, r$ and $j = 0, \ldots, l_i$ satisfying
$$V = \bigoplus_{i=0}^{r} \bigoplus_{j=0}^{l_i} V^i_j,$$
- a symplectic automorphism ι of V such that $\iota \colon V^i_j \xrightarrow{\sim} V^i_{j+1}$ for any $0 \leq i \leq r$ and $0 \leq j \leq l_i$ (here we put $V^i_{l_i+1} := V^i_0$ for convenience),
- a nontrivial character ϑ of $\mathbb{F}_p$.

We write $\mathbb{H}(V^i_j)$ for the finite Heisenberg group associated to the symplectic space V^i_j over $\mathbb{F}_p$. More precisely, $\mathbb{H}(V^i_j)$ is defined to be the set $V^i_j \times \mathbb{F}_p$ equipped with a multiplication law given by
$$(v_1, z_1) \cdot (v_2, z_2) := \bigl(v + w, z_1 + z_2 + \frac{1}{2}\langle v_1, v_2\rangle\bigr),$$
where $\langle -, - \rangle$ denotes the symplectic form on V^i_j. Then, according to the Stone–von Neumann theorem, we have an irreducible representation ω^i_j of $\mathrm{Sp}(V^i_j) \ltimes \mathbb{H}(V^i_j)$ with central character ϑ (called a *Heisenberg–Weil representation*), which is unique up to isomorphism unless $\mathrm{Sp}(V^i_j) \cong \mathrm{SL}_2(\mathbb{F}_3)$. We let W^i_j denote the representation space of ω^i_j:
$$\omega^i_j \colon \mathrm{Sp}(V^i_j) \ltimes \mathbb{H}(V^i_j) \to \mathrm{GL}_{\mathbb{C}}(W^i_j).$$

Let $\mathbb{H}(V)$ denote the Heisenberg group associated to V. Then note that $\mathbb{H}(V)$ is isomorphic to the central product of $\mathbb{H}(V^i_j)$ for $0 \leq i \leq r$ and $0 \leq j \leq l_i$, i.e., the quotient of the product group $\prod_{i,j} \mathbb{H}(V^i_j)$ by the central subgroup
$$\Bigl\{(1, z^i_j)_{i,j} \in \prod_{i,j} \mathbb{H}(V^i_j) \Bigm| \sum_{i,j} z^i_j = 0\Bigr\}.$$
If we put $W := \bigotimes_{i,j} W^i_j$, then W realizes a Heisenberg–Weil representation of $\mathrm{Sp}(V) \ltimes \mathbb{H}(V)$ with central character ϑ, for which we write ω. Furthermore, on the subgroup
$$\Bigl(\prod_{i,j} \mathrm{Sp}(V^i_j)\Bigr) \ltimes \mathbb{H}(V) \subset \mathrm{Sp}(V) \ltimes \mathbb{H}(V),$$
the representation ω is isomorphic to $\bigotimes_{i,j} \omega^i_j$ (see [Gér77, 2.5] for the details).

Since ι is a symplectic automorphism of V, an isomorphism

$$\iota_*\colon \operatorname{Sp}(V) \ltimes \mathbb{H}(V) \xrightarrow{\sim} \operatorname{Sp}(V) \ltimes \mathbb{H}(V); \quad \big(g, (v, z)\big) \mapsto \big({}^{\iota}g, (\iota(v), z)\big)$$

is naturally induced, where ${}^{\iota}g := \iota \circ g \circ \iota^{-1}$. Then, since ι acts on the center part of $\mathbb{H}(V)$ identically, the ι_*-twist of the representation ω (let us write ω^{ι}) is again a Heisenberg–Weil representation of $\operatorname{Sp}(V) \ltimes \mathbb{H}(V)$ with central character ϑ. Hence, by the uniqueness part of the Stone–von Neumann theorem, ω and ω^{ι} are isomorphic as representations of $\operatorname{Sp}(V) \ltimes \mathbb{H}(V)$. Our aim in this section is to construct an intertwiner $\omega \xrightarrow{\sim} \omega^{\iota}$ explicitly by using the symplectic decomposition $V = \bigoplus_{i=0}^{r} \bigoplus_{j=0}^{l_i} V_j^i$ and express the associated twisted character in terms of the intertwiner.

Since the symplectic isomorphism ι maps V_j^i to V_{j+1}^i, the automorphism ι_* of $\operatorname{Sp}(V) \ltimes \mathbb{H}(V)$ induces

$$\iota_*\colon \operatorname{Sp}(V_j^i) \ltimes \mathbb{H}(V_j^i) \xrightarrow{\sim} \operatorname{Sp}(V_{j+1}^i) \ltimes \mathbb{H}(V_{j+1}^i); \quad \big(g, (v, z)\big) \mapsto \big({}^{\iota}g, (\iota(v), z)\big).$$

Therefore the subgroup

$$\Big(\prod_{i,j} \operatorname{Sp}(V_j^i)\Big) \ltimes \mathbb{H}(V) \subset \operatorname{Sp}(V) \ltimes \mathbb{H}(V)$$

is preserved under ι_*. As ω and ω^{ι} are irreducible as representations of the subgroup $(\prod_{i,j} \operatorname{Sp}(V_j^i)) \ltimes \mathbb{H}(V)$ (or even $\mathbb{H}(V)$), any intertwiner between ω and ω^{ι} as representations of $(\prod_{i,j} \operatorname{Sp}(V_j^i)) \ltimes \mathbb{H}(V)$ is automatically an intertwiner as representations of $\operatorname{Sp}(V) \ltimes \mathbb{H}(V)$.

