MODERN ALGEBRA: INTRODUCTION TO REPRESENTATION THEORY (2026 FALL @ NTU)

MASAO OI

Contents

1. Lecture 1: Representation of a group

The reference for the first part of this course (representation theory of finite groups) is [Ser77].

1.1. **What is a representation of a group?** Needless to say, the notion of a *group* is one of the most fundamental and important concepts in mathematics. Therefore, it is a basic motivation to "understand" a given group (say, G) in some sense. However, when can we really say "OK, now we have understood this group G"?

Because this is not a rigorous (mathematically formulated) problem, there is also no rigorous answer; rather, there should be a lot of answers.

For example, one can claim that it is enough to describe the multiplication table for all elements of a given group G. The multiplication law of a group G is nothing but the whole group structure of G, so it indeed characterize the group G. In this sense, it should be a complete answer to the above problem. However, as you can imagine easily, describing the multiplication table is not always so practical to express the nature of the given group G.

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Example 1.1. Let G be the group whose elements consist of a, b, c, d satisfying the following multiplication table:

$\cdot$	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	a	b
d	d	a	b	c

In fact, this group is just $\mathbb{Z}/4\mathbb{Z}$, where $0, 1, 2, 3 \in \mathbb{Z}/4\mathbb{Z}$ are denoted by a, b, c, d, respectively. It is obvious that this way of expressing $\mathbb{Z}/4\mathbb{Z}$ is just making the situation complicated!

Another possible answer should be to describe a set of generators and relations of a given group G. Indeed, this can be a complete "answer", in the sense that a set of generators and relations completely characterize the group G. However, for any given group G, a set of generators and relations may not be unique; sometimes, a bad choice of generators and relations does not make the structure of the given group G transparent. To see this, let us look at the following bad example:

Example 1.2. Let $G := \mathbb{Z}/4\mathbb{Z}$. This group can be written as

$$\langle x, y \mid x^2 = y,\ y^2 = 1,\ xy = yx \rangle.$$

Of course, the relation $x^2 = y$ reduces this to

$$\langle x \mid x^4 = 1 \rangle,$$

which should be the most standard presentation of $\mathbb{Z}/4\mathbb{Z}$.

These observations suggest that there can be many different ways to formulate what it means to "understand a group". *Representation theory*, which we will study in this course, is one way to formulate the meaning of "understand a group".

Here, we caution that although we understand everything about representations of a given group G, it does not mean that we understand everything about G. For example, knowing all about representation theory of G does not necessarily enable us to describe the multiplication table of nor a set of generators and relations of G. What is important is that the notion of representation appears in various area of mathematics (or, even mathematical physics). Of course, representations of groups are interesting objects of study in their own right, but much of their value also comes from their usefulness and applications in many different areas of mathematics.

Now let us define the notion of a representation.

Definition 1.3. Let G be a group. A *representation* of G is a pair (π, V) of a $\mathbb{C}$-vector space V and an action of G on V, i.e., a map

$$G \times V \to V : (g, v) \mapsto g \cdot v$$

such that the induced map

$$\pi : G \to \mathrm{Aut}_{\mathbb{C}}(V) : g \mapsto [\pi(g) : v \mapsto g \cdot v]$$

is a group homomorphism.

It is often said that "a group is a set of symmetries". If we follow this slogan, we can say that "a representation of a group G is a realization of the symmetries contained in G on a vector space". To put it somewhat poetically, a representation of a group is a "shadow" (影) it casts on a vector space.

Remark 1.4. (1) By convention (which is normal in representation theory), we sometimes write only one of π and V to mean a representation (π, V) of a group G. For example, we often say "let π be a representaion of G" or "let V be a representation of G".
(2) Somewhat confusingly, we may also use π to refer to both the action and the underlying vector space. For example, we often say "let π be a representation of G and we take an element $v \in \pi$".
(3) When we only write V for a representation of G, we often use the symbol "$\cdot$" for denoting the action of G on V. For example, we write "let $g \in G$, $v \in V$ and we consider $g \cdot v$".

For a given group G, there are various possible representations of G. For example, we always have the *trivial representation*, which can be considered for any group G:

Example 1.5. Let $V := \mathbb{C}$ and π be the trivial action of G on $V = \mathbb{C}$. We call (V, π) the *trivial representation* of G.

The trivial representation is actually very important; but, of course, we cannot know anything about G only by looking at its trivial representation. The important idea of representation theory is to understand the give group G through its various representations ("what kinds of shadows G can cast on vector spaces"). Accordingly, the most important problem in representation theory is:

classifying all representations of G.

