MODERN ALGEBRA: INTRODUCTION TO REPRESENTATION THEORY (2026 FALL @ NTU)

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1. Lecture 1: Representation of a group

(Main reference for today's lecture: [Ser77, Chapter 1])

1.1. **What is a representation of a group?** Needless to say, the notion of a *group* is one of the most fundamental and important concepts in mathematics. Therefore, it is a basic motivation to "understand" a given group (say, G) in some sense. However, when can we really say "OK, now we have understood this group G"?

Because this is not a rigorous (mathematically formulated) problem, there is also no rigorous answer; rather, there should be a lot of answers.

For example, one can claim that it is enough to describe the multiplication table for all elements of a given group G. The multiplication law of a group G is nothing but the whole group structure of G, so it indeed characterizes the group G. In this sense, it should be a complete answer to the above problem. However, as you can imagine easily, describing the multiplication table is not always a practical way to describe the nature of the given group G.

Example 1.1. Let G be the group whose elements consist of a, b, c, d satisfying the following multiplication table:

$\cdot$	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	a	b
d	d	a	b	c

In fact, this group is just $\mathbb{Z}/4\mathbb{Z}$, where $0, 1, 2, 3 \in \mathbb{Z}/4\mathbb{Z}$ are denoted by a, b, c, d, respectively. It is obvious that this way of expressing $\mathbb{Z}/4\mathbb{Z}$ is just making the situation complicated!

Another possible answer should be to describe a set of generators and relations of a given group G. Indeed, this can be a complete "answer", in the sense that a set of generators and relations completely characterize the group G. However, for any given group G, a set of generators and relations may not be unique; sometimes, a bad choice of generators and relations does not make the structure of the given group G transparent. To see this, let us look at the following bad example:

Example 1.2. Let $G := \mathbb{Z}/4\mathbb{Z}$. This group can be written as

$$\langle x, y \mid x^2 = y,\ y^2 = 1,\ xy = yx \rangle.$$

Of course, the relation $x^2 = y$ reduces this to

$$\langle x \mid x^4 = 1 \rangle,$$

which should be the most standard presentation of $\mathbb{Z}/4\mathbb{Z}$.

These observations suggest that there can be many different ways to formulate what it means to "understand a group". *Representation theory*, which we will study in this course, is one way to formulate the meaning of "understand a group".

Here, we caution that although we understand everything about representations of a given group G, it does not mean that we understand everything about G. For example, knowing all about representation theory of G does not necessarily enable us to describe the multiplication table of G or a set of generators and relations of G. What is important is that the notion of representation appears in various areas of mathematics (or, even mathematical physics). Of course, representations of groups are interesting objects of study in their own right, but much of their value also comes from their usefulness and applications in many different areas of mathematics.

Now let us define the notion of a representation.

Definition 1.3. Let G be a group. A *representation* of G is a pair (π, V) of a $\mathbb{C}$-vector space V and an action of G on V, i.e., a map

$$G \times V \to V \colon (g, v) \mapsto g \cdot v$$

such that the induced map

$$\pi \colon G \to \mathrm{Aut}_{\mathbb{C}}(V) \colon g \mapsto [\pi(g) \colon v \mapsto g \cdot v]$$

is a group homomorphism.

It is often said that "a group is a set of symmetries". If we follow this slogan, we can say that "a representation of a group G is a realization of the symmetries contained in G on a vector space". To put it somewhat poetically, a representation of a group is a "shadow" it casts on a vector space.

Remark 1.4.
(1) By convention (which is normal in representation theory), we sometimes write only one of π and V to mean a representation (π, V) of a group G. For example, we often say "let π be a representation of G" or "let V be a representation of G".
(2) Somewhat confusingly, we may also use π to refer to both the action and the underlying vector space. For example, we often say "let π be a representation of G and we take an element $v \in \pi$".
(3) When we only write V for a representation of G, we often use the symbol "$\cdot$" for denoting the action of G on V. For example, we write "let $g \in G$, $v \in V$ and we consider $g \cdot v$".

For a given group G, there are various possible representations of G. For example, we always have the *trivial representation*, which can be considered for any group G:

Example 1.5. Let $V := \mathbb{C}$ and π be the trivial action of G on $V = \mathbb{C}$. We call (V, π) the *trivial representation* of G.

The trivial representation is actually very important; but, of course, we cannot know anything about G only by looking at its trivial representation. The important idea of representation theory is to understand the given group G through its various representations ("what kinds of shadows G can cast on vector spaces"). Accordingly, the most important problem in representation theory is:

classifying all representations of G.

Example 1.6. Let G be a group and (π, V) be a 1-dimensional representation of G. Then π is identified with a homomorphism $\pi \colon G \to \mathbb{C}^\times$ (since $\mathrm{Aut}_{\mathbb{C}}(V) = \mathbb{C}^\times$).

A homomorphism $G \to \mathbb{C}^\times$ is often referred to as a *character* (or 1-*dimensional character*) of G.

Remark 1.7. (1) In this course, we only consider representations of a group realized on $\mathbb{C}$-vector spaces. However, in general, representation theory heavily depends on the coefficient field of vector spaces. Especially, the whole theory drastically changes when we replace $\mathbb{C}$ with a non-algebraically-closed field or a field of positive characteristic (basically, theory becomes much more difficult).

(2) It is often the case that the group G of our interest is a topological group (e.g., Lie groups, algebraic groups, profinite groups such as infinite Galois groups, etc...). In this case, it is not so meaningful to consider all representations of G that do not take into account the topology of G. If we really want to understand the topological nature of G, it is reasonable to consider, for example, continuous representations of G (i.e., require that $\pi\colon G \to \mathrm{Aut}_{\mathbb{C}}(V)$ or $\pi\colon G \times V \to V$ are continuous). But then, we also encounter a question "which kind of topology should we put on V?" If V is a finite-dimensional $\mathbb{C}$-vector space, then V has a natural topology. However, it is not the case when V is infinite dimensional or the coefficient field is general. In summary, to understand G well, it is important to think about the "appropriate class" of representations of G.

Example 1.8. Let $G := \mathfrak{S}_n$ be the symmetric group consisting of permutations of n letters. By putting $V := \mathbb{C}^{\oplus n} = \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_n$, we can let $\mathfrak{S}_n$ naturally act on V by permuting $\{e_1, \ldots, e_n\}$. The resulting representation is called the *permutation representation* of $\mathfrak{S}_n$ on V.

Example 1.9. Let $G := \mathrm{GL}_n(\mathbb{C})$. Then G naturally acts on $V := \mathbb{C}^{\oplus n}$. We call this representation the *standard representation* of $\mathrm{GL}_n(\mathbb{C})$.

1.2. **Irreducible representations.** Let us think about "classifying all representations" of a given group G. We start with the following notion:

Definition 1.10. Let (π, V) and (π', V') be representations of G.

(1) A *homomorphism* (or also called G-*homomorphism*) from (π, V) to (π', V') is a $\mathbb{C}$-vector space homomorphism $f\colon V \to V'$ which preserves G-actions, that is, f satisfies $f(\pi(g)(v)) = \pi'(g)(f(v))$ for any $g \in G$ and $v \in V$.

(2) An isomorphism (or also called G-*isomorphism*) from (π, V) to (π', V') is a homomorphism which is bijective. If there exists an isomorphism between (π, V) and (π', V'), we say that "(π, V) and (π', V') are isomorphic".

Looking at this definition, it should be reasonable to think that we may identify representations which are isomorphic to each other. So let us think about classifying all isomorphism classes of representations.

Definition 1.11. Let (π, V) be a representation of a group G.

(1) If a subspace $W \subset V$ is closed under the G-action on V, i.e., we have $\pi(g)(w) \in W$ for any $g \in G$ and $w \in W$, we say that W is a *subrepresentation* of (π, V).

(2) Let $W \subset V$ be a subrepresentation of (π, V). By letting $g \in G$ act on V/W by $g \cdot (v + W) := g \cdot v + W$ for $v \in V$, we obtain a representation of G on V/W. We call V/W the *quotient representation* of (π, V) by W.

Lemma 1.12. *Let $f\colon (\pi, V) \to (\pi', V')$ be a homomorphism of representations of G. Then we have the following:*

(1) $\mathrm{Ker}(f) \subset V$ *is a subrepresentation.*
(2) $\mathrm{Im}(f) \subset V'$ *is a subrepresentation.*
(3) *f induces an isomorphism $V/\mathrm{Ker}(f) \cong \mathrm{Im}(f)$ of representations of G.*

Proof. We first show (1). Let $v \in \mathrm{Ker}(f)$. Our task is to show that $\pi(g)(v) \in \mathrm{Ker}(f)$ for any $g \in G$. In other words, it suffices to show that $f(\pi(g)(v)) = 0$ for any $g \in G$. Since f is a homomorphism of representations and $v \in \mathrm{Ker}(f)$, we have

$$f(\pi(g)(v)) = \pi'(g)(f(v)) = \pi'(g)(0) = 0.$$

We next show (2). Let $v' \in \mathrm{Im}(f)$. Our task is to show that $\pi'(g)(v') \in \mathrm{Im}(f)$ for any $g \in G$. Since f is a homomorphism of representations, by writing $v' = f(v)$ with some $v \in V$, we have

$$\pi'(g)(v') = \pi'(g)(f(v)) = f(\pi(g)(v)) \in \mathrm{Im}(f).$$

Finally, by the first isomorphism theorem of linear algebra, we have an isomorphism $V/\mathrm{Ker}(f) \cong \mathrm{Im}(f)$ as $\mathbb{C}$-vector spaces. It can be easily checked that this isomorphism is also an isomorphism of representations of G. □

As this example shows, we can import various notions and results in linear algebra to representation theory.

Definition 1.13. (1) Let (π, V) and (π', V') be representations of G. We define a representation $\pi \oplus \pi'$ of G on $V \oplus V'$ by

$$(\pi \oplus \pi')(g)(v + v') := \pi(g)(v) + \pi'(g)(v')$$

for any $g \in G$, $v \in V$, $v' \in V'$. We call $(\pi \oplus \pi', V \oplus V')$ the *direct sum* of (π, V) and (π', V').

(2) Let (π, V) and (π', V') be representations of G. We define a representation $\pi \otimes \pi'$ of G on $V \otimes V'$ by

$$(\pi \otimes \pi')(g)(v \otimes v') := \pi(g)(v) \otimes \pi'(g)(v')$$

for any $g \in G$, $v \in V$, $v' \in V'$. We call $(\pi \otimes \pi', V \otimes V')$ the *tensor product* of (π, V) and (π', V').

(3) Let (π, V) be a representation of G. We define the dual space $V^\vee := \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ with an action $\pi^\vee$ of G by

$$\langle \pi^\vee(g)(v^\vee), v \rangle := \langle v^\vee, \pi(g^{-1})(v) \rangle$$

for any $g \in G$, $v \in V$, $v^\vee \in V^\vee$, where $\langle -, - \rangle$ denotes the natural pairing

$$V^\vee \times V \to \mathbb{C}\colon (v^\vee, v) \mapsto v^\vee(v).$$

We call $(\pi^\vee, V^\vee)$ the *dual representation* (or *contragredient representation*) of (π, V).

In a more sophisticated language, we can say that the category of representations of a group G has a structure of a "tensor category". Note that, if G is the trivial group $\{1\}$, then a representation of G is just a $\mathbb{C}$-vector space. Therefore, we may understand that representation theory is a generalization of linear algebra, or linear algebra is the simplest case of representation theory.

Now recall from linear algebra that an important theorem is that "any $\mathbb{C}$-vector space has a basis". In other words, this theorem says that any $\mathbb{C}$-vector space is

isomorphic to a direct sum of copies of $\mathbb{C}$ (the trivial representation of the trivial group!). In particular, the isomorphism classes of $\mathbb{C}$-vector spaces are classified only by their dimensions.

Keeping this in mind, let us think about the classification of isomorphism classes of representations of a general group G. Firstly, we need to introduce "atoms" in representation theory, which cannot be decomposed any further like "$\mathbb{C}$" in linear algebra.

Definition 1.14. Let (π, V) be a representation of G. If $V \neq \{0\}$ and there does not exist a subrepresentation of V other than $\{0\}$ and V, we say that (π, V) is *irreducible*.

Ideally, we can hope to have results of the following kind for representations of a group G, which naturally generalizes the classification of $\mathbb{C}$-vector spaces in linear algebra:

(1) Any representation of G is a direct sum of irreducible representations ("*semisimplicity*").
(2) We can give a complete classification of the isomorphism classes of irreducible representations of G ("*classification of irreducible representations*").

If we can really establish such results, we see that the isomorphism classes of representations of G are exactly given by

$$\bigoplus_{\pi} \pi^{\oplus m_\pi},$$

where π runs over the isomorphism classes of irreducible representations of G (which can be further classified by (2)) and $m_\pi \in \mathbb{Z}_{\geq 0}$ is the *multiplicity* of π in the direct sum decomposition.

However, the reality is not so sweet; both (1) and (2) depend heavily on the given group G. Especially, (1) is not necessarily true, and (2) is a very difficult problem in general.

Now let us explain the highlight of the first part of this course, which is the representation theory of finite groups. The main point of representation theory of finite groups consists of the following:

- Any finite-dimensional representation of a finite group is semisimple, i.e., decomposes into the direct sum of irreducible representations (thus (1) of the above expectation is true).
- If a representation ρ of G and an irreducible representation π of G are given, we can establish a uniform manner of determining the multiplicity of π in the irreducible decomposition of ρ.