Let us consider the representation $\omega_{j+1}^{i,\iota}$ given by the pull-back of ω_{j+1}^i via ι_*:

$$\omega_j^{i,\iota}\colon \operatorname{Sp}(V_j^i) \ltimes \mathbb{H}(V_j^i) \xrightarrow{\iota_*} \operatorname{Sp}(V_{j+1}^i) \ltimes \mathbb{H}(V_{j+1}^i) \to \mathrm{GL}_{\mathbb{C}}(W_{j+1}^i).$$

Here, similarly to the notation $V_{l_i+1}^i := V_0^i$, we put $\omega_{l_i+1}^i := \omega_0^i$ for convenience. Then, since ι preserves the center part of $\mathbb{H}(V_j^i)$ identically, $\omega_{j+1}^{i,\iota}$ is a Heisenberg–Weil representation of $\operatorname{Sp}(V_j^i) \ltimes \mathbb{H}(V_j^i)$ with central character ϑ. In particular, by the uniqueness part of the Stone–von Neumann theorem, $\omega_{j+1}^{i,\iota}$ is isomorphic to ω_j^i as a representation of $\operatorname{Sp}(V_j^i) \ltimes \mathbb{H}(V_j^i)$. Let us fix an intertwiner I_j^i between these two representations, i.e., an isomorphism

$$I_j^i\colon (\omega_j^i, W_j^i) \xrightarrow{\sim} (\omega_{j+1}^{i,\iota}, W_{j+1}^i)$$

making the following diagram commutative for any $(g, h) \in \operatorname{Sp}(V_j^i) \ltimes \mathbb{H}(V_j^i)$:

$$\begin{array}{ccc} W_j^i & \xrightarrow{I_j^i} & W_{j+1}^i \\ {\scriptstyle \omega_j^i(g,h)}\downarrow & & \downarrow{\scriptstyle \omega_{j+1}^{i,\iota}(g,h) = \omega_{j+1}^i(\iota_*(g,h))} \\ W_j^i & \xrightarrow[I_j^i]{} & W_{j+1}^i \end{array}$$

If we put $V^i := \bigoplus_{j=0}^{l_i} V_j^i$, then $\mathbb{H}(V^i)$ is isomorphic to the central product of $\mathbb{H}(V_j^i)$ for $0 \leq j \leq l_i$. The automorphism ι_* of $\operatorname{Sp}(V) \ltimes \mathbb{H}(V)$ preserves the subgroup

$$\Big(\prod_{j=0}^{l_i} \operatorname{Sp}(V_j^i)\Big) \ltimes \mathbb{H}(V^i)$$

and its action is described on this subgroup by

$$\iota_* \colon \big((g_0 \dots, g_{l_i}), (v, z)\big) \mapsto \big(({}^{\iota}g_{l_i}, {}^{\iota}g_0, \dots, {}^{\iota}g_{l_i-1}), (\iota(v), z)\big).$$

We put $W^i := \bigotimes_{j=0}^{l_i} W_j^i$ and define an $\mathbb{C}$-linear automorphism I^i on W^i by

$$I^i \colon v_0 \otimes \cdots \otimes v_{l_i-1} \otimes v_{l_i} \mapsto I_{l_i}^i(v_{l_i}) \otimes I_0^i(v_0) \otimes \cdots \otimes I_{l_i-1}^i(v_{l_i-1}).$$

We write (ω^i, W^i) for the tensored Heisenberg–Weil representation $(\bigotimes_j \omega_j^i, \bigotimes_j W_j^i)$ of $\mathrm{Sp}(V^i) \ltimes \mathbb{H}(V^i)$. Then we can easily check that I^i gives an intertwiner between (ω^i, W^i) and its ι_*-twist $(\omega^{i,\iota}, W^i)$ as representation of $(\prod_{j=0}^{l_i} \mathrm{Sp}(V_j^i)) \ltimes \mathbb{H}(V^i)$, that is, the following diagram is commutative for any $((g_0, \dots, g_{l_i}), h) \in (\prod_{j=0}^{l_i} \mathrm{Sp}(V_j^i)) \ltimes \mathbb{H}(V^i)$:

$$\begin{array}{ccc} W^i & \xrightarrow{I^i} & W^i \\ {\scriptstyle \omega^i((g_0,\dots,g_{l_i}),h)} \downarrow & & \downarrow {\scriptstyle \omega^{i,\iota}((g_0,\dots,g_{l_i}),h) = \omega(\iota_*((g_0,\dots,g_{l_i}),h))} \\ W^i & \xrightarrow[I^i]{} & W^i \end{array}$$

Now we define a $\mathbb{C}$-linear isomorphism I of $W = \bigotimes_{i=0}^{r} W^i$ by $I := \bigotimes_{i=0}^{r} I^i$, i.e.,

$$I \colon v^0 \otimes \cdots \otimes v^r \mapsto I^0(v^0) \otimes \cdots \otimes I^r(v^r).$$

Then I is an intertwiner between ω $(= \bigotimes_i \omega^i)$ and its ι_*-twist ω^{ι} $(= \bigotimes_i \omega^{i,\iota})$.