Example 1.6. Let G be a group and (π, V) be a 1-dimensional representation of G. Then π is identified with a homomorphism $\pi\colon G \to \mathbb{C}^\times$ (since $\mathrm{Aut}_{\mathbb{C}}(V) = \mathbb{C}^\times$). A homomorphism $G \to \mathbb{C}^\times$ is often referred to as a *character* (or 1-*dimensional character*) of G.

Remark 1.7. (1) In this course, we only consider representations of a group realized on $\mathbb{C}$-vector spaces. However, in general, representation theory heavily depends on the coefficient field of vector spaces. Especially, the whole theory drastically changes when we replace $\mathbb{C}$ with a non-algebraically-closed field or a field of positive characteristic (basically, theory becomes much more difficult).
(2) It is often the case that the group G of our interest is a topological group (e.g., Lie groups, algebraic groups, profinite groups such as infinite Galois groups, etc...). In this case, it is not so meaningful to consider all representations of G which does not take into account the topology of G. If we really want to understand the topological nature of G, it is reasonable to consider, for example, continuous representations of G (i.e., require that $\pi\colon G \to \mathrm{Aut}_{\mathbb{C}}(V)$ or $\pi\colon G \times V \to V$ are continuous). But then, we also encounter a question "which kind of topology should we put on V?" If V if a finite-dimensional $\mathbb{C}$-vector space, then V has a natural topology. However, it is not the case when V is infinite dimensional or the coefficient field is general. In summary, to understand G well, it is important to think about the "appropriate class" of representations of G.

Example 1.8. Let $G := \mathfrak{S}_n$ be the symmetric group consisting of permutations of n letters. By putting $V := \mathbb{C}^{\oplus n} = \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_n$, we can let $\mathfrak{S}_n$ naturally act on

V by permuting $\{e_1, \ldots, e_n\}$. The resulting representation is called the *standard representation* of $\mathfrak{S}_n$ on V.

Example 1.9. Let $G := \mathrm{GL}_n(\mathbb{C})$. Then G naturally acts on $V := \mathbb{C}^{\oplus n}$. We call this representation the *standard representation* of $\mathrm{GL}_n(\mathbb{C})$.

1.2. **Irreducible representations.** Let us think about "classifying all representations" of a given group G. We start with the following notion:

Definition 1.10. Let (π, V) and (π', V') be representations of G.

(1) A *homomorphism* (or also called *G-homomorphism*) from (π, V) to (π', V') is a $\mathbb{C}$-vector space homomorphism $f\colon V \to V'$ which preserves G-actions, that is, f satisfies $f(\pi(g)(v)) = \pi'(g)(f(v))$ for any $g \in G$ and $v \in V$.

(2) An isomorphism (or also called *G-isomorphism*) from (π, V) to (π', V') is a homomorphism which is bijective. If there exists an isomorphism between (π, V) and (π', V'), we say that "(π, V) and (π', V') are isomorphic".

Looking at this definition, it should be reasonable to think that we may identify representations which are isomorphic to each other. So let us think about classifying all isomorphism classes of representations.

Definition 1.11. Let (π, V) be a representation of a group G.

(1) If a subspace $W \subset V$ is closed under the G-action on V, i.e., we have $\pi(g)(w) \in W$ for any $g \in G$ and $w \in W$, we say that W is a *subrepresentation* of (π, V).

(2) Let $W \subset V$ be a subrepresentation of (π, V). By letting $g \in G$ act on V/W by $g \cdot (v + W) := g \cdot v + W$ for $v \in V$, we obtain a representation of G on V/W. We call V/W the *quotient representation* of (π, V) by W.

Lemma 1.12. *Let $f\colon (\pi, V) \to (\pi', V')$ be a homomorphism of representations of G. Then we have the following:*

(1) $\mathrm{Ker}(f) \subset V$ *is a subrepresentation.*

(2) $\mathrm{Im}(f) \subset V'$ *is a subrepresentation.*

(3) f *induces an isomorphism* $V/\mathrm{Ker}(f) \cong \mathrm{Im}(f)$ *of representations of* G.