Here, we remark that the problem of classifying irreducible representations is still a difficult problem in general, even in the case of finite groups. For example, it is known that irreducible representations of symmetric groups can be classified by Young diagrams, which are very explicit combinatorial objects. However, such a nice classification exists only for very few groups (probably), and basically depends on each individual group. In other words, we can interpret that the "general theory of representation theory of finite groups" is a claim that "classification of all representations can be completely reduced to classification of irreducible representations".

1.3. **Semisimplicity.**

Definition 1.15. Let (π, V) be a finite-dimensional representation of G. If the $\mathbb{C}$-vector space V is equipped with a Hermitian inner product $(-,-)\colon V \times V \to \mathbb{C}$ which is G-invariant, i.e., the equality

$$(\pi(g)(v), \pi(g)(v')) = (v, v')$$

is satisfied by any $g \in G$ and $v, v' \in V$, we say that (π, V) is a *unitary representation* of G.

Lemma 1.16. *Let G be a group and (π, V) be a unitary finite-dimensional representation of G. Then, for any subrepresentation $W \subset V$, there exists a subrepresentation $W' \subset V$ such that $V = W \oplus W'$.*

Proof. We let $W^\perp$ be the orthogonal complement of W in V with respect to the inner product $(-,-)$, i.e.,

$$W^\perp := \{v \in V \mid (v, w) = 0 \text{ for any } w \in W\}.$$

Then, since $(-,-)$ is an inner product, we have $V = W \oplus W^\perp$. Thus it is enough to check that $W^\perp$ is a subrepresentation. For any $v \in W^\perp$ and $g \in G$, the G-invariance of $(-,-)$ implies that

$$(\pi(g)(v), w) = (v, \pi(g^{-1})w) = 0$$

for any $w \in W$ (note that $\pi(g^{-1})w \in W$ since W is closed under G-action). In other words, $\pi(g)v$ again belongs to $W^\perp$. □

Lemma 1.17. *Any unitary finite-dimensional representation of a group G is semisimple.*

Proof. Let (π, V) be a unitary finite-dimensional representation of G. If V is already irreducible, then π itself gives an irreducible decomposition of π. Thus let us suppose that V is not irreducible in the following. Then, by definition, there exists a non-zero proper subrepresentation $\{0\} \subsetneq W \subsetneq V$. By applying Lemma 1.16 to W, we can choose a subrepresentation $W' \subset V$ such that $V = W \oplus W'$. Since the restriction of the inner product of V to W' gives an inner product of W', we see that W' is again a unitary finite-dimensional representation of G. By repeating this procedure, we eventually arrive at an irreducible decomposition of V since V is finite-dimensional. □

Proposition 1.18. *Any finite-dimensional representation of a finite group G is semisimple.*

Proof. By Lemma 1.17, it is enough to prove that any finite-dimensional representation of a finite group G is automatically unitary. Let $(-,-)\colon V \times V \to \mathbb{C}$ be any inner product (which may not necessarily be G-invariant). We modify $(-,-)$ by "averaging" it with respect to the G-action:

$$(-,-)'\colon V \times V \to \mathbb{C}; \quad (v, v')' := \frac{1}{|G|} \sum_{g \in G} (\pi(g)(v), \pi(g)(v')).$$

This is obviously again an inner product (positive-definite Hermitian inner product) of V. Moreover, it is G-invariant; indeed, for any $h \in G$ and $v, v' \in V$, we have

$$\begin{aligned}(\pi(h)(v), \pi(h)(v'))' &= \frac{1}{|G|} \sum_{g \in G} (\pi(gh)(v), \pi(gh)(v')) \\ &= \frac{1}{|G|} \sum_{g \in G} (\pi(g)(v), \pi(g)(v')) = (v, v')'.\end{aligned}$$

□

1.4. **Exercises.**

Exercise 1.1. Let G be a finite group. Prove that any irreducible representation of G is necessarily finite-dimensional.

Exercise 1.2. Let $G := \mathbb{Z}$. We consider the 2-dimensional representation (π, V) of G defined by $V = \mathbb{C}^{\oplus 2}$ and

$$\pi \colon \mathbb{Z} \to \mathrm{GL}_2(\mathbb{C}) \colon n \mapsto \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}.$$

Prove that π is not semisimple.

2. Lecture 2: Characters of finite-dimensional representations

(Main reference for today's lecture: [Ser77, Chapter 2])

2.1. Characters of representations.

Definition 2.1. Let (π, V) be a finite-dimensional representation of a group G. We define a function $\Theta_\pi \colon G \to \mathbb{C}$ by

$$\Theta_\pi \colon G \to \mathbb{C} \colon g \mapsto \mathrm{Tr}(\pi(g) \mid V).$$

We call Θ_π the *character* (or *trace character*) of (π, V).

Practically, the trace character can be computed by expressing $\pi(g)$ as a square matrix with respect to any $\mathbb{C}$-basis of V (recall that the trace of a $\mathbb{C}$-linear endomorphism is independent of the choice of a $\mathbb{C}$-basis).

Remark 2.2. Recall that we also call a 1-dimensional representation of G a "character" of G. So these terminologies are confusing, but I believe there will not be a serious problem. (If (π, V) is a 1-dimensional character, then its trace character $\Theta_\pi \colon G \to \mathbb{C}$ is nothing but $\pi \colon G \to \mathbb{C}^\times$ itself.)

Let us investigate basic properties of characters of representations.

Lemma 2.3. *Let (π, V) be an n-dimensional representation of a finite group G. Then we have the following:*

1. $\Theta_\pi(1) = n$.
2. $\Theta_\pi(g^{-1}) = \overline{\Theta_\pi(g)}$ *(the right-hand side denotes the complex conjugate).*
3. Θ_π *is conjugation-invariant, i.e., we have* $\Theta_\pi(hgh^{-1}) = \Theta_\pi(g)$ *for any* $g \in G$ *and* $h \in G$.

Proof. By choosing a $\mathbb{C}$-basis of V, let us think of π as a homomorphism $G \to \mathrm{GL}_n(\mathbb{C})$.

Since $\pi(1)$ is the identity matrix I_n of size n, its trace is given by n. Thus we get (1).

We check (2). Let $\alpha_1, \dots, \alpha_n$ be the set of eigenvalues of $\pi(g)$. Since G is a finite group, we have $g^r = 1$ for some $r \in \mathbb{Z}$. Then we have $\pi(g)^r = \pi(g^r) = I_n$. In particular, this implies that all $\alpha_1, \dots, \alpha_n$ are r-th roots of unity. On the other hand, since $\pi(g^{-1}) = \pi(g)^{-1}$, the eigenvalues of $\pi(g^{-1})$ are given by $\alpha_1^{-1}, \dots, \alpha_n^{-1}$. For any root of unity, its inverse is nothing but its complex conjugate. Hence we get

$$\Theta_\pi(g^{-1}) = \alpha_1^{-1} + \cdots + \alpha_n^{-1} = \overline{\alpha_1} + \cdots + \overline{\alpha_n} = \overline{\alpha_1 + \cdots + \alpha_n} = \overline{\Theta_\pi(g)}.$$

Finally, (3) follows from a general property of the trace of a matrix, that is, the trace is invariant under conjugation:

$$\Theta_\pi(hgh^{-1}) = \mathrm{Tr}(\pi(hgh^{-1})) = \mathrm{Tr}(\pi(h)\pi(g)\pi(h)^{-1}) = \mathrm{Tr}(\pi(g)) = \Theta_\pi(g).$$

□

Definition 2.4. For a group G, we call any function $f \colon G \to \mathbb{C}$ which is invariant under G-conjugation a *class function on* G. We write $C(G)$ for the set of class functions on G (note that $C(G)$ naturally has a structure of a $\mathbb{C}$-vector space).

Note that, with this definition, we may understand that Lemma 2.3 (3) asserts that the character Θ_π of any representation π of G belongs to $C(G)$.

Lemma 2.5. *Let (π, V) be a finite-dimensional representation of a group G. Then we have $\Theta_\pi(g) = \Theta_{\pi^\vee}(g^{-1})$ for any $g \in G$.*

Proof. Recall that, by definition, the contragredient representation $\pi^\vee$ is realized on the dual space $V^\vee$ by

$$\langle \pi^\vee(g)(v^\vee), v\rangle := \langle v^\vee, \pi(g^{-1})(v)\rangle.$$

The assertion immediately follows from this. □

Lemma 2.6. *Let (π_1, V_1) and (π_2, V_2) be finite-dimensional representations of a group G. Then we have $\Theta_{\pi_1 \oplus \pi_2} = \Theta_{\pi_1} + \Theta_{\pi_2}$.*

Proof. This is almost obvious from the definition; if we fix $\mathbb{C}$-bases of V_1 and V_2 and express $\pi_1(g)$ and $\pi_2(g)$ as square matrices, then $(\pi_1 \oplus \pi_2)(g)$ is expressed as the block matrix with block-diagonal entries $\pi_1(g)$ and $\pi_2(g)$. □

Example 2.7. Let $G := \mathfrak{S}_3 = \{1, (1\,2), (2\,3), (3\,1), (1\,2\,3), (3\,2\,1)\}$. Let $(\pi, V := \mathbb{C}^{\oplus 3})$ be the permutation representation of $\mathfrak{S}_3$. Recall that the conjugacy classes of the symmetric groups are classified by the "cycle types". In the case of $\mathfrak{S}_3$, we have 3 conjugacy classes:

$$\{1\}, \quad \{(1\,2), (2\,3), (3\,1)\}, \quad \{(1\,2\,3), (3\,2\,1)\}.$$

Let us compute the character Θ_π at each conjugacy class.

(1) If $g = 1$, then we have $\Theta_\pi(g) = \dim(V) = 3$.
(2) If $g = (1\,2)$, then the representation matrix of $\pi(g)$ is given by

$$\begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix}.$$

Hence we get $\Theta_\pi(g) = 1$.
(3) If $g = (1\,2\,3)$, then the representation matrix of $\pi(g)$ is given by

$$\begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix}.$$

Hence we get $\Theta_\pi(g) = 0$.

2.2. **Schur's lemma.** Let G be a finite group.

Lemma 2.8. *Let (π_1, V_1) and (π_2, V_2) be irreducible representations of G which are not isomorphic. Then $\mathrm{Hom}_G(V_1, V_2) = \{0\}$. In other words, any homomorphism $f\colon V_1 \to V_2$ as a representation of G is necessarily the zero map.*

Proof. Let $f\colon V_1 \to V_2$ be a homomorphism as representations of G. Then its kernel $\mathrm{Ker}(f)$ is a subrepresentation of V_1 (Lemma 1.12 (1)). Since V_1 is irreducible, we must have either $\mathrm{Ker}(f) = \{0\}$ or $\mathrm{Ker}(f) = V_1$. If $\mathrm{Ker}(f) = V_1$, then it means that $f = 0$. Thus let us suppose that $\mathrm{Ker}(f) = \{0\}$ for the sake of contradiction. By Lemma 1.12 (3), this implies that $V_1 \cong V_1/\mathrm{Ker}(f) \cong \mathrm{Im}(f) \subset V_2$. However, as $\mathrm{Im}(f)$ is also a subrepresentation of V_2 (Lemma 1.12 (2)) and nonzero, we necessarily have $\mathrm{Im}(f) = V_2$. Accordingly, we get $V_1 \cong V_2$, hence a contradiction. □

Theorem 2.9 (Schur's lemma)**.** *Let (π, V) be an irreducible finite-dimensional*[1] *representation of G. Then $\mathrm{End}_G(V) \cong \mathbb{C}$. In other words, any endomorphism $f\colon V \to V$ as a representation of G is necessarily of the form $f = \lambda \cdot \mathrm{id}_V$ with $\lambda \in \mathbb{C}$.*

Proof. Any $\mathbb{C}$-linear endomorphism of V of the form $\lambda \cdot \mathrm{id}_V$ is obviously an endomorphism of V as a representation of G. Hence our task is to show its converse.

Let $f\colon V \to V$ be a nonzero endomorphism of (π, V) as a representation of G. Let $v \in V \smallsetminus \{0\}$ be an eigenvector with eigenvalue $\lambda \in \mathbb{C}$, i.e., $f(v) = \lambda \cdot v$. Then $f - \lambda \cdot \mathrm{id}_V \in \mathrm{End}_{\mathbb{C}}(V)$ again belongs to $\mathrm{End}_G(V)$. Since the kernel of any homomorphism as G-representations is a subrepresentation (Lemma 1.12 (1)), we see that $\mathrm{Ker}(f - \lambda \cdot \mathrm{id}_V) \subset V$ is a subrepresentation. Since V is irreducible and $\mathrm{Ker}(f - \lambda \cdot \mathrm{id}_V) \neq 0$ (as it at least contains $v \neq 0$), we necessarily have $\mathrm{Ker}(f - \lambda \cdot \mathrm{id}_V) = V$. In other words, $f = \lambda \cdot \mathrm{id}_V$. □

Corollary 2.10. *Let (π_1, V_1) and (π_2, V_2) be irreducible representations of a group G. For any $f \in \mathrm{Hom}_{\mathbb{C}}(V_1, V_2)$, we define $f^0 \in \mathrm{Hom}_{\mathbb{C}}(V_1, V_2)$ by*

$$f^0 := \frac{1}{|G|} \sum_{g \in G} \pi_2(g)^{-1} \circ f \circ \pi_1(g).$$

Then we have

$$f^0 = \begin{cases} 0 & \text{if } \pi_1 \ncong \pi_2, \\ \frac{\mathrm{Tr}(f)}{\dim(V)} \cdot \mathrm{id}_V & \text{if } \pi_1 \cong \pi_2, \end{cases}$$

where, in the second case, we assume that $\pi_1 = \pi_2$ and $V_1 = V_2 =: V$.