Lemma A.1. *Let $W_0', \dots, W_l'$ be finite-dimensional $\mathbb{C}$-vector spaces equipped with $\mathbb{C}$-linear isomorphisms $I_j' \colon W_j' \xrightarrow{\sim} W_{j+1}'$ for $1 \leq j \leq l$, where we put $W_{l+1}' := W_0'$. We define an automorphism I' of $W' := W_0' \otimes \cdots \otimes W_l'$ by*

$$I' \colon v_0 \otimes \cdots \otimes v_l \mapsto I_l'(v_l) \otimes I_0'(v_0) \otimes \cdots \otimes I_{l-1}'(v_{l-1}).$$

Then we have

$$\mathrm{tr}(I' \mid W') = \mathrm{tr}(I_l' \circ \cdots \circ I_0' \mid W_0').$$

Proof. We take a $\mathbb{C}$-basis $\{e_1^{(0)}, \dots, e_n^{(0)}\}$ of W_0' and define a $\mathbb{C}$-basis $\{e_1^{(i)}, \dots, e_n^{(i)}\}$ of each W_i' $(1 \leq i \leq l)$ by $e_j^{(i)} := I_{i-1}' \circ \cdots \circ I_0'(e_j^{(0)})$. Then, by the definition of the trace, we have

$$\mathrm{tr}(I' \mid W') = \sum_{1 \leq j_0, \dots, j_l \leq n} \langle I'(e_{j_0}^{(0)} \otimes \cdots \otimes e_{j_l}^{(l)}), e_{j_0}^{(0)} \otimes \cdots \otimes e_{j_l}^{(l)} \rangle_{W'},$$

where $\langle -, - \rangle_{W'}$ denotes the standard $\mathbb{C}$-bilinear pairing on $W' \times W'$ given by

$$\langle e_{j_0}^{(0)} \otimes \cdots \otimes e_{j_l}^{(l)}, e_{j_0'}^{(0)} \otimes \cdots \otimes e_{j_l'}^{(l)} \rangle_V = \delta_{j_0, j_0'} \cdots \delta_{j_l, j_l'}$$

for any $1 \leq j_0, \dots, j_l \leq n$ and $1 \leq j_0', \dots, j_l' \leq n$, where $\delta_{-,-}$ denotes the Kronecker delta. By the definition of I', we have

$$I'(e_{j_0}^{(0)} \otimes \cdots \otimes e_{j_l}^{(l)}) = (I_l' \circ \cdots \circ I_0'(e_{j_l}^{0})) \otimes e_{j_0}^{(1)} \otimes \cdots \otimes e_{j_{l-1}}^{(l)}$$

Hence the summand of the above formula for the trace of I' is not zero only when $j_0 = \cdots = j_l$. Moreover, in this case (let us put $j := j_1 = \cdots = j_l$), we have

$$\langle I'(e_{j_0}^{(0)} \otimes \cdots \otimes e_{j_l}^{(l)}), e_{j_0}^{(0)} \otimes \cdots \otimes e_{j_l}^{(l)} \rangle_{W'} = \langle I_l' \circ \cdots \circ I_0'(e_j^{(0)}), e_j^{(0)} \rangle_{W_0'},$$

where $\langle -,- \rangle_{W_0'}$ denotes the standard $\mathbb{C}$-bilinear pairing on $W_0' \times W_0'$ satisfying $\langle e_j^{(0)}, e_{j'}^{(0)} \rangle_{W_0'} = \delta_{j,j'}$ for any $1 \leq j \leq n$ and $1 \leq j' \leq n$. Thus we get

$$\operatorname{tr}(I' \mid V') = \sum_{j=1}^{n} \langle I_l' \circ \cdots \circ I_0'(e_j^{(0)}), e_j^{(0)} \rangle_{W_0'}.$$

This is nothing but the trace of $I_0' \circ \cdots \circ I_l'$ on W_0'. □

Proposition A.2. *Let $g := (g_j^i)_{i,j} \in \prod_{i,j} \operatorname{Sp}(V_j^i)$. Then the trace of $\omega(g) \circ I$ on W is given by*

$$\prod_{i=0}^{r} \operatorname{tr}\Big(\omega_0^i\big(g_0^i \circ \iota_*(g_{l_i}^i) \circ \cdots \circ \iota_*^{l_i}(g_1^i)\big) \circ I_{l_i}^i \circ \cdots \circ I_0^i \,\Big|\, W_0^i\Big)$$

Proof. We put $g^i := (g_j^i)_j$. Recall that $W = \bigotimes_{i=0}^r W^i$, $\omega(g) = \bigotimes_{i=0}^r \omega^i(g^i)$, and $I = \bigotimes_{i=0}^r I^i$. Hence we have

$$\operatorname{tr}(\omega(g) \circ I \mid W) = \prod_{i=0}^{r} \operatorname{tr}(\omega^i(g^i) \circ I^i \mid W^i).$$

Let us compute each $\operatorname{tr}(\omega^i(g^i) \circ I^i \mid W^i)$. Recall that $W^i = \bigotimes_{j=0}^{l_i} W_j^i$, $\omega^i(g^i) = \bigotimes_{j=0}^{l_i} \omega_j^i(g_j^i)$, and an automorphism I^i of W^i is defined by

$$I^i \colon v_0 \otimes \cdots \otimes v_{l_i-1} \otimes v_{l_i} \mapsto I_{l_i}^i(v_{l_i}) \otimes I_0^i(v_0) \otimes \cdots \otimes I_{l_i-1}^i(v_{l_i-1}).$$

Hence the automorphism $\omega^i(g^i) \circ I^i$ of W^i is given by

$$v_0 \otimes \cdots \otimes v_{l_i-1} \otimes v_{l_i} \mapsto I_{l_i}^{i,\prime}(v_{l_i}) \otimes I_0^{i,\prime}(v_0) \otimes \cdots \otimes I_{l_i-1}^{i,\prime}(v_{l_i-1}),$$

where we put

$$I_j^{i,\prime} := \omega_{j+1}^i(g_{j+1}^i) \circ I_j^i \colon W_j^i \xrightarrow{\sim} W_{j+1}^i.$$

Thus, by Lemma A.1, we get

$$\operatorname{tr}(\omega^i(g^i) \circ I^i \mid W^i) = \operatorname{tr}(I_{l_i}^{i,\prime} \circ \cdots \circ I_0^{i,\prime} \mid W_0^i).$$