Proof. We first show (1). Let $v \in \mathrm{Ker}(f)$. Our task is to show that $\pi(g)(v) \in \mathrm{Ker}(f)$ for any $g \in G$. In other words, it suffices to show that $f(\pi(g)(v)) = 0$ for any $g \in G$. Since f is a homomorphism of representations and $v \in \mathrm{Ker}(f)$, we have

$$f(\pi(g)(v)) = \pi'(g)(f(v)) = \pi'(g)(0) = 0.$$

We next show (2). Let $v' \in \mathrm{Im}(f)$. Our task is to show that $\pi'(g)(v') \in \mathrm{Im}(f)$ for any $g \in G$. Since f is a homomorphism of representations, by writing $v' = f(v)$ with some $v \in V$, we have

$$\pi'(g)(v') = \pi'(g)(f(v)) = f(\pi(g)(v)) \in \mathrm{Im}(f).$$

Finally, by the first isomorphism theorem of linear algebra, we have an isomorphism $V/\mathrm{Ker}(f) \cong \mathrm{Im}(f)$ as $\mathbb{C}$-vector spaces. It can be easily checked that this isomorphism is also an isomorphism of representations of G. $\square$

As this example shows, we can import various notions and results in linear algebra to representation theory.

Definition 1.13. (1) Let (π, V) and (π', V') be representations of G. We define an representation $\pi \oplus \pi'$ of G on $V \oplus V'$ by

$$(\pi \oplus \pi')(g)(v + v') := \pi(g)(v) + \pi'(g)(v')$$

for any $g \in G$, $v \in V$, $v' \in V'$. We call $(\pi \oplus \pi', V \oplus V')$ the *direct sum* of (π, V) and (π', V').

(2) Let (π, V) and (π', V') be representations of G. We define an representation $\pi \otimes \pi'$ of G on $V \otimes V'$ by

$$(\pi \otimes \pi')(g)(v \otimes v') := \pi(g)(v) \otimes \pi'(g)(v')$$

for any $g \in G$, $v \in V$, $v' \in V'$. We call $(\pi \otimes \pi', V \otimes V')$ the *tensor product* of (π, V) and (π', V').

(3) Let (π, V) be a representation of G. We define the dual space $V^\vee := \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ with an action $\pi^\vee$ of G by

$$\langle \pi^\vee(g)(v^\vee), v \rangle := \langle v^\vee, \pi(g^{-1})(v) \rangle$$

for any $g \in G$, $v \in V$, $v^\vee \in V^\vee$, where $\langle -, - \rangle$ denotes the natural pairing

$$V^\vee \times V \to \mathbb{C} \colon (v^\vee, v) \mapsto v^\vee(v).$$

We call $(\pi^\vee, V^\vee)$ the *dual representation* (or *contragredient representation*) of (π, V).

In a more sophisticated language, we can say that the category of representations of a group G has a structure of a "tensor category". Note that, if G is the trivial group $\{1\}$, then a representation of G is just a $\mathbb{C}$-vector space. Therefore, we may understand that representation theory is a generalization of linear algebra, or linear algebra is the simplest case of representation theory.

Now recall from linear algebra that an important theorem is that "any $\mathbb{C}$-vector space has a basis". In other words, this theorem says that any $\mathbb{C}$-vector space is isomorphic to a direct sum of copies of $\mathbb{C}$ (the trivial representation of the trivial group!). In particular, the isomorphism classes of $\mathbb{C}$-vector spaces are classified only by their dimensions.

Keeping this in mind, let us think about the classification of isomorphism classes of representations of a general group G. Firstly, we need to introduce "atoms" in representation theory, which cannot be decomposed any further like "$\mathbb{C}$" in linear algebra.

Definition 1.14. Let (π, V) be a representation of G. If $V \neq \{0\}$ and there does not exist a subrepresentation of V other than $\{0\}$ and V, we say that (π, V) is *irreducible*.

Ideally, we can hope to have the following kind of results for representations of a group G, which naturally generalizes the classification of $\mathbb{C}$-vector spaces in linear algebra:

(1) Any representation of G is a direct sum of irreducible representations ("*semisimplicity*").
(2) We can give a complete classification of the isomorphism classes of irreducible representations of G ("*classification of irreducible representations*").

If we can really establish such results, we see that the isomorphism classes of representations of G are exactly given by

$$\bigoplus_{\pi} \pi^{\oplus m_\pi},$$

where π runs over the isomorphism classes of irreducible representations of G (which can be further classified by (2)) and $m_\pi \in \mathbb{Z}_{\geq 0}$ is the *multiplicity* of π in the direct sum decomposition.

However, the reality is not so sweet; both (1) and (2) depend heavily on the given group G. Especially, (1) is not necessarily true, and (2) is a very difficult problem in general.

Now let us explain the highlight of the first part of this course, which is the representation theory of finite groups. The main point of representation theory of finite groups consists of the following:

- Any finite-dimensional representation of a finite group is semisimple, i.e., decomposes into the direct sum of irreducible representations (thus (1) of the above expectation is true).
- If a representation ρ of G and an irreducible representation π of G are give, we can establish a uniform manner of determining the multiplicity of π in the irreducible decomposition of ρ.