Proof. We note that f^0 is a homomorphism as representations of G. Indeed, for any $h \in G$, we have

$$\begin{aligned} f^0 \circ \pi_1(h) &= \frac{1}{|G|} \sum_{g \in G} \pi_2(g)^{-1} \circ f \circ \pi_1(g) \circ \pi_1(h) \\ &= \frac{1}{|G|} \sum_{g \in G} \pi_2(g)^{-1} \circ f \circ \pi_1(gh) \\ &= \frac{1}{|G|} \sum_{g \in G} \pi_2(gh^{-1})^{-1} \circ f \circ \pi_1(g) \\ &= \frac{1}{|G|} \sum_{g \in G} \pi_2(h) \circ \pi_2(g)^{-1} \circ f \circ \pi_1(g) = \pi_2(h) \circ f^0, \end{aligned}$$

where, in the third equality, we replaced gh with g. Therefore, by Lemma 2.8 and Theorem 2.9, we get

$$f^0 = \begin{cases} 0 & \text{if } \pi_1 \ncong \pi_2, \\ \lambda \cdot \mathrm{id}_V & \text{if } \pi_1 \cong \pi_2, \end{cases}$$

with some scalar $\lambda \in \mathbb{C}$. Thus our remaining task is to determine the scalar λ when $\pi_1 \cong \pi_2$. We may assume $\pi_1 = \pi_2$ and write π for $\pi_1 = \pi_2$. Then

$$\mathrm{Tr}(f^0) = \frac{1}{|G|} \sum_{g \in G} \mathrm{Tr}(\pi(g)^{-1} \circ f \circ \pi(g)) = \frac{1}{|G|} \sum_{g \in G} \mathrm{Tr}(f) = \mathrm{Tr}(f).$$

[1] This finite-dimensionality condition is in fact unnecessary; see Exercise of Lecture 1.

As $f^0 = \lambda \cdot \mathrm{id}_V$, we also have $\mathrm{Tr}(f^0) = \lambda \cdot \dim(V)$. Hence we get $\lambda = \mathrm{Tr}(f)/\dim(V)$. □

Let us furthermore rewrite this corollary in terms of matrix representations. By fixing $\mathbb{C}$-bases of V_1 and V_2, we write $\pi_2(g^{-1})$ and $\pi_1(g)$ as

$$(\pi_2(g^{-1})_{ij})_{ij} \quad \text{and} \quad (\pi_1(g)_{pq})_{pq},$$

respectively.

Corollary 2.11. (1) *If $\pi_1 \ncong \pi_2$, then, for any i, j, p, q,*

$$\sum_{g \in G} \pi_2(g^{-1})_{ij} \cdot \pi_1(g)_{pq} = 0.$$

(2) *If $\pi_1 \cong \pi_2$ (assume $\pi_1 = \pi_2$), then, for any i, j, p, q,*

$$\sum_{g \in G} \pi_2(g^{-1})_{ij} \cdot \pi_1(g)_{pq} = \begin{cases} \frac{|G|}{\dim(V)} & \text{if } i = q \text{ and } j = p, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let us take any $f \in \mathrm{Hom}_{\mathbb{C}}(V_1, V_2)$ and write it as $f = (f_{kl})_{kl}$ using the fixed $\mathbb{C}$-bases of V_1 and V_2. Then each entry of the representation matrix $(f^0_{iq})_{iq}$ of f^0 is given by:

$$f^0_{iq} = \frac{1}{|G|} \sum_{g \in G} \sum_{j,p} \pi_2(g^{-1})_{ij} \cdot f_{jp} \cdot \pi_1(g)_{pq}.$$

We first consider the case where $\pi_1 \ncong \pi_2$. In this case, by the above corollary, we have $(f^0_{iq})_{iq} = 0$ for any $f = (f_{kl})_{kl}$. Hence, the above equality implies that

$$\frac{1}{|G|} \sum_{g \in G} \sum_{j,p} \pi_2(g^{-1})_{ij} \cdot f_{jp} \cdot \pi_1(g)_{pq} = 0$$

for each i, q (for any $f = (f_{kl})_{kl}$). Since f can vary arbitrarily, by looking at the coefficients of each f_{jp} on the both hand sides of this equality, we get

$$\sum_{g \in G} \pi_2(g^{-1})_{ij} \cdot \pi_1(g)_{pq} = 0$$

for each i, j, p, q.

We next consider the case where $\pi_1 = \pi_2$. In this case, for any $f = (f_{kl})_{kl}$, f^0 is represented by the scalar matrix with diagonal entries $\mathrm{Tr}(f)/\dim(V)$. In other words, we have

$$\frac{1}{|G|} \sum_{g \in G} \sum_{j,p} \pi_2(g^{-1})_{ij} \cdot f_{jp} \cdot \pi_1(g)_{pq} = \begin{cases} \frac{1}{\dim(V)}(f_{11} + \cdots + f_{nn}) & i = q, \\ 0 & i \neq q, \end{cases}$$

where $n = \dim(V)$. Therefore, we get

$$\sum_{g \in G} \pi_2(g^{-1})_{ij} \cdot \pi_1(g)_{pq} = \begin{cases} \frac{|G|}{\dim(V)} & i = q \quad \text{and} \quad j = p, \\ 0 & \text{otherwise.} \end{cases}$$

□

2.3. **Orthogonality relation.** In the following, we continue to assume that G is a finite group. Also, by a "representation of G", we always mean a finite-dimensional representation. Let $\mathbb{C}[G]$ denote the space of $\mathbb{C}$-valued functions on G. We define an inner product $(-,-)_G$ on $\mathbb{C}[G]$ by

$$(f_1, f_2)_G := \frac{1}{|G|} \sum_{g \in G} f_1(g) \cdot \overline{f_2(g)}$$

for any $f_1, f_2 \in \mathbb{C}[G]$. (It is obvious that this indeed defines an inner product; $(-,-)_G$ is $\mathbb{C}$-linear in f_1, anti-$\mathbb{C}$-linear in f_2, and satisfies $(f,f)_G > 0$ for any nonzero f.) As long as G is fixed and the choice of G is clear from the context, we often simply write $(-,-)$ instead of $(-,-)_G$.

Theorem 2.12. *Let (π_1, V_1) and (π_2, V_2) be irreducible representations of G. Then we have*

$$(\Theta_{\pi_1}, \Theta_{\pi_2}) = \begin{cases} 0 & \text{if } \pi_1 \not\cong \pi_2, \\ 1 & \text{if } \pi_1 \cong \pi_2. \end{cases}$$

Proof. By fixing $\mathbb{C}$-bases of V_1 and V_2, we write $\pi_2(g^{-1})$ and $\pi_1(g)$ as

$$(\pi_2(g^{-1})_{ij})_{ij} \quad \text{and} \quad (\pi_1(g)_{pq})_{pq},$$

respectively. Then we have

$$\begin{aligned}(\Theta_{\pi_1}, \Theta_{\pi_2}) &= \frac{1}{|G|} \sum_{g \in G} \Theta_{\pi_1}(g) \cdot \overline{\Theta_{\pi_2}(g)} = \frac{1}{|G|} \sum_{g \in G} \Theta_{\pi_1}(g) \cdot \Theta_{\pi_2}(g^{-1}) \\ &= \frac{1}{|G|} \sum_{g \in G} \Big(\sum_i \pi_1(g)_{ii} \Big) \cdot \Big(\sum_p \pi_2(g^{-1})_{pp} \Big)\end{aligned}$$

If $\pi_1 \not\cong \pi_2$, then Corollary 2.11 (1) immediately implies that this is equal to 0. Thus let us suppose that $\pi_1 = \pi_2$. Again by Corollary 2.11 (2), we get

$$\frac{1}{|G|} \sum_{g \in G} \Big(\sum_i \pi_1(g)_{ii} \Big) \cdot \Big(\sum_p \pi_2(g^{-1})_{pp} \Big) = \frac{1}{|G|} \sum_i \frac{|G|}{\dim(V)} = 1.$$

□

Corollary 2.13. *Let (π, V) be a representation of G and let $\pi = \pi_1 \oplus \cdots \oplus \pi_r$ be a decomposition into irreducible representations (recall that there always exists such a decomposition by Proposition 1.18). Then, for any irreducible representation π_0 of G, the number of π_i's which are isomorphic to π_0 is given by $(\Theta_\pi, \Theta_{\pi_0})$. In particular, a decomposition of any representation of G into irreducible representations is always unique up to permutation and isomorphism of irreducible factors.*

Proof. Since $\Theta_\pi = \Theta_{\pi_1} + \cdots + \Theta_{\pi_r}$, we have

$$(\Theta_\pi, \Theta_{\pi_0}) = (\Theta_{\pi_1} + \cdots + \Theta_{\pi_r}, \Theta_{\pi_0}) = \sum_{i=1}^{r} (\Theta_{\pi_i}, \Theta_{\pi_0}) = |\{1 \le i \le r \mid \pi_i \cong \pi_0\}|$$

by Theorem 2.12. □

Corollary 2.14. *Let (π, V) and (π', V') be representations of G. Then $\pi \cong \pi'$ if and only if $\Theta_\pi = \Theta_{\pi'}$.*

Proof. The only if direction is trivial, so let us prove the if direction. Let us take irreducible decompositions of π and π':

$$\pi \cong \pi_1 \oplus \cdots \oplus \pi_r \quad \text{and} \quad \pi' \cong \pi'_1 \oplus \cdots \oplus \pi'_s.$$

Then, for each irreducible representation π_0 of G, Corollary 2.13 implies that

$$|\{1 \le i \le r \mid \pi_i \cong \pi_0\}| = (\Theta_\pi, \Theta_{\pi_0}) = (\Theta_{\pi'}, \Theta_{\pi_0}) = |\{1 \le j \le s \mid \pi'_j \cong \pi_0\}|.$$

This shows that the above decompositions of π and π' coincide up to permutations of indices and up to isomorphism. □

At this point, we can clearly understand why the notion of the character of a representation is important in representation theory of finite groups. Any (finite-dimensional) representation of a finite group is uniquely determined by its character, so *characters tell all*.

Another important point is that the character of a representation is a complex-valued function on the group, which is a much more concrete object to work with. Representations have the freedom to "move": two isomorphic representations can be realized on completely different vector spaces. This freedom makes representation theory so flexible and useful, but, at the same time, also makes things more complicated. For example, when we want to compare two different representations of a group, we first must think about how to relate those two totally different (seemingly) spaces. On the other hand, by passing to their characters, the problem of comparing representations is reduced to the much more concrete problem of comparing complex-valued functions.

Corollary 2.15. *Let π be a representation of G. Then π is irreducible if and only if $(\Theta_\pi, \Theta_\pi) = 1$.*

Proof. We consider the irreducible decomposition

$$\pi \cong \bigoplus_{\pi_0} \pi_0^{\oplus m_{\pi_0}},$$

where the sum runs over isomorphism classes of irreducible representations of G. Then, by Corollary 2.13, we have

$$(\Theta_\pi, \Theta_\pi) = \sum_{\pi_0} (m_{\pi_0} \cdot \Theta_{\pi_0}, m_{\pi_0} \cdot \Theta_{\pi_0}) = \sum_{\pi_0} m_{\pi_0}^2.$$

Hence $(\Theta_\pi, \Theta_\pi) = 1$ if and only if $\sum_{\pi_0} m_{\pi_0}^2 = 1$, which is equivalent to that only one π_0 appears in the irreducible decomposition of π, whose multiplicity $m_{\pi_0} = 1$. □

Let us also present a practical consequence of our discussions so far:

Corollary 2.16. *For any two finite-dimensional representations (π_1, V_1) and (π_2, V_2) of G, we have*

$$\dim \mathrm{Hom}_G(\pi_1, \pi_2) = (\Theta_{\pi_1}, \Theta_{\pi_2}).$$

Proof. By taking irreducible decompositions of π_1 and π_2, the identity follows from Theorems 2.9 and 2.12. □

Example 2.17. Let us consider the permutation representation (π, V) of $G := \mathfrak{S}_3$ again (see Example 2.7). Recall that we have three conjugacy classes of G:

$$\{1\}, \quad \{(1\,2), (2\,3), (3\,1)\}, \quad \{(1\,2\,3), (3\,2\,1)\}.$$

The characters $\Theta_\pi(g)$ on these conjugacy classes are given by 3, 1, 0, respectively. Therefore, we get

$$(\Theta_\pi, \Theta_\pi) = \frac{1}{|\mathfrak{S}_3|} \sum_{g \in \mathfrak{S}_3} \Theta_\pi(g) \cdot \overline{\Theta_\pi(g)} = \frac{1}{6}(1 \cdot 3^2 + 3 \cdot 1^2 + 2 \cdot 0^2) = 2.$$

Hence, at least Corollary 2.15 implies that π is not irreducible. More precisely, if we let $\pi \cong \bigoplus_{\pi_0} \pi_0^{\oplus m_{\pi_0}}$ be the irreducible decomposition, then we have

$$\sum_{\pi_0} m_{\pi_0}^2 = 2.$$

Thus the only possibility is that we have exactly two irreducible representations appearing in the decomposition of π and that the multiplicities are both 1.

Furthermore, if we write $V = \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \mathbb{C}e_3$, then the subspace $\mathbb{C}(e_1 + e_2 + e_3)$ is a subrepresentation of V, on which G acts trivially. Therefore, we see that π decomposes into the sum of the trivial representation and a 2-dimensional irreducible representation, both of which have multiplicity 1. The 2-dimensional irreducible subrepresentation can be concretely given by $\{x_1e_1 + x_2e_2 + x_3e_3 \in V \mid x_1 + x_2 + x_3 = 0\}$.