Then, by the intertwining property of I_j^i, i.e., $\omega_{j+1}^i(\iota_*(-)) \circ I_j^i = I_j^i \circ \omega_j^i(-)$,

$$\begin{aligned} I_{l_i}^{i,\prime} \circ \cdots \circ I_0^{i,\prime} &= \big(\omega_{l_i+1}^i(g_{l_i+1}^i) \circ I_{l_i}^i\big) \circ \cdots \circ \big(\omega_1^i(g_1^i) \circ I_0^i\big) \\ &= \omega_0^i\big(g_0^i \circ \iota_*(g_{l_i}^i) \circ \cdots \circ \iota_*^{l_i}(g_1^i)\big) \circ I_{l_i}^i \circ \cdots \circ I_0^i. \end{aligned}$$

Hence we get the assertion. □

A.2. **Gérardin's character formulas of Weil representations.** In this subsection, we review character formulas of Weil representations established by Gérardin [Gér77]. We let V be a finite-dimensional vector space over $\mathbb{F}_p$ equipped with a symplectic pairing $\langle -,- \rangle \colon V \times V \to \mathbb{F}_p$. Let (ω_V, W_V) be a Heisenberg–Weil representation of $\operatorname{Sp}(V) \ltimes \mathbb{H}(V)$ with central character, say, $\vartheta \colon \mathbb{F}_p \hookrightarrow \mathbb{C}^\times$.

We introduce some notation following [Gér77, Section 4]. Suppose that T is an $\mathbb{F}_p$-rational maximal torus of $\operatorname{Sp}(V)$. Then, since T acts on V, we have a decomposition

$$V_{\overline{\mathbb{F}}_p} \, (:= V \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p) = \bigoplus_{\epsilon \in P(V,T)} V_{\overline{\mathbb{F}}_p}^{\epsilon},$$

where $P(V,T)$ denotes the set of weights of T in $V_{\overline{\mathbb{F}}_p}$ and $V_{\overline{\mathbb{F}}_p}^{\epsilon}$ denotes the weight space with respect to $\epsilon \in P(V,T)$. As the action of T on V is $\mathbb{F}_p$-rational, the set

$P(V,T)$ is equipped with an action of $\Gamma_{\mathbb{F}_p} = \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$. Furthermore, by putting $\Sigma_{\mathbb{F}_p} := \Gamma_{\mathbb{F}_p} \times \{\pm 1\}$, $\Sigma_{\mathbb{F}_p}$ also acts on $P(V,T)$ (-1 acts via $\epsilon \mapsto -\epsilon$). We say that a $\Gamma_{\mathbb{F}_p}$-orbit ω in $P(V,T)$ is symmetric (resp. asymmetric) if $-\omega = \omega$ (resp. $-\omega \neq \omega$). For each $\Omega \in P(V,T)/\Sigma_{\mathbb{F}_p}$ such that $\epsilon \in \Omega$, we define a quadratic character χ^T_Ω of $T(\mathbb{F}_p)$ by

$$\chi^T_\Omega(t) := \begin{cases} \epsilon(t)^{\frac{1-q_\Omega}{2}} & \text{if an(y) } \omega \subset \Omega \text{ is asymmetric,} \\ \epsilon(t)^{\frac{1+q_\Omega}{2}} & \text{if an(y) } \omega \subset \Omega \text{ is symmetric,} \end{cases}$$

for $t \in T(\mathbb{F}_p)$, where we put $q_\Omega := p^{\frac{1}{2}|\Omega|}$. we define a quadratic character χ^T of $T(\mathbb{F}_p)$ by

$$\chi^T := \prod_{\Omega \in P(V,T)/\Sigma_{\mathbb{F}_p}} \chi^T_\Omega.$$

Proposition A.3 ([Gér77, Corollary 4.8.1])**.** *For any $t \in T \subset \mathrm{Sp}(V)$, we have*

$$\Theta_{\omega_V}(t) = (-1)^{l(V,T;t)} \cdot p^{N(V;t)} \cdot \chi^T(t),$$

where

- $l(V,T;t) := |\{\omega \in P(V,T)/\Gamma_{\mathbb{F}_p} \mid \epsilon(t) \neq 1$ *for an(y)* $\epsilon \in \omega\}|$,
- $N(V;t) := \frac{1}{2}\dim_{\mathbb{F}_p} V^t$ *(hence $p^{N(V;t)} = |V^t|^{\frac{1}{2}}$).*

In order to apply Proposition A.3 to a given semisimple element $g \in \mathrm{Sp}(V)$, we have to pick an $\mathbb{F}_p$-rational maximal torus T of $\mathrm{Sp}(V)$ containing g and analyze the structure of the set of weights $P(V,T)$ including its Galois action. The following lemmas are useful for this:

Lemma A.4. *Let $g, t \in \mathrm{Sp}(V)$ be semisimple elements. If g and t have the same (multi-)sets of eigenvalues, then they are* $\mathrm{Sp}(V)$*-conjugate.*

Proof. The proof of this lemma should be standard, but we explain it for the sake of completeness. Note that $\mathrm{Sp}(V) \subset \mathrm{Sp}(V_{\overline{\mathbb{F}}_p}) \subset \mathrm{GL}(V_{\overline{\mathbb{F}}_p})$. The assumption that g and t have the same eigenvalues implies that g and t are conjugate in $\mathrm{GL}(V_{\overline{\mathbb{F}}_p})$. It is known that this furthermore implies that g and t are conjugate in $\mathrm{Sp}(V_{\overline{\mathbb{F}}_p})$ (for example, see [SS70, 275 page, Exercises 2.15 (ii)]).