Here, we remark that the problem of classifying irreducible representations is still a difficult problem in general, even in the case of finite groups. For example, it is known that irreducible representations of symmetric groups can be classified by Young diagrams, which are very explicit combinatorial objects. However, such a nice classification exists only for very few groups (probably), and basically depends on each individual group. In other words, we can interpret that the "general theory of representation theory of finite groups" is a claim that "classification of all representations can be completely reduced to classification of irreducible representations".

1.3. **Semisimplicity.**

Definition 1.15. Let (π, V) be a finite-dimensional representation of G. If the $\mathbb{C}$-vector space V is equipped with a Hermitian inner product $(-,-)\colon V \times V \to \mathbb{C}$ which is G-invariant, i.e., the equality

$$(\pi(g)(v), \pi(g)(v')) = (v, v')$$

is satisfied by any $g \in G$ and $v, v' \in V$, we say that (π, V) is a *unitary representation* of G.

Lemma 1.16. *Let G be a group and (π, V) be a unitary finite-dimensional representation of G. Then, for any subrepresentation $W \subset V$, there exists a subrepresentation $W' \subset V$ such that $V = W \oplus W'$.*

Proof. We let $W^\perp$ be the orthogonal complement of W in V with respect to the inner product $(-,-)$, i.e.,

$$W^\perp := \{v \in V \mid (v, w) = 0 \text{ for any } w \in W\}.$$

Then, since $(-,-)$ is an inner product, we have $V = W \oplus W^\perp$. Thus it is enough to check that $W^\perp$ is a subrepresentation. For any $v \in W^\perp$ and $g \in G$, the G-invariance of $(-,-)$ implies that

$$(\pi(g)(v), w) = (v, \pi(g^{-1})w) = 0$$

for any $w \in W$ (note that $\pi(g^{-1})w \in W$ since W is closed under G-action). In other words, $\pi(g)v$ again belongs to $W^{\perp}$. □

Lemma 1.17. *Any unitary finite-dimensional representation of a group G is semisimple.*

Proof. Let (π, V) be a unitary finite-dimensional representation of G. If V is already irreducible, then π itself gives an irreducible decomposition of π. Thus let us suppose that V is not irredcuible in the following. Then, by definition, there exists a non-zero proper subrepresentation $\{0\} \subsetneq W \subsetneq V$. By applying Lemma 1.16 to W, we can choose a subrepresentaion $W' \subset V$ such that $V = W \oplus W'$. Since the restriction of the inner product of V to W' gives an inner product of W', we see that W' is again a unitary finite-dimensional representation of G. By repeating this procedure, we eventually arrive at an irreducible decomposition of V since V is finite-dimensional. □

Proposition 1.18. *Any finite-dimensional representaion of a finite group G is semisimple.*

Proof. By Lemma 1.17, it is enough to prove that any finite-dimensional representation of a finite group G is automatically unitary. Let $(-,-)\colon V \times V \to \mathbb{C}$ be any inner product (which may not necessarily be G-invariant). We modify $(-,-)$ by "averaging" it with respect to the G-action:

$$(-,-)'\colon V \times V \to \mathbb{C}; \quad (v, v')' := \frac{1}{|G|} \sum_{g \in G} (\pi(g)(v), \pi(g)(v')).$$

This is obviously again an inner product (positive-definite Hermitian inner product) of V. Moreover, it is G-invariant; indeed, for any $h \in G$ and $v, v' \in V$, we have

$$\begin{aligned}(\pi(h)(v), \pi(h)(v'))' &= \frac{1}{|G|} \sum_{g \in G} (\pi(gh)(v), \pi(gh)(v')) \\ &= \frac{1}{|G|} \sum_{g \in G} (\pi(g)(v), \pi(g)(v')) = (v, v')'.\end{aligned}$$

□

1.4. **Exercises.**

Exercise 1.1. Let G be a finite group. Prove that any irreducible representation of G is necessarily finite-dimensional.

Exercise 1.2. Let $G := \mathbb{Z}$. We consider the 2-dimensional representation (π, V) of G defined by $V = \mathbb{C}^{\oplus 2}$ and

$$\pi\colon \mathbb{Z} \to \mathrm{GL}_2(\mathbb{C})\colon n \mapsto \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}.$$

Prove that π is not semisimple.

Appendix A. Regulations for exercise sessions and grading

This course has Exercise Sessions once every four lectures. Thus it will be like:

$$2\text{h lecture} \to 2\text{h lecture} \to 2\text{h lecture} \to 2\text{h Exercise Session}$$
$$\to 2\text{h lecture} \to 2\text{h lecture} \to 2\text{h lecture} \to 2\text{h Exercise Session} \to \cdots$$

The grade for this course will be determined by

(1) written reports and
(2) oral presentations.