2.4. **The case of abelian groups.**

Proposition 2.18. *Let G be a finite abelian group. Then any irreducible representation of G is 1-dimensional.*

Proof. Let (π, V) be an irreducible representation of G. If we fix $g \in G$, then $\pi(g) \in \mathrm{End}_{\mathbb{C}}(V)$ is in fact an endomorphism as G-representations. Indeed, for any $h \in G$, we have $\pi(g) \circ \pi(h) = \pi(gh) = \pi(hg) = \pi(h) \circ \pi(g)$. By Schur's lemma (Theorem 2.9), this implies that $\pi(g) = z_g \cdot \mathrm{id}_V$ for some $z_g \in \mathbb{C}^\times$. Namely, any $g \in G$ acts on V via a scalar multiplication. Therefore, so that V is irreducible, we necessarily have that V is 1-dimensional. □

2.5. **Exercises.** In the following exercises, "to describe the irreducible decomposition" means to write a given representation π as

$$\pi \cong \bigoplus_{\pi_0} \pi_0^{m_{\pi_0}},$$

where the sum is over isomorphism classes of irreducible representations, with explicitly determining each m_{π_0}. (You do not have to find an explicit description of each subspace π_0 in π.)

Exercise 2.1.
(1) Let (π_1, V_1) and (π_2, V_2) be finite-dimensional representations of a group G. Prove that $\Theta_{\pi_1 \otimes \pi_2} = \Theta_{\pi_1} \cdot \Theta_{\pi_2}$.
(2) Let (π, V) be the 2-dimensional irreducible representation of $\mathfrak{S}_3$ considered in Example 2.17. Describe the irreducible decomposition of $\pi^{\otimes 2} = \pi \otimes \pi$.

Exercise 2.2. Let X be the set of subsets of $\{1, 2, 3, 4\}$ consisting of two elements:

$$X = \{\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}\}.$$

Then the permutation group $\mathfrak{S}_4$ naturally acts on X. Accordingly, if we let $V := \bigoplus_{x \in X} \mathbb{C}e_x$ (the $\mathbb{C}$-vector space with basis labeled by elements of X), we get a representation (π, V) of $\mathfrak{S}_4$. Describe the irreducible decomposition of π.

Exercise 2.3. Consider the dihedral group of order 10:

$$D_{10} := \langle r, s \mid r^5 = s^2 = 1,\ srs = r^{-1} \rangle.$$

The group D_{10} naturally acts on the set of five vertices of the pentagon. By identifying these vertices with the five standard basis vectors of $\mathbb{C}^5$, we obtain a representation $(\pi, \mathbb{C}^5)$ of D_{10}. Describe the irreducible decomposition of π.

3. Lecture 3: Basic results on irreducible representations

(Main reference for today's lecture: [Ser77, Chapter 2])

3.1. **Square-sum formula for dimensions.** Let G be a finite group. Recall that $\mathbb{C}[G]$ denotes the $\mathbb{C}$-vector space of $\mathbb{C}$-valued functions on G. We let G act on $\mathbb{C}[G]$ by, for $f \in \mathbb{C}[G]$ and $g \in G$,

$$(R(g) \cdot f)(x) := f(xg).$$

It is not difficult to see that this is indeed a well-defined group action, hence defines a representation $(R, \mathbb{C}[G])$ of G.

Definition 3.1. We call $(R, \mathbb{C}[G])$ the *regular representation* (or *right regular representation*) of G.

The irreducible decomposition of $(R, \mathbb{C}[G])$ has a particularly interesting feature as we will examine from now on.

First, note that $\mathbb{C}[G]$ has the following $\mathbb{C}$-basis:

$$\{\mathbb{1}_g \mid g \in G\},$$

where $\mathbb{1}_g$ denotes the characteristic function of $\{g\} \subset G$, i.e.,

$$\mathbb{1}_g(x) = \begin{cases} 1 & \text{if } x = g, \\ 0 & \text{if } x \neq g. \end{cases}$$

(Hence, in particular, $\dim \mathbb{C}[G] = |G|$.)

Lemma 3.2. *The character Θ_R of the regular representation R of G is given by*

$$\Theta_R(g) = \begin{cases} |G| & \text{if } g = 1, \\ 0 & \text{if } g \neq 1. \end{cases}$$

Proof. Since R is $|G|$-dimensional, we have $\Theta_R(1) = \mathrm{Tr}(R(1)) = |G|$.

On the other hand, for any $g \in G$ and $x \in G$, we have

$$R(g)(\mathbb{1}_x) = \mathbb{1}_{xg^{-1}}.$$

Hence, whenever $g \neq 1$, the representation matrix of $R(g)$ with respect to the $\mathbb{C}$-basis $\{\mathbb{1}_g \mid g \in G\}$ is given by a permutation matrix whose diagonal entries are all zero. In particular, we have $\Theta_R(g) = \mathrm{Tr}(R(g)) = 0$. □

Proposition 3.3. *If (π, V) is an irreducible representation of G, then π is contained in the regular representation R with multiplicity $\dim \pi$. Especially, the irreducible decomposition of R is given by*

$$R \cong \bigoplus_{\pi} \pi^{\oplus \dim \pi},$$

where the sum is over all isomorphism classes of irreducible representations of G.

Proof. Recall that (Corollary 2.13) the multiplicity of π in R is given by the inner product of their characters (Θ_π, Θ_R). By Lemma 3.2,

$$(\Theta_\pi, \Theta_R) = \frac{1}{|G|} \sum_{g \in G} \Theta_\pi(g) \cdot \overline{\Theta_R(g)} = \frac{1}{|G|} \dim \pi \cdot |G| = \dim \pi.$$

□

Corollary 3.4. *There are only finitely many isomorphism classes of irreducible representations of G. More precisely, we have*

$$\sum_{\pi}(\dim \pi)^2 = |G|,$$

where the sum is over the isomorphism classes of irreducible representations of G.

Proof. By Proposition 3.3, we obtain

$$|G| = \dim R = \sum_{\pi} \dim(\pi^{\oplus \dim \pi}) = \sum_{\pi}(\dim \pi)^2.$$

□

3.2. **Number of irreducible representations.** Recall that we call an element $f \in \mathbb{C}[G]$ which is G-conjugation-invariant a *class function*. We write $C(G)$ for the $\mathbb{C}$-subspace of $\mathbb{C}[G]$ of class functions.

$$C(G) := \{f \in \mathbb{C}[G] \mid f \text{ is } G\text{-conjugation-invariant}\} \subset \mathbb{C}[G].$$

Also recall that the character of a representation of G is always a class function. Furthermore, the orthogonality relation (Theorem 2.12) shows that the characters of irreducible representations of G are orthonormal to each other with respect to the inner product on $C(G) \subset \mathbb{C}[G]$. Our aim here is to show the following:

Theorem 3.5. *The characters of irreducible representations of G form an orthonormal basis of $C(G)$. More precisely, if we let $\{\pi_1, \dots, \pi_r\}$ be a set of representatives of isomorphism classes of irreducible representations of G, then $\{\Theta_{\pi_1}, \dots, \Theta_{\pi_r}\}$ is a $\mathbb{C}$-basis of $C(G)$ satisfying*

$$(\Theta_{\pi_i}, \Theta_{\pi_j}) = \delta_{ij},$$

where δ_{ij} is the Kronecker delta.

Before we prove this theorem, we introduce some notation. Suppose that (π, V) is a representation of G. Then, for any class function $f \in C(G)$, we define an endomorphism $\pi(f) \in \mathrm{End}_{\mathbb{C}}(V)$ by

$$\pi(f) := \sum_{g \in G} f(g) \cdot \pi(g)$$

("averaging π with weight f").

Lemma 3.6. *With the above notation, we have $\pi(f) \in \mathrm{End}_G(V)$, i.e., $\pi(f)$ is an endomorphism of V as a representation of G.*

Proof. It suffices to show that, for any $h \in G$, we have $\pi(f) \circ \pi(h) = \pi(h) \circ \pi(f)$, or equivalently, $\pi(f) = \pi(h) \circ \pi(f) \circ \pi(h)^{-1}$. We have

$$\begin{aligned}
\pi(h) \circ \pi(f) \circ \pi(h)^{-1} &= \sum_{g \in G} f(g) \cdot \pi(h) \circ \pi(g) \circ \pi(h)^{-1} \\
&= \sum_{g \in G} f(g) \cdot \pi(hgh^{-1}) \\
&= \sum_{g \in G} f(h^{-1}gh) \cdot \pi(g) = \sum_{g \in G} f(g) \cdot \pi(g) = \pi(f),
\end{aligned}$$

where we replaced hgh^{-1} with g in the third equality and used the G-conjugation-invariance of f in the fourth equality. □

Proof of Theorem 3.5. We already saw that $\{\Theta_{\pi_1}, \dots, \Theta_{\pi_r}\}$ is an orthonormal subset. Thus our remaining task is to prove that $\{\Theta_{\pi_1}, \dots, \Theta_{\pi_r}\}$ is a $\mathbb{C}$-basis of $C(G)$. To show this, it is enough to check that any $f \in C(G)$ satisfying $(f, \Theta_{\pi_i}) = 0$ for any $1 \leq i \leq r$ is necessarily zero.

Let $f \in C(G)$ be such a function. For each irreducible representation (π_i, V_i), let us compute the endomorphism $\pi_i(f) \in \mathrm{End}_G(V_i)$. By Schur's lemma (Theorem 2.9), $\pi_i(f) = c_i \cdot \mathrm{id}_{V_i}$ with some constant $c_i \in \mathbb{C}$. By the definition of the operator $\pi_i(f)$, we have

$$\mathrm{Tr}(\pi_i(f)) = \sum_{g \in G} f(g) \cdot \mathrm{Tr}(\pi_i(g)) = |G| \cdot (f, \Theta_{\pi_i^\vee})$$

(recall that $\Theta_{\pi_i^\vee}(g) = \overline{\Theta_{\pi_i}(g)}$). On the other hand, we also have $\mathrm{Tr}(\pi_i(f)) = c_i \cdot \dim(V_i)$. This implies that

$$c_i = \frac{|G|}{\dim(V_i)} (f, \Theta_{\pi_i^\vee}).$$

Now we use the assumption on f, i.e., $(f, \Theta_{\pi_i}) = 0$ for all i's. By noting that $\pi_i^\vee$ is irreducible, we see that $c_i = 0$ for all i's. Hence, $\pi_i(f) = 0$ for all i's. Since the regular representation $(R, \mathbb{C}[G])$ of G has the decomposition (Proposition 3.3)

$$R \cong \bigoplus_{\pi_i} \pi_i^{\oplus \dim \pi_i},$$

we also know that $R(f) = 0$. However, by the definition of $R(f)$, we have

$$R(f)(\mathbb{1}_1) = \sum_{g \in G} f(g) \cdot R(g)(\mathbb{1}_1) = \sum_{g \in G} f(g) \cdot (\mathbb{1}_{g^{-1}}).$$

Especially, the equality $R(f)(\mathbb{1}_1) = 0$ implies that $f(g) = 0$ at all $g \in G$. □

Corollary 3.7. *The number of isomorphism classes of irreducible representations of G is equal to the number of conjugacy classes of G.*

Proof. By Theorem 3.5, the number of isomorphism classes of irreducible representations of G is given by $\dim C(G)$. If we let $\{C_1, \dots, C_r\}$ be the set of all conjugacy classes of G, then their characteristic functions

$$\{\mathbb{1}_{C_1}, \dots, \mathbb{1}_{C_r}\}$$

form a $\mathbb{C}$-basis of $C(G)$. In particular, $\dim C(G)$ is equal to the number of conjugacy classes of G. □

Recall that any irreducible representation of a finite abelian group is 1-dimensional (Proposition 2.18). Using the facts we proved so far, we can upgrade Proposition 2.18 as follows:

Proposition 3.8. *Let G be a finite group. Then G is abelian if and only if any irreducible representation of G is 1-dimensional.*

Proof. Let G be a finite group of order n.

The "only if" part was already proved in Proposition 2.18, but here let us also see how it follows from our discussion in this subsection. Suppose that G is abelian. Then, for any $g \in G$, the conjugacy class containing g is the singleton $\{g\}$:

$$\{xgx^{-1} \in G \mid x \in G\} = \{g\}.$$

Hence we have n conjugacy classes of G. If we let $\pi_1, \dots, \pi_n$ be the irreducible representations, then Corollaries 3.4 and 3.7 imply that

$$n = (\dim \pi_1)^2 + \cdots + (\dim \pi_n)^2.$$

Hence we must have $\dim \pi_1 = \cdots = \dim \pi_n = 1$.

Let us show the "if" part. Let r be the number of conjugacy classes in G. To show that G is abelian, it suffices to check that $r = n$. Let $\pi_1, \dots, \pi_r$ be the set of irreducible representations of G, which must be all 1-dimensional by assumption. Then Corollary 3.4 implies that

$$n = (\dim \pi_1)^2 + \cdots + (\dim \pi_r)^2 = r.$$

This completes the proof. □

Example 3.9. Let us classify the irreducible representations of $\mathfrak{S}_3$. Since $\mathfrak{S}_3$ has three conjugacy classes, we must have three irreducible representations up to isomorphism by Corollary 3.7. Let π_1, π_2, π_3 be three (isomorphism classes of) irreducible representations. By Corollary 3.4, we have

$$|\mathfrak{S}_3| = 6 = (\dim \pi_1)^2 + (\dim \pi_2)^2 + (\dim \pi_3)^2.$$

The only possible triple of positive integers whose square sum equals 6 is $(1, 1, 2)$ up to permutation; thus there are two 1-dimensional irreducible representations and one 2-dimensional representation. These can be explicitly realized as follows:

- The trivial representation is obviously a 1-dimensional irreducible representation of $\mathfrak{S}_3$.
- The sign character $\mathrm{sgn}\colon \mathfrak{S}_3 \to \{\pm 1\} \subset \mathbb{C}^\times$ is also a 1-dimensional irreducible representation of $\mathfrak{S}_3$, which is obviously not equal to the trivial representation. Here, recall that sgn is given by

$$\mathrm{sgn}(1) = 1,$$
$$\mathrm{sgn}((12)) = \mathrm{sgn}((23)) = \mathrm{sgn}((31)) = -1,$$
$$\mathrm{sgn}((123)) = \mathrm{sgn}((132)) = 1.$$

- Recall from Example 2.17 that the 3-dimensional permutation representation (π, V) of $\mathfrak{S}_3$ decomposes into the sum of two irreducible representations, one of which is the trivial representation (write $\mathbb{1}$). Hence, the quotient representation $\pi/\mathbb{1}$ gives a 2-dimensional irreducible representation of $\mathfrak{S}_3$.