Let $x \in \mathrm{Sp}(V_{\overline{\mathbb{F}}_p})$ be an element such that $g = xtx^{-1}$. Then we have $xtx^{-1} = g = \sigma(g) = \sigma(x)t\sigma(x)^{-1}$ for any $\sigma \in \Gamma_{\mathbb{F}_p}$. In other words, by putting H to be the centralizer of t in $\mathrm{Sp}(V_{\overline{\mathbb{F}}_p})$, we have $\sigma(x)^{-1}x \in H$. Hence we obtain a 1-cocycle $z_\sigma \in Z^1(\Gamma_{\mathbb{F}_p}, H)$ given by $z_\sigma = \sigma(x)^{-1}x$.

We note that H is connected (as an algebraic group) since $\mathrm{Sp}(V_{\overline{\mathbb{F}}_p})$ is simply-connected (for example, see [Hum95, Section 2.11]). Thus we have $H^1(\Gamma_{\mathbb{F}_p}, H) = 1$ by Lang's theorem. This means that there exists an element $h \in H$ satisfying $z_\sigma = \sigma(h)^{-1}h$. In particular, xh^{-1} is $\mathbb{F}_p$-rational, i.e., an element of $\mathrm{Sp}(V)$. As we have $(xh^{-1})t(xh^{-1})^{-1} = xtx^{-1} = g$, we obtain the assertion. □

Lemma A.5. *Let $2n := \dim_{\mathbb{F}_p}(V)$. Let $k_1^\circ, \dots, k_l^\circ$ be finite extensions of $\mathbb{F}_p$ satisfying $[k_1^\circ : \mathbb{F}_p] + \cdots + [k_l^\circ : \mathbb{F}_p] \leq n$. Let k_i be the quadratic extension of k_i° for $1 \leq i \leq l$. Then there exists an $\mathbb{F}_p$-maximal torus T of* $\mathrm{Sp}(V)$ *of the form*

$$\prod_{i=1}^{l} \mathrm{Ker}(\mathrm{Nr}_{k_i/k_i^\circ} \colon \mathrm{Res}_{k_i/\mathbb{F}_p} \mathbb{G}_{\mathrm{m}} \to \mathrm{Res}_{k_i^\circ/\mathbb{F}_p} \mathbb{G}_{\mathrm{m}}) \times \mathbb{G}_{\mathrm{m}}^r,$$

where $r := n - ([k_1^\circ : \mathbb{F}_p] + \cdots + [k_l^\circ : \mathbb{F}_p])$. Moreover, we have the following:

(1) *The set of weights* $P(V,T)$ *is of the form* $\bigsqcup_{i=1}^{l} \Omega_i \sqcup \bigsqcup_{j=1}^{r}\{\pm\epsilon_j\}$, *where*
   - Ω_j *is a finite set of order* $[k_l : \mathbb{F}_p]$ *on which* $\mathrm{Gal}(k_l/\mathbb{F}_p)$ *acts simply transitively* (Γ_{k_l} *acts trivially) and the unique nontrivial element of* $\mathrm{Gal}(k_l/k_l^{\circ})$ *acts via negation,*
   - ϵ_j *is a weight on which* $\Gamma_{\mathbb{F}_p}$ *acts trivially.*

(2) *If* $t = (t_1, \dots, t_l, t_{l+1}, \dots, t_{l+r}) \in T(\mathbb{F}_p) \cong \prod_{i=1}^{l} k_i^1 \times (\mathbb{F}_p^{\times})^r$, *then the (multi-)set of eigenvalues of* t *is given by*

$$\bigsqcup_{i=1}^{l}\{\sigma(t_i)\}_{\sigma\in\mathrm{Gal}(k_i/\mathbb{F}_p)} \sqcup \bigsqcup_{j=1}^{r}\{t_{i+j}, t_{i+j}^{-1}\}.$$

Proof. If we let τ_i be the unique nontrivial element of $\mathrm{Gal}(k_i/k_i^{\circ})$, then we can define an $\mathbb{F}_p$-symplectic form on k_i by

$$(x,y) \mapsto \mathrm{Tr}_{k_i/\mathbb{F}_p}(x\tau_i(y) - \tau_i(x)y).$$

Since the action of k_i^1 on k_i preserves this symplectic form, we see that the symplectic group $\mathrm{Sp}(k_i)$ (as an algebraic group over $\mathbb{F}_p$) contains an $\mathbb{F}_p$-rational torus $\mathrm{Ker}(\mathrm{Nr}_{k_i/k_i^{\circ}}\colon \mathrm{Res}_{k_i/\mathbb{F}_p}\mathbb{G}_{\mathrm{m}} \to \mathrm{Res}_{k_i^{\circ}/\mathbb{F}_p}\mathbb{G}_{\mathrm{m}})$. Since its rank is given by $[k_i^{\circ} : \mathbb{F}_p]$, which is the half of $\dim_{\mathbb{F}_p}(k_i)$, it is a maximal torus. On the other hand, obviously $\mathbb{G}_{\mathrm{m}}$ is realized as a split maximal torus of the rank one symplectic group. Hence the torus as in the statement can be realized in $\prod_{i=1}^{l}\mathrm{Sp}_{[k_i:\mathbb{F}_p]} \times \prod_{j=1}^{r}\mathrm{Sp}_2$, which can be embedded in $\mathrm{Sp}_{2n} \cong \mathrm{Sp}(V)$. Again by looking at the rank, we see that it gives an $\mathbb{F}_p$-rational maximal torus of $\mathrm{Sp}(V)$. The remaining assertions immediately follows from this explicit realization. □

Proposition A.6 ([Gér77, Theorem 4.9.1 (a), (c)]). *Let* $g \in \mathrm{Sp}(V)$.