Lecture notes will be uploaded for each lecture, and exercise problems will be included in the notes. Solve these problems and submit your solutions. Basically, you can receive up to **4 points** for each problem. Your final score will be the total number of points you earn. Please collect as many points as you want!

However, I also would like to introduce the following special rules:

- You can receive up to **80 points** in total by submitting written reports.
- If you give an oral presentation of your solution to any problem at the blackboard during an Exercise Session, you can receive up to **20 points**.
- To pass the course, you must give an oral presentation **at least once**.

Example A.1. (1) [oral $\varnothing$] + [written 80 pts] $\rightsquigarrow$ [No credit].
(2) [oral 5 pts] + [written 80 pts] $\rightsquigarrow$ [85 pts].
(3) [oral 15 pts] + [oral 20 pts] + [written 70 pts] $\rightsquigarrow$ [100 pts].
(4) [oral 20 pts]$\times 5$ +[written $\varnothing$] $\rightsquigarrow$ [100 pts].

The detailed regulations are explained below.

A.1. Written reports.

(1) Please submit your written solution through NTU COOL or by email to me.
(2) You may submit your reports at any time during the semester.
(3) After each Exercise Session, the reports submitted up to that point will be returned to you with your score and brief feedback.
(4) After getting a feedback, if your score is less than 4, you can modify your solution following the feedback and **re-submit up to once**. If your re-submitted solution is improved, you may get more points.

A.2. Oral presentations.

(1) During an Exercise Session, students will present solutions problems they chose at the blackboard.
(2) Each presentation must be given **within 15 minutes**, and then a 5 minutes question/comment time will be given. (So, at maximum, 6 students can give oral presentations in one Exercise Session.)
(3) Before each Exercise Session, students can express which problem they want to present (through the google spread sheet for this course).
    - The list of students and their preferred problems will be visible to everyone.
    - Each student can select more than one problems.
    - If two or more students wish to present the same problem, priority will be given to the student who has given fewer oral presentations. (If they

have given the same number of oral presentations, the presenter will be selected randomly.)
  - Within the day before an Exercise Session, the list of presenters will be announced.
  - I would like to do my best so that presentation opportunities will be allocated to every student equally.

(4) If you are selected as a presenter, please prepare a 15-minute presentation of your solution. You should try to explain your argument so that the other students, TA (Hao-An Wu), and I can follow it. Please regard this as an opportunity to develop your presentation skills.

(5) In addition to the oral presentation, **the presenter must submit a written solution** to the problem by the Exercise Session.

(6) You will receive feedback on your written solution and may revise and resubmit it afterward. An improved written solution may result in a higher score as well. However, the quality of your oral presentation has most weights. For example;
  - Suppose that you have prepared your presentation seriously, but during the presentation a mistake in your argument is discovered and the proof breaks down. This is not necessarily a serious problem. If you understand the issue through the subsequent discussion and successfully correct it in your revised written solution, you may still receive a high score.
  - On the other hand, suppose that your written solution is perfect, but it is clear that you have not prepared for the presentation. Similarly, if you are unable to explain your own argument or answer basic questions about your solution, this may result in a low score even if the written solution itself is excellent.

A.3. **Q & A.**

Question 1. Can I give an oral presentation of a problem which were already presented by someone at the blackboard?

Answer 1. No. Each problem can be presented at the brackboard at most once.

Question 2. Can I submit a written solution to a problem which were already presented by someone at the blackboard?

Answer 2. Yes. In this case, probably you already got to know how to solve the problem. However, what is important in writing a report is not only its correctness, but also how it is organized/explained. Please try to write a report in your words with your understanding even if you already have listened to other student's oral presentation on the problem.

Question 3. Can I submit all written solutions together at the end of the course?

Answer 3. Yes, in principle. However, please keep in mind that it will be more difficult for you to know how many points you can get precisely in advance.

Question 4. Is there any other necessary condition to get the course credit (e.g., attendance)?

Answer 4. No, but please do not forget to give an oral presentation at least once.

References

[Ser77] J.-P. Serre, *Linear representations of finite groups*, Springer-Verlag, New York-Heidelberg, 1977, Translated from the second French edition by Leonard L. Scott, Graduate Texts in Mathematics, Vol. 42.

Department of Mathematics, National Taiwan University, Astronomy Mathematics Building 5F, No. 1, Sec. 4, Roosevelt Rd., Taipei 10617, Taiwan.

Email address: masaooi@ntu.edu.tw