3.3. **Exercises.**

Exercise 3.1. Let G be the alternating group $\mathfrak{A}_5$ on five letters, i.e.,

$$\mathfrak{A}_5 := \mathrm{Ker}(\mathrm{sgn}\colon \mathfrak{S}_5 \to \{\pm 1\}).$$

Determine the dimensions of all irreducible representations of G.

Exercise 3.2. Let $G := \mathfrak{S}_4$. We consider the following 3-dimensional subrepresentation of the permutation representation of G:

$$W := \{(x_1, x_2, x_3, x_4) \in \mathbb{C}^{\oplus 4} \mid x_1 + x_2 + x_3 + x_4 = 0\}.$$

Let $\mathrm{sgn}\colon \mathfrak{S}_4 \to \{\pm 1\}$ denote the sign character of G.

(1) Prove that W and $W \otimes \mathrm{sgn}$ are irreducible and are not isomorphic to each other.
(2) Determine the dimensions of all irreducible representations of G.

Exercise 3.3. Let G be the quaternion group Q_8, i.e.,

$$G := Q_8 := \{\pm 1, \pm i, \pm j, \pm k\}, \qquad i^2 = j^2 = k^2 = ijk = -1.$$

(1) Find all 1-dimensional representations of G.
(2) Determine the dimensions of all irreducible representations of G.

4. Lecture 4: Induced representations

(Main reference for today's lecture: [Ser77, Chapter 7])

4.1. **Induced representations.** Let G be a group and H its subgroup. If a representation (π, V) of G is given, then we may obtain a representation of H by restricting π to H. The resulting representation is called the *restriction* of (π, V), and denoted by $(\pi|_H, V)$ or $(\mathrm{Res}^G_H \pi, V)$.

What we want to investigate today is the "converse" of this procedure. That is, let us suppose that a representation (ρ, W) of H is given first. Then, is there any natural way to get a representation of G using it? The answer is yes; it is called the *induction* explained below.

We consider the space $\mathrm{Ind}^G_H W$ of W-valued functions satisfying the "left-(H, ρ)-invariance condition" as follows:

$$\mathrm{Ind}^G_H W := \{f \colon G \to W \mid f(hx) = \rho(h)(f(x)) \text{ for any } x \in G, h \in H\}.$$

We define an action $\mathrm{Ind}^G_H \rho$ of G on $\mathrm{Ind}^G_H W$ by the right-translation of W-valued functions, i.e., for any $g \in G$ and $f \in \mathrm{Ind}^G_H W$, we define $g \cdot f \in \mathrm{Ind}^G_H W$ by

$$(g \cdot f)(x) := f(xg).$$

(To be more precise, $g \cdot f$ should be written as $(\mathrm{Ind}^G_H \rho)(g)(f)$, but we avoid it since it is too heavy.) Then we obtain a representation $(\mathrm{Ind}^G_H \rho, \mathrm{Ind}^G_H W)$ of G. We call $(\mathrm{Ind}^G_H \rho, \mathrm{Ind}^G_H W)$ the *induction* of (ρ, W) from H to G.

Example 4.1. Let G be a finite group and $H := \{1\}$. We take a representation (ρ, W) of H to be the trivial representation $(\mathbb{1}, \mathbb{C})$. In this case, the space $\mathrm{Ind}^G_H W$ simply equals the $\mathbb{C}$-vector space of all $\mathbb{C}$-valued functions on G, i.e., $\mathbb{C}[G]$. Therefore, the induction $(\mathrm{Ind}^G_{\{1\}} \mathbb{1}, \mathrm{Ind}^G_{\{1\}} \mathbb{C})$ is nothing but the right regular representation of G on $\mathbb{C}[G]$.

Lemma 4.2. *We have* $\dim(\mathrm{Ind}^G_H W) = [G : H] \cdot \dim W$.

Proof. We put $n := \dim W$ and choose a $\mathbb{C}$-basis $\{v_1, \ldots, v_n\}$ of W. We fix a set of representatives $\{g_1, \ldots, g_r\}$ of $H \backslash G$. For each g_i and v_i, we define a W-valued function $f_{i,j} \colon G \to W$ by

$$f_{i,j}(g) := \begin{cases} \rho(h) v_i & \text{if } g = h g_i \text{ for some } h \in H, \\ 0 & \text{otherwise.} \end{cases}$$

Then it is easy to see that this is an element of $\mathrm{Ind}^G_H W$ and also that $\{f_{i,j}\}_{i,j}$ forms a $\mathbb{C}$-basis of $\mathrm{Ind}^G_H W$. In particular, $\dim(\mathrm{Ind}^G_H W) = [G : H] \cdot \dim W$. □

4.2. **Frobenius character formula.**

Theorem 4.3 (Frobenius character formula)**.** *Let G be a finite group and H a subgroup. Let (ρ, W) be a finite-dimensional representation of H. We put $(\pi, V) := (\mathrm{Ind}^G_H \rho, \mathrm{Ind}^G_H W)$. Then, for any $g \in G$,*

$$\Theta_\pi(g) = \sum_{\substack{x \in H \backslash G \\ x g x^{-1} \in H}} \Theta_\rho(x g x^{-1}).$$

Proof. We use the same notation as in the proof of Lemma 4.2. To be more precise, we fix a set of representatives $\{g_1, \dots, g_r\}$ of $H\backslash G$ and a $\mathbb{C}$-basis $\{v_1, \dots, v_n\}$ of W to define a $\mathbb{C}$-basis $\{f_{i,j}\}_{i,j}$ of V. For each $i = 1, \dots, r$, we put

$$V_i := \mathrm{Span}_{\mathbb{C}}\{f_{i,j}\}_j$$

so that

$$V = \bigoplus_{i=1}^{r} V_i.$$

Note that an element $f \in V$ belongs to V_i if and only if $\mathrm{supp}(f) \subset Hg_i$.

Let $g \in G$. Then, for any $f \in V$, we have $\mathrm{supp}(\pi(g)f) = \mathrm{supp}(f)g^{-1}$. Thus, for each i, if we let i' be the unique index such that $Hg_ig^{-1} = Hg_{i'}$, then $\pi(g)$ maps V_i to $V_{i'}$. Hence,

$$\Theta_\pi(g) = \mathrm{Tr}(\pi(g) \mid V) = \sum_{\substack{i=1,\dots,r \\ Hg_ig^{-1}=Hg_i}} \mathrm{Tr}(\pi(g) \mid V_i).$$

Let us compute each summand by fixing i such that $Hg_ig^{-1} = Hg_i$, or equivalently, $g \in g_i^{-1}Hg_i$. For each $j = 1, \dots, n$, we write

$$\pi(g)f_{i,j} = \sum_{l=1}^{n} a_l^{(j)}(g) f_{i,l}$$

with coefficients $a_l^{(j)}(g) \in \mathbb{C}$. With this notation, $\mathrm{Tr}(\pi(g) \mid V_i)$ is given by

$$\mathrm{Tr}(\pi(g) \mid V_i) = \sum_{j=1}^{n} a_j^{(j)}(g).$$

On the other hand, if we write $g = g_i^{-1}hg_i$ with an element $h \in H$, then, by evaluating the identity $\pi(g)f_{i,j} = \sum_{l=1}^{n} a_l^{(j)}(g) f_{i,l}$ at g_i, we get

$$f_{i,j}(hg_i) = \sum_{l=1}^{n} a_l^{(j)}(g) f_{i,l}(g_i),$$

which implies

$$\rho(h)v_j = \sum_{l=1}^{n} a_l^{(j)}(g) v_l.$$

This identity shows that we also have

$$\mathrm{Tr}(\rho(h) \mid W) = \sum_{j=1}^{n} a_j^{(j)}(g).$$

Therefore, in summary, we have obtained

$$\begin{aligned} \Theta_\pi(g) &= \sum_{\substack{i=1,\dots,r \\ Hg_ig^{-1}=Hg_i}} \mathrm{Tr}(\pi(g) \mid V_i) \\ &= \sum_{\substack{x \in H\backslash G \\ xgx^{-1} \in H}} \mathrm{Tr}(\rho(xgx^{-1}) \mid W) = \sum_{\substack{x \in H\backslash G \\ xgx^{-1} \in H}} \Theta_\rho(xgx^{-1}). \end{aligned}$$

This completes the proof. □

Corollary 4.4. *Let G be a finite group and H a normal subgroup. Let (ρ, W) be a finite-dimensional representation of H. We put $(\pi, V) := (\mathrm{Ind}_H^G \rho, \mathrm{Ind}_H^G W)$. Then, for any $g \in G$,*

$$\Theta_\pi(g) = \begin{cases} 0 & \text{if } g \notin H, \\ \sum_{x \in H\backslash G} \Theta_\rho(xgx^{-1}) & \text{if } g \in H. \end{cases}$$

4.3. **Frobenius reciprocity.**

Theorem 4.5. *Let G be a finite group and H its subgroup. Let (π, V) be a finite-dimensional representation of G. Let (ρ, W) be a finite-dimensional representation of H. Then we have*

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G \rho) \cong \mathrm{Hom}_H(\mathrm{Res}_H^G \pi, \rho).$$

Proof. We construct $\mathbb{C}$-linear maps between two homomorphism sets which are inverses to each other.

We first construct a $\mathbb{C}$-linear map

$$F\colon \mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G \rho) \to \mathrm{Hom}_H(\mathrm{Res}_H^G \pi, \rho).$$

For any $\phi \in \mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G \rho)$, we define a $\mathbb{C}$-linear homomorphism $F(\phi)\colon V \to W$ by

$$F(\phi)(v) := \phi(v)(1)$$

for $v \in V$ (note that $\phi(v)$ is an element of $\mathrm{Ind}_H^G W$, hence a W-valued function on G). Then $F(\phi)$ is also an H-homomorphism. Indeed, using that ϕ is a G-homomorphism, we have

$$F(\phi)(\pi(h)v) = \phi(\pi(h)v)(1) = \big((\mathrm{Ind}_H^G \rho)(h)(\phi(v))\big)(1) = \phi(v)(h).$$

Since $\phi(v)$ is an element of $\mathrm{Ind}_H^G W$, we get

$$\phi(v)(h) = \rho(h)\phi(v)(1) = \rho(h)F(\phi)(v).$$

We next construct a $\mathbb{C}$-linear map

$$F'\colon \mathrm{Hom}_H(\mathrm{Res}_H^G \pi, \rho) \to \mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G \rho).$$

For any $\psi \in \mathrm{Hom}_H(\mathrm{Res}_H^G \pi, \rho)$, we define a $\mathbb{C}$-linear homomorphism $F'(\psi)\colon V \to \mathrm{Ind}_H^G W$ by

$$F'(\psi)(v)(x) := \psi(\pi(x)v)$$

for $v \in V$. Then $F'(\psi)$ is a G-homomorphism. Indeed, we have

$$F'(\psi)(\pi(g)v)(x) = \psi(\pi(x)\pi(g)v) = \psi(\pi(xg)v) = F'(\psi)(v)(xg) = \big((\mathrm{Ind}_H^G \rho)(g)F'(\psi)(v)\big)(x)$$

(recall that the action $\mathrm{Ind}_H^G \rho$ is defined to be the right-translation for W-valued functions).

It is routine to check that these maps are inverses to each other, so let me omit it. □

Remark 4.6. (1) By Corollary 2.16, we may also interpret Theorem 4.5 as the identity

$$(\Theta_\pi, \Theta_{\mathrm{Ind}_H^G \rho})_G = (\Theta_\pi|_H, \Theta_\rho)_H.$$

(2) In the more sophisticated language of category theory, we may understand that Theorem 4.5 says that the functor

$$\mathrm{Res}_H^G \colon \mathrm{Rep}(G) \to \mathrm{Rep}(H)$$

between the categories of finite-dimensional representations of G and H is the "left adjoint" to the functor

$$\mathrm{Ind}_H^G \colon \mathrm{Rep}(H) \to \mathrm{Rep}(G).$$

4.4. **Exercises.**

Exercise 4.1. Let G be a finite group and H a normal subgroup of index 2. Let $\chi \colon H \to \mathbb{C}^\times$ be a character of H (1-dimensional representation). Determine the irreducible decomposition of $\mathrm{Ind}_H^G \chi$.

Exercise 4.2. Let $n \in \mathbb{Z}_{>1}$. Prove that $\mathrm{Ind}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n} \mathbb{1}$ decomposes into the direct sum of the trivial representation and an $(n-1)$-dimensional irreducible representation.

5. Lecture 5: Mackey decomposition

(Main reference for today's lecture: [Ser77, Chapter 7])

5.1. **Restriction problem.** Let G be a finite group and H its subgroup. Recall that, for a representation (π, V) of G, we may consider its restriction $(\mathrm{Res}^G_H \pi, V)$. In fact, it is generally not so easy to understand how representations of G behave under the restriction to H.