(1) *Suppose that* g *has no nonzero fixed point in* V. *If* V' *is a maximal* g*-invariant totally isotropic subspace of* V, *then we have*

$$\Theta_{\omega_V}(g) = \mathrm{sgn}_{\mathbb{F}_p^{\times}}\bigl((-1)^{\frac{\dim V_0}{2}} \cdot \det(g \mid V') \cdot \det(g-1 \mid V_0)\bigr),$$

where $V_0 := V'^{\perp}/V'$.

(2) *Suppose that* g *fixes pointwise a line* $L \subset V$. *If* V_0 *is a* g*-invariant subspace of* $L^{\perp}$ *such that* $L^{\perp} = L \oplus V_0$, *then we have*

$$\Theta_{\omega_V}(g) = \Theta_{\omega_{V_0}}(g) \sum_{v\in V_0^{\perp}/L} \vartheta(\langle gv, v\rangle),$$

where ω_{V_0} *is a Heisenberg–Weil representation of* $\mathrm{Sp}(V_0) \ltimes \mathbb{H}(V_0)$ *with central character* ϑ.

Lemma A.7. *Let* $V = V_+ \oplus V_-$ *be a polarization of* V *(i.e.,* V_+ *and* V_- *are totally isotropic subspaces). Let* $g \in \mathrm{Sp}(V)$ *be a semisimple element and we suppose that* V_+ *and* V_- *are invariant under* g. *Then, for any line* $L_+ \subset V_+$ *fixed by* g *pointwise, there exist* g*-invariant decompositions* $V_+ = L_+ \oplus M_+$ *and* $V_- = L_- \oplus M_-$ *such that* L_- *is a line fixed by* g *pointwise and we have* $L_+^{\perp} = V_+ \oplus M_-$ *and* $L_-^{\perp} = V_- \oplus M_+$.

Proof. Let l be an nonzero element of the line L_+. We put $M_- := \{v \in V_- \mid \langle l, v\rangle = 0\}$. Since $V = V_+ \oplus V_-$ is a polarization, we can find an element $w \in V_-$ satisfying $\langle l, w\rangle \neq 0$. Note that, as g is semisimple, the order of g (say p') is prime to p. Thus, since g stabilizes the subspace V_-, the averaged element $w' := \frac{1}{p'}\sum_{i=0}^{p'-1} g^i(w)$ belongs to V_-. Moreover, w' is g-invariant and satisfies $\langle l, w'\rangle = \langle l, w\rangle \neq 0$. By putting $L_- := \mathbb{F}_p w'$, we get $V_- = L_- \oplus M_-$. By applying the same construction to V_+ using w' instead of l, we get $V_+ = L_+ \oplus M_+$. These decompositions satisfy the conditions as desired. □

Corollary A.8. *Let $g \in \mathrm{Sp}(V)$ be a semisimple element and we suppose that we have a g-invariant polarization $V = V_+ \oplus V_-$ of V. Then we have*

$$\Theta_{\omega_V}(g) = \mathrm{sgn}_{\mathbb{F}_p^\times}(\det(g \mid V_+)) \cdot |V^g|^{\frac{1}{2}}.$$

Proof. When V has no nonzero point fixed by g, then Proposition A.6 (1) can be applied to $V' = V_+$. Then, as $V = V_+ \oplus V_-$ is a polarization, we have $V'^{\perp} = V_+$, hence $V_0 = 0$. Thus we get

$$\Theta_{\omega_V}(g) = \mathrm{sgn}_{\mathbb{F}_p^\times}(\det(g \mid V_+)).$$

We next suppose that V has a nonzero point v fixed by g. We write $v = v_+ + v_-$ according to the polarization $V = V_+ \oplus V_-$ ($v_+ \in V_+$ and $v_- \in V_-$). Then, since the decomposition $V = V_+ \oplus V_-$ is g-invariant, both of v_+ and v_- are fixed by g. Since $v \neq 0$, either v_+ or v_- is not zero. We may assume that v_+ is not zero without loss of generality. Then the $L_+ := \mathbb{F}_p v_+ \subset V_+$ is fixed by g pointwise.

We take g-invariant decompositions $V_+ = L_+ \oplus M_+$ and $V_- = L_- \oplus M_-$ as in Lemma A.7. We use Proposition A.6 (2) with $L = L_+$; then $L^\perp = V_+ \oplus M_- = L_+ \oplus M_+ \oplus M_-$, hence V_0 can be taken to be $M_+ \oplus M_-$. Hence we get

$$\Theta_{\omega_V}(g) = \Theta_{\omega_{V_0}}(g) \sum_{v \in V_0^\perp / L} \vartheta(\langle gv, v\rangle).$$

Since $V_0^\perp$ is given by $L_+ \oplus L_-$ and g fixes L_- pointwise, we have

$$\sum_{v \in V_0^\perp / L} \vartheta(\langle gv, v\rangle) = \sum_{v \in L_-} \vartheta(\langle gv, v\rangle) = \sum_{v \in L_-} \vartheta(\langle v, v\rangle) = \sum_{v \in L_-} 1 = p.$$

Then the same argument can be applied to V_0. By repeating this procedure and using the result on the case where V has no nonzero point fixed by g, which is already proved in the first paragraph, we eventually get

$$\Theta_{\omega_V} = \mathrm{sgn}_{\mathbb{F}_p^\times}(\det(g \mid V_+/V_+^g)) \cdot p^{\dim_{\mathbb{F}_p}(V_+^g)}.$$

By noting that

$$\det(g \mid V_+) = \det(g \mid V_+^g) \cdot \det(g \mid V_+/V_+^g) = \det(g \mid V_+/V_+^g)$$

and $p^{\dim_{\mathbb{F}_p}(V_+^g)} = |V_+^g| = |V^g|^{\frac{1}{2}}$, we get the desired result. □

References

[Art13] J. Arthur, *The endoscopic classification of representations: Orthogonal and symplectic groups*, American Mathematical Society Colloquium Publications, vol. 61, American Mathematical Society, Providence, RI, 2013.