For example, even if π is irreducible as a representation of G, its restriction to H may not be irreducible in general. So it is natural to ask when $\mathrm{Res}^G_H \pi$ remains irreducible. If $\mathrm{Res}^G_H \pi$ is reducible, then the next question is how to describe the irreducible decomposition of $\mathrm{Res}^G_H \pi$. Which irreducible representations of H do appear in $\mathrm{Res}^G_H \pi$? What are their multiplicities?

All these problems do heavily depend on the setting, i.e., which groups G and H we consider, and also which kind of representations of G we look at.

In this lecture, we study this *restriction problem* in the case of induced representations.

5.2. **Mackey decomposition.** Let G be a finite group. For two subgroups H and K of G, we consider the following problem:

- Let (ρ, W) be a representation of H.
- Let $(\mathrm{Ind}^G_H \rho, \mathrm{Ind}^G_H W)$ be its induction to G.
- If we restrict it to K, then how can $\mathrm{Res}^G_K \mathrm{Ind}^G_H \rho$ be described?

The *Mackey decomposition* theorem provides an answer to this problem. To explain it, we introduce some notation.

For $g \in G$, we write $H^g := g^{-1}Hg$. For a representation (ρ, W) of H, we write (ρ^g, W^g) for the representation of H^g on $W^g := W$ defined by

$$\rho^g(g^{-1}hg) := \rho(h)$$

for $h \in H$.

We first fix a complete set of representatives S of the double quotient $H\backslash G/K$. For $s \in S$, we consider the representation (ρ^s, W^s) of H^s defined as above. Moreover, by putting $H_s := H^s \cap K$, we define a representation (ρ_s, W_s) by

$$\rho_s := \mathrm{Res}^{H^s}_{H_s} \rho^s.$$

Theorem 5.1 (Mackey decomposition)**.** *We have a decomposition as representations of K:*

$$\mathrm{Res}^G_K(\mathrm{Ind}^G_H \rho) \cong \bigoplus_{s \in S} \mathrm{Ind}^K_{H_s} \rho_s.$$

Proof. Let us write $(\pi, V) := (\mathrm{Ind}^G_H \rho, \mathrm{Ind}^G_H W)$. Recall that, by definition,

$$V = \mathrm{Ind}^G_H W := \{f \colon G \to W \mid f(hx) = \rho(h)(f(x)) \text{ for any } x \in G, h \in H\}.$$

For each $s \in S$, we define a subspace $V(s)$ of V to be the set of functions $f \colon G \to W$ such that

- $\mathrm{supp}(f) \subset HsK$, and
- $f(hx) = \rho(h)(f(x))$ for any $x \in G, h \in H$.

Then we obviously have a decomposition as a $\mathbb{C}$-vector space

$$V = \bigoplus_{s \in S} V(s).$$

Moreover, since the G-action π on V is defined by the right-translation for functions $G \to W$, we can also see that $V(s)$ is stable under the action of K. In other words, as a representation of K, we have

$$\mathrm{Res}^G_K(\mathrm{Ind}^G_H W) \cong \bigoplus_{s \in S} V(s).$$

Hence it is enough to see that each $V(s)$ is in fact isomorphic to $\mathrm{Ind}^K_{H_s} \rho_s$. We define a homomorphism

$$V(s) \to \mathrm{Ind}^K_{H_s} W_s \colon f \mapsto f_K$$

as representations of K by

$$f_K(k) := f(sk)$$

for $k \in K$. This is well-defined. Indeed, obviously the map $f \mapsto f_K$ preserves the K-actions (right-translation). Moreover, for any $s^{-1}hs = k' \in H_s = s^{-1}Hs \cap K$, we have

$$f_K(k' \cdot k) = f(sk'k) = f(hsk) = \rho(h)f(sk) = \rho^s(s^{-1}hs)f_K(k).$$

In other words, the function $f_K \colon K \to W_s$ belongs to $\mathrm{Ind}^K_{H_s} W_s$.

Conversely, we define a map of opposite direction

$$\mathrm{Ind}^K_{H_s} W_s \to V(s) \colon f_K \mapsto f$$

by the conditions $\mathrm{supp}(f) \subset HsK$ and

$$f(hsk) := \rho(h) f_K(k)$$

for $h \in H$ and $k \in K$. It is straightforward to check that this is well-defined and gives the inverse to the above map. Hence we conclude that $V(s)$ is isomorphic to $\mathrm{Ind}^K_{H_s} W_s$ as representations of K. □

5.3. **Mackey's irreducibility criterion.** Let us consider the special case where $K = H$.

Theorem 5.2. *Let G be a finite group and H a subgroup of G. Let (ρ, W) be an irreducible representation of H. Then its induction $\pi := \mathrm{Ind}^G_H \rho$ is irreducible if and only if the representations ρ_s and $\mathrm{Res}^H_{H_s} \rho$ of H_s have no common irreducible constituent for any $s \in G \smallsetminus H$.*

Proof. Recall that π is irreducible if and only if the self-inner product of its character $(\Theta_\pi, \Theta_\pi)_G$ is equal to 1. By the Frobenius reciprocity, we have

$$(\Theta_\pi, \Theta_\pi)_G = (\Theta_{\mathrm{Ind}^G_H \rho}, \Theta_{\mathrm{Ind}^G_H \rho})_G = (\Theta_{\mathrm{Res}^G_H \mathrm{Ind}^G_H \rho}, \Theta_\rho)_H.$$

On the other hand, by the Mackey decomposition, we have

$$\mathrm{Res}^G_H(\mathrm{Ind}^G_H \rho) \cong \bigoplus_{s \in H \backslash G / H} \mathrm{Ind}^H_{H_s} \rho_s.$$

In particular, this implies that, as a function on H, we have

$$\Theta_{\mathrm{Res}^G_H \mathrm{Ind}^G_H \rho} = \sum_{s \in H \backslash G / H} \Theta_{\mathrm{Ind}^H_{H_s} \rho_s}.$$

Hence we get

$$(\Theta_{\mathrm{Res}^G_H \mathrm{Ind}^G_H \rho}, \Theta_\rho)_H = \sum_{s \in H\backslash G/H} (\Theta_{\mathrm{Ind}^H_{H_s} \rho_s}, \Theta_\rho)_H.$$

Again using the Frobenius reciprocity, we furthermore get

$$\sum_{s \in H\backslash G/H} (\Theta_{\mathrm{Ind}^H_{H_s} \rho_s}, \Theta_\rho)_H = \sum_{s \in H\backslash G/H} (\Theta_{\rho_s}, \Theta_\rho)_{H_s}.$$

When s represents the trivial coset H, we may choose s to be 1. In this case, we simply have $H_s = H^s \cap H = H$ and $\rho_s = \rho$. Thus we get $(\Theta_{\rho_s}, \Theta_\rho)_{H_s} = 1$. Therefore, in summary, $(\Theta_\pi, \Theta_\pi)_G = 1$ if and only if we have $(\Theta_{\rho_s}, \Theta_\rho)_{H_s} = 0$ for any nontrivial coset representatives $s \in S$. The condition that $(\Theta_{\rho_s}, \Theta_\rho)_{H_s} = 0$ is equivalent to the condition that ρ_s and $\mathrm{Res}^H_{H_s} \rho$ do not have the same irreducible constituent. □

Let us think about some particular cases which would be convenient.

Corollary 5.3. *Let G be a finite group and H a normal subgroup of G. Let (ρ, W) be an irreducible representation of H. Then its induction $\pi := \mathrm{Ind}^G_H \rho$ is irreducible if and only if the representations ρ_s and ρ of H_s are not isomorphic for any $s \in G \smallsetminus H$.*

Proof. Since H is normal in G, we have

$$H_s := s^{-1}Hs \cap H = H$$

for any $s \in G$. Moreover, since ρ is irreducible, the representations ρ_s and ρ of H have no common irreducible constituent if and only if they are not isomorphic. Hence the corollary is just a special case of the Mackey criterion. □

Corollary 5.4. *Let G be a finite group and H a subgroup of G. Let $(\rho, \mathbb{C})$ be a 1-dimensional irreducible representation of H. Then its induction $\pi := \mathrm{Ind}^G_H \rho$ is irreducible if and only if the representations ρ_s and $\mathrm{Res}^H_{H_s} \rho$ of H_s are not isomorphic for any $s \in G \smallsetminus H$.*

Proof. The representations ρ_s and $\mathrm{Res}^H_{H_s} \rho$ of H_s are both 1-dimensional. Therefore, they have no common irreducible constituent if and only if they are not isomorphic. Hence the corollary is just a special case of the Mackey criterion. □

5.4. **Example: Heisenberg groups.** As an application of the results we have learned so far, let us study representation theory of finite Heisenberg groups.

Before we introduce a Heisenberg group, let us define the notion of central character:

Lemma 5.5. *Let G be a finite group with center Z. Then, for any irreducible representation (π, V) of G, there exists a character $\chi\colon Z \to \mathbb{C}^\times$ such that*

$$\pi(z)v = \chi(z)v$$

for any $z \in Z$ and $v \in V$ (or, equivalently, $\mathrm{Res}^G_Z \pi \cong \chi^{\oplus \dim V}$).

Proof. Let $z \in Z$. Then we have $\pi(g)\pi(z) = \pi(gz) = \pi(zg) = \pi(z)\pi(g)$ for any $g \in G$. This means that $\pi(z) \in \mathrm{End}_{\mathbb{C}}(V)$ is in fact a G-homomorphism, i.e., $\pi(z) \in \mathrm{End}_G(V)$. By Schur's lemma, we know that $\mathrm{End}_G(V) = \mathbb{C} \cdot \mathrm{id}_V$. Hence there exists an element $\chi(z) \in \mathbb{C}^\times$ such that $\pi(z) = \chi(z) \cdot \mathrm{id}_V$. It can be easily checked that the association $z \mapsto \chi(z)$ is a homomorphism from Z to $\mathbb{C}^\times$. □

Definition 5.6. For any irreducible representation (π, V) of a finite group G, the character of the center of G associated to (π, V) as in the above lemma is called the *central character* of (π, V).

Let $\mathbb{F}_q$ be a finite field of characteristic $p > 0$. We assume that p is odd.

We consider a finite-dimensional nonzero $\mathbb{F}_q$-vector space $\mathbb{V}$ equipped with a symplectic form $\omega\colon \mathbb{V} \times \mathbb{V} \to \mathbb{F}_q$, that is, ω is a non-degenerate alternating bilinear form on $\mathbb{V}$. Note that, $\dim \mathbb{V}$ must be even (say $2n > 0$) so that such ω exists.

We consider the product set $\mathbb{H} := \mathbb{V} \times \mathbb{F}_q$. For any $(v_1, z_1), (v_2, z_2) \in \mathbb{H}$, we put

$$(v_1, z_1) \cdot (v_2, z_2) := \left(v_1 + v_2, z_1 + z_2 + \frac{1}{2}\omega(v_1, v_2)\right) \in \mathbb{H}.$$

Then we can easily check that $\mathbb{H}$ forms a group with respect to this operation. We call this group $\mathbb{H}$ the *Heisenberg group* associated to the symplectic space $(\mathbb{V}, \omega)$. The following can be also easily checked:

(1) The order of $\mathbb{H}$ is q^{2n+1}.
(2) The center $Z(\mathbb{H})$ of $\mathbb{H}$ is $\{0\} \times \mathbb{F}_q$.
(3) The commutator subgroup $[\mathbb{H}, \mathbb{H}]$ is equal to $Z(\mathbb{H})$.
(4) The quotient $\mathbb{H}/Z(\mathbb{H})$ is isomorphic to $\mathbb{V}$.

Remark 5.7. It is well-known that any symplectic forms on the same $\mathbb{F}_q$-vector space are equivalent to each other. Hence, by choosing a suitable $\mathbb{F}_q$-basis of $\mathbb{V}$, you may assume that the representation matrix of ω is of a form you prefer: for example,

$$\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \quad \begin{pmatrix} & & & 1 \\ & & -1 & \\ & \iddots & & \\ -1 & & & \end{pmatrix}, \quad \text{and so on.}$$

Let us try to classify all irreducible representations of $\mathbb{H}$.

We first consider characters (1-dimensional representations) of $\mathbb{H}$. Any 1-dimensional representation can be thought of as a homomorphism $\mathbb{H} \to \mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^\times$, hence must factor through the maximal abelian quotient of $\mathbb{H}$, i.e., $\mathbb{H}/[\mathbb{H}, \mathbb{H}]$. Since $\mathbb{H}/Z(\mathbb{H}) \cong \mathbb{V}$, such a homomorphism is identified with a character of $\mathbb{V}$. Conversely, any character of $\mathbb{V}$ can be regarded as a character of $\mathbb{H}$. Since $\mathbb{V}$ is of order q^{2n}, we conclude that there are q^{2n} characters of $\mathbb{H}$ in total.

Next we consider higher-dimensional irreducible representations.

Theorem 5.8 (Stone–von Neumann's theorem)**.** *For any nontrivial character $\psi\colon \mathbb{F}_q \to \mathbb{C}^\times$, there exists a unique (up to isomorphism) irreducible representation π_ψ of $\mathbb{H}$ with central character ψ. Moreover, its dimension is given by q^n.*

Proof. We fix a polarization $\mathbb{V} = \mathbb{V}_1 \oplus \mathbb{V}_2$. Here, a "polarization" means that both $\mathbb{V}_1$ and $\mathbb{V}_2$ are Lagrangian subspaces (maximal totally isotropic subspaces) of the symplectic space $\mathbb{V}$ in the sense that $\mathbb{V}_i$ is a maximal subspace satisfying $\omega|_{\mathbb{V}_i \times \mathbb{V}_i} = 0$ for $i = 1, 2$. For example, if we choose an $\mathbb{F}_q$-basis $v_1, \ldots, v_{2n}$ of $\mathbb{V}$ so that the representation matrix of ω is $\left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right)$, then we may choose $\mathbb{V}_1$ and $\mathbb{V}_2$ to be the subspaces spanned by $\{v_1, \ldots, v_n\}$ and $\{v_{n+1}, \ldots, v_{2n}\}$, respectively.

We put $\mathbb{H}' := \mathbb{V}_1 \times \mathbb{F}_q \subset \mathbb{H}$. Let us show that this is a normal subgroup of $\mathbb{H}$. Since $\mathbb{H}$ is generated by $\mathbb{V}_1 \times \{0\}$, $\mathbb{V}_2 \times \{0\}$, and $Z(\mathbb{H}) = \{0\} \times \mathbb{F}_q$, it is enough to check that $\mathbb{H}'$ is closed under the conjugation by these subgroups. It is obvious

that $\mathbb{H}'$ is closed under $Z(\mathbb{H})$-conjugation, so let us consider $\mathbb{V}_i \times \{0\}$-conjugation. For any $(v,0) \in \mathbb{V}_i \times \{0\}$, its inverse is given by $(-v,0)$ since $\mathbb{V}_i$ is a Lagrangian subspace.[2] Hence, for any $(v',z) \in \mathbb{H}' = \mathbb{V}_1 \times \mathbb{F}_q$, we have

$$\begin{aligned}(v,0)\cdot(v',z)\cdot(v,0)^{-1} &= (v,0)\cdot(v',z)\cdot(-v,0)\\ &= (v,0)\cdot\Big(v'-v, z+\frac{1}{2}\omega(v',-v)\Big)\\ &= \Big(v', z+\frac{1}{2}\omega(v',-v)+\frac{1}{2}\omega(v,v'-v)\Big)\\ &= \Big(v', z+\omega(v,v')\Big) \in \mathbb{H}'.\end{aligned}$$

We note that the subgroup $\mathbb{H}'$ is abelian since $\mathbb{V}_1$ is a Lagrangian subspace. In particular, for any nontrivial character $\psi\colon \mathbb{F}_q \to \mathbb{C}^\times$, we may find its extension $\tilde{\psi}\colon \mathbb{H}' \to \mathbb{C}^\times$ (this is a consequence of the classification theorem of finite abelian groups). We fix any such $\tilde{\psi}$ and put

$$\pi_{\tilde{\psi}} := \mathrm{Ind}_{\mathbb{H}'}^{\mathbb{H}} \tilde{\psi}.$$

We claim that $\pi_{\tilde{\psi}}$ is an irreducible representation of $\mathbb{H}$ with central character ψ. It is obvious that the central character is ψ. To show the irreducibility, by Corollary 5.3, it is enough to verify that $\tilde{\psi}^s \neq \tilde{\psi}$ for any $s \in \mathbb{H} \smallsetminus \mathbb{H}'$. Let us write $s = (v_1+v_2, z)$, where $v_i \in \mathbb{V}_i$, $z \in \mathbb{F}_q$ such that $v_2 \neq 0$. As we have $\tilde{\psi}^s = \tilde{\psi}^{(v_2,0)}$, it suffices to check that $\tilde{\psi}^{(v_2,0)} \neq \tilde{\psi}$. For any $(v',z') \in \mathbb{H}' = \mathbb{V}_1 \times \mathbb{F}_q$, we have

$$\begin{aligned}\tilde{\psi}^{(v_2,0)}((v',z')) &= \tilde{\psi}\big((v_2,0)\cdot(v',z')\cdot(v_2,0)^{-1}\big)\\ &= \tilde{\psi}\big((v',z'+\omega(v_2,v'))\big)\\ &= \tilde{\psi}\big((v',z')\big)\cdot\psi(\omega(v_2,v')).\end{aligned}$$

(see the previous computation). As $\mathbb{V} = \mathbb{V}_1 \oplus \mathbb{V}_2$ is a polarization, the map $\mathbb{V}_1 \to \mathbb{F}_q\colon v' \to \omega(v_2,v')$ is surjective. Thus, recalling that ψ is a nontrivial character of $\mathbb{F}_q$, we see that there exists $v' \in \mathbb{V}_1$ such that $\psi(\omega(v_2,v')) \neq 1$. In particular, we see that $\tilde{\psi}^{(v_2,0)} \neq \tilde{\psi}$ as a character of $\mathbb{H}'$.

Now, we have obtained an irreducible representation $\pi_{\tilde{\psi}}$ with central character ψ for each nontrivial character $\psi\colon \mathbb{F}_q \to \mathbb{C}^\times$. Let us do a counting argument. By the dimension formula for induced representation, we have $\dim \pi_{\tilde{\psi}} = [\mathbb{H} : \mathbb{H}'] = q^n$. By the square-sum formula for dimensions of irreducible representations, we have

$$\sum_{\pi \in \mathrm{Irr}(\mathbb{H})} (\dim \pi)^2 = |\mathbb{H}| = q^{2n+1}.$$

So far, at least, we have constructed

- q^{2n} many 1-dimensional irreducible representations, and
- $(q-1)$ many q^n-dimensional irreducible representations ($q-1$ is the number of nontrivial characters of $\mathbb{F}_q$).

However, we have

$$q^{2n}\cdot 1^2 + (q-1)\cdot(q^n)^2 = q^{2n+1}.$$

[2] This observation is actually not essential here; the subgroup $\mathbb{H}'$ is a normal subgroup even if $\mathbb{V}_1$ is replaced with any subspace of $\mathbb{V}$.

In other words, the square sum of the irreducible representations we have constructed already reaches $|\mathbb{H}|$. This means that there are no more irreducible representations of $\mathbb{H}$. In particular, we conclude that $\pi_{\tilde{\psi}}$ is the unique irreducible representation of $\mathbb{H}$ with central character ψ (hence $\pi_{\tilde{\psi}}$ is actually independent of the choice of $\tilde{\psi}$). □

5.5. **Exercises.**

Exercise 5.1. Let $G := \mathrm{GL}_2(\mathbb{F}_q)$. We consider the subgroup B of G consisting of upper-triangular matrices, i.e.,

$$B := \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \,\middle|\, a, d \in \mathbb{F}_q^\times,\, b \in \mathbb{F}_q \right\} \subset \mathrm{GL}_2(\mathbb{F}_q)$$

For characters (1-dimensional representations) $\chi_1 \colon \mathbb{F}_q^\times \to \mathbb{C}^\times$ and $\chi_2 \colon \mathbb{F}_q^\times \to \mathbb{C}^\times$ of $\mathbb{F}_q^\times$, we consider the following character of B:

$$\chi \colon B \to \mathbb{C}^\times; \quad \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto \chi_1(a) \cdot \chi_2(d).$$

Describe the irreducible decomposition of $\mathrm{Ind}_B^G \chi$.

Exercise 5.2. Let $G := \mathrm{GL}_2(\mathbb{F}_q)$. We consider the following subgroups of G:

$$M := \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \,\middle|\, a \in \mathbb{F}_q^\times,\, b \in \mathbb{F}_q \right\} \subset \mathrm{GL}_2(\mathbb{F}_q),$$

$$N := \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \,\middle|\, b \in \mathbb{F}_q \right\} \subset M,$$

For a nontrivial character $\psi \colon N \cong \mathbb{F}_q \to \mathbb{C}^\times$, we put

$$\Pi_\psi = \mathrm{Ind}_N^M \psi.$$

(1) Show that Π_ψ is irreducible.
(2) Classify all irreducible representations of M.

Exercise 5.3. Let $\mathbb{H}$ be the Heisenberg group associated to a $2n$-dimensional symplectic space $(\mathbb{V}, \omega)$ over $\mathbb{F}_q$ (of odd characteristic). We fix a nontrivial character $\psi \colon \mathbb{F}_q \to \mathbb{C}^\times$ and consider an irreducible representation π_ψ with central character ψ, which uniquely exists by the Stone–von Neumann theorem.

(1) Compute the character $\Theta_{\pi_\psi}(v, z)$ for any $(v, z) \in \mathbb{H}$.
(2) Determine the irreducible decomposition of the representation $\pi_\psi \otimes \pi_\psi^\vee$.

Exercise 5.4. Let G be a nontrivial p-group, i.e., a finite group whose order is a power of a prime number p. Suppose that there exists an irreducible representation (π, V) of G which is *faithful*, that is, $\pi \colon G \to \mathrm{Aut}_{\mathbb{C}}(V)$ is injective. Prove that the center Z of G is a nontrivial cyclic group.

6. Lecture 6: Brauer's induction theorem

(Main reference for today's lecture: [Ser77, Chapters 9, 10])

6.1. **More about induced representations.** Here we list a few more important properties of induction.

Proposition 6.1 (chain rule)**.** *Let G be a finite group and $H' \subset H$ be subgroups of G. For any representation (σ, W) of H',*

$$\mathrm{Ind}_H^G(\mathrm{Ind}_{H'}^H \sigma) \cong \mathrm{Ind}_{H'}^G \sigma.$$

Proof. To show the assertion, it is enough to show that the Hom spaces

$$\mathrm{Hom}_G\big(\pi, \mathrm{Ind}_H^G(\mathrm{Ind}_{H'}^H \sigma)\big)$$

and

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_{H'}^G \sigma)$$

are isomorphic for any representation π of G. By using Frobenius reciprocity twice, we get

$$\begin{aligned}\mathrm{Hom}_G\big(\pi, \mathrm{Ind}_H^G(\mathrm{Ind}_{H'}^H \sigma)\big) &\cong \mathrm{Hom}_H(\mathrm{Res}_H^G \pi, \mathrm{Ind}_{H'}^H \sigma) \\ &\cong \mathrm{Hom}_{H'}\big(\mathrm{Res}_{H'}^H(\mathrm{Res}_H^G \pi), \sigma\big).\end{aligned}$$

Obviously we have $\mathrm{Res}_{H'}^H(\mathrm{Res}_H^G(-)) = \mathrm{Res}_{H'}^G(-)$. Hence, again by using the Frobenius reciprocity, we get

$$\begin{aligned}\mathrm{Hom}_{H'}\big(\mathrm{Res}_{H'}^H(\mathrm{Res}_H^G \pi), \sigma\big) &\cong \mathrm{Hom}_{H'}(\mathrm{Res}_{H'}^G \pi, \sigma) \\ &\cong \mathrm{Hom}_G(\pi, \mathrm{Ind}_{H'}^G \sigma).\end{aligned}$$

This completes the proof. □

Proposition 6.2 (projection formula)**.** *Let G be a finite group and $H \subset G$ a subgroup. For any representation σ of H and any representation π of G,*

$$\mathrm{Ind}_H^G\big(\sigma \otimes (\mathrm{Res}_H^G \pi)\big) \cong (\mathrm{Ind}_H^G \sigma) \otimes \pi.$$

Proof. Similarly to the proof above, it is enough to show that the Hom spaces

$$\mathrm{Hom}_G\Big(\pi', \mathrm{Ind}_H^G\big(\sigma \otimes (\mathrm{Res}_H^G \pi)\big)\Big)$$

and

$$\mathrm{Hom}_G\big(\pi', (\mathrm{Ind}_H^G \sigma) \otimes \pi\big)$$

are isomorphic for any representation π' of G. By Frobenius reciprocity,

$$\mathrm{Hom}_G\Big(\pi', \mathrm{Ind}_H^G\big(\sigma \otimes (\mathrm{Res}_H^G \pi)\big)\Big) \cong \mathrm{Hom}_H\big(\mathrm{Res}_H^G \pi', \sigma \otimes (\mathrm{Res}_H^G \pi)\big).$$

By the adjunction between Hom and $\otimes$, we have

$$\mathrm{Hom}_H\big(\mathrm{Res}_H^G \pi', \sigma \otimes (\mathrm{Res}_H^G \pi)\big) \cong \mathrm{Hom}_H\big(\mathrm{Res}_H^G \pi' \otimes (\mathrm{Res}_H^G \pi)^\vee, \sigma\big)$$

(this adjunction has not been explained in this course, but can be proved in a similar manner to the case of $\mathbb{C}$-vector space homomorphisms). Thus, again using Frobenius reciprocity, we get

$$\begin{aligned}\mathrm{Hom}_H\big(\mathrm{Res}_H^G \pi' \otimes (\mathrm{Res}_H^G \pi)^\vee, \sigma\big) &= \mathrm{Hom}_H\big(\mathrm{Res}_H^G(\pi' \otimes \pi^\vee), \sigma\big) \\ &\cong \mathrm{Hom}_G(\pi' \otimes \pi^\vee, \mathrm{Ind}_H^G \sigma).\end{aligned}$$

Again using the Hom-$\otimes$ adjunction, we get

$$\mathrm{Hom}_G(\pi' \otimes \pi^\vee, \mathrm{Ind}_H^G \sigma) \cong \mathrm{Hom}_G\big(\pi', (\mathrm{Ind}_H^G \sigma) \otimes \pi\big).$$

This completes the proof. □

6.2. **Virtual representations.**

Definition 6.3. Let $\mathrm{Irr}(G)$ denote the set of isomorphism classes of irreducible representations of G. Let $\mathbb{Z}[\mathrm{Irr}(G)]$ denote the free abelian group generated by $\mathrm{Irr}(G)$, i.e.,

$$\mathbb{Z}[\mathrm{Irr}(G)] := \left\{ \sum_{\pi \in \mathrm{Irr}(G)} n_\pi \pi \;\middle|\; n_\pi \in \mathbb{Z} \text{ for each } \pi \in \mathrm{Irr}(G) \right\}.$$

We call an element of $\mathbb{Z}[\mathrm{Irr}(G)]$ a *virtual representation* of G. For any $\mathbb{Z}$-algebra R, we write $R[\mathrm{Irr}(G)] := \mathbb{Z}[\mathrm{Irr}(G)] \otimes_{\mathbb{Z}} R$.