[AS08] J. D. Adler and L. Spice, *Good product expansions for tame elements of p-adic groups*, Int. Math. Res. Pap. IMRP (2008), no. 1, Art. ID rp. 003, 95.

[AS09] ______, *Supercuspidal characters of reductive p-adic groups*, Amer. J. Math. **131** (2009), no. 4, 1137–1210.

[BH06] C. J. Bushnell and G. Henniart, *The local Langlands conjecture for* GL(2), Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 335, Springer-Verlag, Berlin, 2006.

[Clo91] L. Clozel, *Invariant harmonic analysis on the Schwartz space of a reductive p-adic group*, Harmonic analysis on reductive groups (Brunswick, ME, 1989), Progr. Math., vol. 101, Birkhäuser Boston, Boston, MA, 1991, pp. 101–121.

[DL76] P. Deligne and G. Lusztig, *Representations of reductive groups over finite fields*, Ann. of Math. (2) **103** (1976), no. 1, 103–161.

[DR09] S. DeBacker and M. Reeder, *Depth-zero supercuspidal L-packets and their stability*, Ann. of Math. (2) **169** (2009), no. 3, 795–901.

[DS18] S. DeBacker and L. Spice, *Stability of character sums for positive-depth, supercuspidal representations*, J. Reine Angew. Math. **742** (2018), 47–78.

[Fin21] J. Fintzen, *Tame tori in p-adic groups and good semisimple elements*, Int. Math. Res. Not. IMRN (2021), no. 19, 14882–14904.

[FKS23] J. Fintzen, T. Kaletha, and L. Spice, *A twisted Yu construction, Harish-Chandra characters, and endoscopy*, Duke Math. J. **172** (2023), no. 12, 2241–2301.

[FS21] J. Fintzen and S. W. Shin, *Congruences of algebraic automorphic forms and supercuspidal representations*, Camb. J. Math. **9** (2021), no. 2, 351–429.

[Gér77] P. Gérardin, *Weil representations associated to finite fields*, J. Algebra **46** (1977), no. 1, 54–101.

[Hak18] J. Hakim, *Constructing tame supercuspidal representations*, Represent. Theory **22** (2018), 45–86.

[Hal93] T. C. Hales, *A simple definition of transfer factors for unramified groups*, Representation theory of groups and algebras, Contemp. Math., vol. 145, Amer. Math. Soc., Providence, RI, 1993, pp. 109–134.

[HC70] Harish-Chandra, *Harmonic analysis on reductive p-adic groups*, Lecture Notes in Mathematics, Vol. 162, Springer-Verlag, Berlin-New York, 1970, Notes by G. van Dijk.

[HM08] J. Hakim and F. Murnaghan, *Distinguished tame supercuspidal representations*, Int. Math. Res. Pap. IMRP (2008), no. 2, Art. ID rpn005, 166.

[HT01] M. Harris and R. Taylor, *The geometry and cohomology of some simple Shimura varieties*, Annals of Mathematics Studies, vol. 151, Princeton University Press, Princeton, NJ, 2001, With an appendix by Vladimir G. Berkovich.

[Hum95] J. E. Humphreys, *Conjugacy classes in semisimple algebraic groups*, Mathematical Surveys and Monographs, vol. 43, American Mathematical Society, Providence, RI, 1995.

[Kal13] T. Kaletha, *Simple wild L-packets*, J. Inst. Math. Jussieu **12** (2013), no. 1, 43–75.

[Kal15] ———, *Epipelagic L-packets and rectifying characters*, Invent. Math. **202** (2015), no. 1, 1–89.

[Kal16] ———, *Rigid inner forms of real and p-adic groups*, Ann. of Math. (2) **184** (2016), no. 2, 559–632.

[Kal19a] ———, *On L-embeddings and double covers of tori over local fields*, preprint, `arXiv:1907.05173v2`, 2019.

[Kal19b] ———, *Regular supercuspidal representations*, J. Amer. Math. Soc. **32** (2019), no. 4, 1071–1170.

[Kal19c] ———, *Supercuspidal L-packets*, preprint, `arXiv:1912.03274`, 2019.

[Kal21a] ———, *Errata*, unpublished, available at `http://www-personal.umich.edu/~kaletha/errata.pdf`, 2021.

[Kal21b] ———, *Lectures on the stable trace formula, with emphasis on* SL_2, unpublished, available at `http://www-personal.umich.edu/~kaletha/errata.pdf`, 2021.

[Kot82] R. E. Kottwitz, *Rational conjugacy classes in reductive groups*, Duke Math. J. **49** (1982), no. 4, 785–806.

[Kot86] ———, *Stable trace formula: elliptic singular terms*, Math. Ann. **275** (1986), no. 3, 365–399.

[KP23] T. Kaletha and G. Prasad, *Bruhat-Tits theory—a new approach*, New Mathematical Monographs, vol. 44, Cambridge University Press, Cambridge, 2023.

[KS99] R. E. Kottwitz and D. Shelstad, *Foundations of twisted endoscopy*, Astérisque (1999), no. 255, vi+190.

[KS12] ———, *On splitting invariants and sign conventions in endoscopic transfer*, preprint, `arXiv:1201.5658`, 2012.

[Lab04] J.-P. Labesse, *Stable twisted trace formula: elliptic terms*, J. Inst. Math. Jussieu **3** (2004), no. 4, 473–530.

[Lan89] R. P. Langlands, *On the classification of irreducible representations of real algebraic groups*, Representation theory and harmonic analysis on semisimple Lie groups, Math. Surveys Monogr., vol. 31, Amer. Math. Soc., Providence, RI, 1989, pp. 101–170.

[LH17] B. Lemaire and G. Henniart, *Représentations des espaces tordus sur un groupe réductif connexe* $\mathfrak{p}$*-adique*, Astérisque (2017), no. 386, ix+366.

[Li13] W.-W. Li, *On a pairing of Goldberg-Shahidi for even orthogonal groups*, Represent. Theory **17** (2013), 337–381.

[LS87] R. P. Langlands and D. Shelstad, *On the definition of transfer factors*, Math. Ann. **278** (1987), no. 1-4, 219–271.

[LS90] R. Langlands and D. Shelstad, *Descent for transfer factors*, The Grothendieck Festschrift, Vol. II, Progr. Math., vol. 87, Birkhäuser Boston, Boston, MA, 1990, pp. 485–563.

[Mez13] P. Mezo, *Character identities in the twisted endoscopy of real reductive groups*, Mem. Amer. Math. Soc. **222** (2013), no. 1042, vi+94.

[Mok15] C. P. Mok, *Endoscopic classification of representations of quasi-split unitary groups*, Mem. Amer. Math. Soc. **235** (2015), no. 1108, vi+248.

[MW16] C. Mœglin and J.-L. Waldspurger, *Stabilisation de la formule des traces tordue. Vol. 1*, Progress in Mathematics, vol. 316, Birkhäuser/Springer, Cham, 2016.

[MW18] ———, *La formule des traces locale tordue*, Mem. Amer. Math. Soc. **251** (2018), no. 1198, v+183.

[Ngô10] B. C. Ngô, *Le lemme fondamental pour les algèbres de Lie*, Publ. Math. Inst. Hautes Études Sci. (2010), no. 111, 1–169.

[OT21] M. Oi and K. Tokimoto, *Local Langlands correspondence for regular supercuspidal representations of* $GL(n)$, Int. Math. Res. Not. IMRN (2021), no. 3, 2007–2073.

[Poo17] B. Poonen, *Rational points on varieties*, Graduate Studies in Mathematics, vol. 186, American Mathematical Society, Providence, RI, 2017.

[Pra99] D. Prasad, *Some remarks on representations of a division algebra and of the Galois group of a local field*, J. Number Theory **74** (1999), no. 1, 73–97.

[Spi08] L. Spice, *Topological Jordan decompositions*, J. Algebra **319** (2008), no. 8, 3141–3163.

[Spi18] ———, *Explicit asymptotic expansions for tame supercuspidal characters*, Compos. Math. **154** (2018), no. 11, 2305–2378.

[Spi21] ———, *Explicit asymptotic expansions in p-adic harmonic analysis II*, preprint, `arXiv:2108.12935`, 2021.

[SS70] T. A. Springer and R. Steinberg, *Conjugacy classes*, Seminar on Algebraic Groups and Related Finite Groups (The Institute for Advanced Study, Princeton, N.J., 1968/69), Lecture Notes in Mathematics, Vol. 131, Springer, Berlin, 1970, pp. 167–266.

[Ste68] R. Steinberg, *Endomorphisms of linear algebraic groups*, Memoirs of the American Mathematical Society, No. 80, American Mathematical Society, Providence, R.I., 1968.

[Tam16] K.-F. Tam, *Admissible embeddings of L-tori and the essentially tame local Langlands correspondence*, Int. Math. Res. Not. IMRN (2016), no. 6, 1695–1775.

[Tat79] J. Tate, *Number theoretic background*, Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979, pp. 3–26.

[Tok23] K. Tokimoto, *Local Jacquet–Langlands correspondence for regular supercuspidal representations*, `arXiv:2303.02354`, preprint, 2023.

[Wal95] J.-L. Waldspurger, *Une formule des traces locale pour les algèbres de Lie p-adiques*, J. Reine Angew. Math. **465** (1995), 41–99.

[Wal97] ———, *Le lemme fondamental implique le transfert*, Compositio Math. **105** (1997), no. 2, 153–236.

[Wal06] ———, *Endoscopie et changement de caractéristique*, J. Inst. Math. Jussieu **5** (2006), no. 3, 423–525.

[Wal08] ———, *L'endoscopie tordue n'est pas si tordue*, Mem. Amer. Math. Soc. **194** (2008), no. 908, x+261.

[Wal09] ———, *Errata*, unpublished, available at `https://webusers.imj-prg.fr/~jean-loup.waldspurger/errata03112009.pdf`, 2009.

[Wal10] ———, *Les facteurs de transfert pour les groupes classiques: un formulaire*, Manuscripta Math. **133** (2010), no. 1-2, 41–82.

[Yu01] J.-K. Yu, *Construction of tame supercuspidal representations*, J. Amer. Math. Soc. **14** (2001), no. 3, 579–622.

[Yu09] ———, *On the local Langlands correspondence for tori*, Ottawa lectures on admissible representations of reductive p-adic groups, Fields Inst. Monogr., vol. 26, Amer. Math. Soc., Providence, RI, 2009, pp. 177–183.

[Zha20] C. Zhang, *Distinguished regular supercuspidal representations*, Math. Ann. **376** (2020), no. 3-4, 1561–1598.

Department of Mathematics (Hakubi center), Kyoto University, Kitashirakawa, Oiwake-cho, Sakyo-ku, Kyoto 606-8502, Japan.

Email address: masaooi@math.kyoto-u.ac.jp