Remark 6.4. (1) Let $\mathrm{Rep}(G)$ denote the set of isomorphism classes of finite-dimensional representations of G. By the semisimplicity of representations of a finite group, $\mathrm{Rep}(G)$ is identified with the subset of $\mathbb{Z}[\mathrm{Irr}(G)]$ consisting of elements with non-negative coefficients:

$$\mathrm{Rep}(G) = \left\{ \sum_{\pi \in \mathrm{Irr}(G)} n_\pi \pi \in \mathbb{Z}[\mathrm{Irr}(G)] \;\middle|\; n_\pi \in \mathbb{Z}_{\geq 0} \text{ for each } \pi \in \mathrm{Irr}(G) \right\}.$$

Using this isomorphism, we often loosely identify a virtual representation of G with its trace character.

(2) Recall that $C(G)$ denotes the set of class functions on G. Also recall that the characters of irreducible representations of G form an orthonormal basis of $C(G)$. This means that taking the character of a representation $[\pi \mapsto \Theta_\pi]$ induces an isomorphism of $\mathbb{C}$-vector spaces

$$\mathbb{C}[\mathrm{Irr}(G)] \xrightarrow{\cong} C(G) \colon \sum_{\pi \in \mathrm{Irr}(G)} c_\pi \pi \mapsto \sum_{\pi \in \mathrm{Irr}(G)} c_\pi \Theta_\pi.$$

(3) The abelian group $\mathbb{Z}[\mathrm{Irr}(G)]$ has a natural ring structure whose multiplication is given by the tensor product of representations. (The unit is the trivial representation.) Recall that $\Theta_{\pi_1 \otimes \pi_2} = \Theta_{\pi_1} \cdot \Theta_{\pi_2}$ (see Exercise 2.1); this implies that the above isomorphism is a $\mathbb{C}$-algebra isomorphism.

(4) Be careful that the image of $\mathbb{Z}[\mathrm{Irr}(G)]$ under the above isomorphism (i.e., the $\mathbb{Z}$-submodule generated by Θ_π for $\pi \in \mathrm{Irr}(G)$) is **not** necessarily equal to the set of $\mathbb{Z}$-valued class functions on G. For example, when $G = \mathbb{Z}/d\mathbb{Z}$, an irreducible representation of G is given by a character $\chi \colon G \to \mathbb{C}^\times$ such that $\chi(1)$ is a d-th root of unity.

Definition 6.5. A representation of G is called *monomial* if it is of the form $\mathrm{Ind}_H^G \chi$ for some subgroup $H \subset G$ and a character (1-dimensional representation) χ of H.

Our goal is the following:

Theorem 6.6 (Brauer's induction theorem)**.** *The abelian group $\mathbb{Z}[\mathrm{Irr}(G)]$ is generated by monomial representations of G. In particular (or, "equivalently"), for*

any irreducible representation π of G, there exist subgroups $H_1, \dots, H_r$ of G and characters χ_i of H_i such that

$$\pi = \sum_{i=1}^{r} n_i \operatorname{Ind}_{H_i}^G \chi_i \in \mathbb{Z}[\operatorname{Irr}(G)].$$

6.3. **Artin's theorem.** Let G be a finite group and $H \subset G$ a subgroup. Then the induction $\operatorname{Ind}_H^G(-)$ induces a $\mathbb{Z}$-linear homomorphism

$$\operatorname{Ind}_H^G \colon \mathbb{Z}[\operatorname{Irr}(H)] \to \mathbb{Z}[\operatorname{Irr}(G)]; \quad \sum_{\pi \in \operatorname{Irr}(H)} n_\pi \pi \mapsto \sum_{\pi \in \operatorname{Irr}(H)} n_\pi \operatorname{Ind}_H^G \pi.$$

For any set $\mathcal{X}$ of subgroups of G, we define a $\mathbb{Z}$-linear homomorphism $\operatorname{Ind}_{\mathcal{X}}^G$ by taking the sum of Ind_H^G for all $H \in \mathcal{X}$:

$$\operatorname{Ind}_{\mathcal{X}}^G := \bigoplus_{H \in \mathcal{X}} \operatorname{Ind}_H^G \colon \bigoplus_{H \in \mathcal{X}} \mathbb{Z}[\operatorname{Irr}(H)] \to \mathbb{Z}[\operatorname{Irr}(G)].$$

Theorem 6.7 (Artin's theorem)**.** *The following are equivalent:*

(1) $G = \bigcup_{H \in \mathcal{X}} \bigcup_{g \in G} gHg^{-1}$.
(2) $\operatorname{Cok}(\operatorname{Ind}_{\mathcal{X}}^G)$ *is finite.*

Proof. The condition (2) is equivalent to the condition that the map

$$\operatorname{Ind}_{\mathcal{X},\mathbb{Q}}^G := \operatorname{Ind}_{\mathcal{X}}^G \otimes_{\mathbb{Z}} \mathbb{Q} \colon \bigoplus_{H \in \mathcal{X}} \mathbb{Q}[\operatorname{Irr}(H)] \to \mathbb{Q}[\operatorname{Irr}(G)]$$

is surjective, which is further equivalent to the condition that the map

$$\operatorname{Ind}_{\mathcal{X},\mathbb{C}}^G := \operatorname{Ind}_{\mathcal{X}}^G \otimes_{\mathbb{Z}} \mathbb{C} \colon \bigoplus_{H \in \mathcal{X}} C(H) \to C(G)$$

is surjective. We consider the dual map of this map with respect to the inner products on $C(G)$ and $C(H)$ for $H \in \mathcal{X}$:

$$(\operatorname{Ind}_{\mathcal{X},\mathbb{C}}^G)^\vee \colon C(G) \to \bigoplus_{H \in \mathcal{X}} C(H).$$

More precisely, this map is characterized by the property that

$$((\operatorname{Ind}_{\mathcal{X},\mathbb{C}}^G)^\vee(f), f_H)_H = (f, \operatorname{Ind}_{H,\mathbb{C}}^G(f_H))_G$$

for any $f \in C(G)$ and any $f_H \in C(H)$. However, by Frobenius reciprocity, we have

$$(f, \operatorname{Ind}_{H,\mathbb{C}}^G(f_H))_G = (\operatorname{Res}_{H,\mathbb{C}}^G(f), f_H)_H,$$

where $\operatorname{Res}_{H,\mathbb{C}}^G(-)$ denotes the restriction map $C(G) \to C(H)$. Hence, we conclude that $(\operatorname{Ind}_{\mathcal{X},\mathbb{C}}^G)^\vee$ is nothing but the direct sum of the restriction maps:

$$(\operatorname{Ind}_{\mathcal{X},\mathbb{C}}^G)^\vee = \operatorname{Res}_{\mathcal{X},\mathbb{C}}^G := \bigoplus_{H \in \mathcal{X}} \operatorname{Res}_{H,\mathbb{C}}^G \colon C(G) \to \bigoplus_{H \in \mathcal{X}} C(H).$$

Therefore, we see that the surjectivity of $\operatorname{Ind}_{\mathcal{X},\mathbb{C}}^G$ is equivalent to the injectivity of $\operatorname{Res}_{\mathcal{X},\mathbb{C}}^G$.

Now our task is to check that the injectivity of $\operatorname{Res}_{\mathcal{X},\mathbb{C}}^G$ is equivalent to the condition (1). Suppose that $f \in C(G)$. Then f belongs to $\operatorname{Ker}(\operatorname{Res}_{\mathcal{X},\mathbb{C}}^G)$ if and only if $f|_H = 0$ for any $H \in \mathcal{X}$. Since f is a class function, this is equivalent to the condition that $f|_{gHg^{-1}} = 0$ for any $H \in \mathcal{X}$ and any $g \in G$. Therefore, if (1) holds,

then such f must be 0. Conversely, if (1) does not hold, then there exists $g \in G$ which is not contained in any conjugate of any $H \in \mathcal{X}$. Then the characteristic function $\mathbb{1}_{C_g}$ of the conjugacy class C_g of g is a nonzero element of $\mathrm{Ker}(\mathrm{Res}^G_{\mathcal{X},\mathbb{C}})$. □

6.4. **Reduction to "pre-Brauer".** In the following, the symbol p always denotes a prime number.

Definition 6.8. (1) A finite group is called *p-elementary* if it is of the form $C \times P$, where C is a cyclic group of order prime-to-p and P is a p-group (i.e., a group of order a power of p).
(2) A finite group is called *elementary* if it is p-elementary for some prime number p.

For a fixed finite group G, we let

- $\mathcal{E}_p$ denote the set of p-elementary subgroups of G, and
- $\mathcal{E}$ denote the set of elementary subgroups of G.

Note that any $g \in G$ is contained in the cyclic subgroup $\langle g \rangle \subset G$ generated by g, which is of course a p-elementary group for any prime number p. Hence, $\mathcal{E}_p$ satisfies the condition (1) in Artin's theorem (Theorem 6.7). Let us write V_p the image of $\mathrm{Ind}^G_{\mathcal{E}_p}$ in $\mathbb{Z}[\mathrm{Irr}(G)]$, which is of finite index by Artin's theorem:

$$V_p := \mathrm{Im}\Big(\mathrm{Ind}^G_{\mathcal{E}_p} : \bigoplus_{H \in \mathcal{E}_p} \mathbb{Z}[\mathrm{Irr}(H)] \to \mathbb{Z}[\mathrm{Irr}(G)]\Big) \subset \mathbb{Z}[\mathrm{Irr}(G)].$$

Furthermore, note that V_p is an ideal of the ring $\mathbb{Z}[\mathrm{Irr}(G)]$; this is a consequence of the projection formula (Proposition 6.2).

The key step in the proof of Brauer's induction theorem is to prove the following theorem. For our convenience, let us call this theorem the "pre-Brauer" theorem.

Theorem 6.9 ("pre-Brauer")**.** *The index $[\mathbb{Z}[\mathrm{Irr}(G)] : V_p]$ is finite. More precisely, if we write $|G| = p^r \cdot m$ with $r \in \mathbb{Z}_{\geq 0}$ and $m \in \mathbb{Z}_{>0}$ prime-to-p, then the index $[\mathbb{Z}[\mathrm{Irr}(G)] : V_p]$ divides a power of m.*

Before we prove the pre-Brauer theorem, let us see how the Brauer induction theorem follows from it.

Theorem 6.9 $\Longrightarrow$ Theorem 6.6. We consider the finite-index subgroup $V_p \subset \mathbb{Z}[\mathrm{Irr}(G)]$ for each prime number p. By Theorem 6.9, the index $[\mathbb{Z}[\mathrm{Irr}(G)] : V_p]$ is prime-to-p. We let V be the submodule of $\mathbb{Z}[\mathrm{Irr}(G)]$ generated by V_p for all prime numbers p. Since $V_p \subset V$, the index $[\mathbb{Z}[\mathrm{Irr}(G)] : V]$ must divide $[\mathbb{Z}[\mathrm{Irr}(G)] : V_p]$. Hence $[\mathbb{Z}[\mathrm{Irr}(G)] : V]$ is prime-to-p for any prime number p, which implies $[\mathbb{Z}[\mathrm{Irr}(G)] : V] = 1$. In other words, any virtual representation of G can be written as a $\mathbb{Z}$-linear combination of induced representations from characters of elementary subgroups.

Here, we appeal to a nontrivial fact that any irreducible representation of an elementary group is monomial (see [Ser77, §8.5]; this is a consequence of a fact that any irreducible representation of a p-group is monomial). If we write a virtual representation of G as

$$\sum_{i=1}^{r} n_i \,\mathrm{Ind}^G_{H_i} \pi_i,$$

where $n_i \in \mathbb{Z}$, $H_i \in \mathcal{E}$, and $\pi_i \in \mathrm{Irr}(H_i)$ then we can write each π_i as $\mathrm{Ind}_{H_i'}^{H_i} \chi_i$ with a subgroup $H_i' \subset H_i$ and a 1-dimensional representation χ_i of H_i', hence we get

$$\sum_{i=1}^{r} n_i \,\mathrm{Ind}_{H_i}^{G} \pi_i = \sum_{i=1}^{r} n_i \,\mathrm{Ind}_{H_i}^{G}(\mathrm{Ind}_{H_i'}^{H_i} \chi_i) = \sum_{i=1}^{r} n_i \,\mathrm{Ind}_{H_i'}^{G} \chi_i$$

by the chain rule of induction. □

6.5. **Proof of "pre-Brauer".** We first introduce several supplementary notions.

Definition 6.10. Let G be a finite group. Let $g \in G$.

- We call g a *p'-element* if the order of g is prime-to-p.
- We call g a *p-element* if the order of g is p-power.

Lemma 6.11. *Any $g \in G$ can be uniquely written as $g = su$, where s is a p'-element and u is a p-element such that $su = us$.*

Proof. Let $d = p^r \cdot m$ be the order of g, where $r \in \mathbb{Z}_{\geq 0}$ and $m \in \mathbb{Z}_{>0}$ is a prime-to-p number. Then g^{p^r} is a p'-element and g^m is a p-element. Since p^r and m are coprime, we can find integers a and b such that $ap^r + bm \equiv 1 \pmod{d}$, which implies that

$$g = g^{ap^r + bm} = (g^{p^r})^a \cdot (g^m)^b.$$

Thus, if we put $s := (g^{p^r})^a$ and $u := (g^m)^b$, then s and u satisfy the desired conditions.

To check the uniqueness, suppose that we have another such pair (s', u'). By taking the p^r-th power of $su = s'u'$, we get $s^{p^r} = s'^{p^r}$. Since s and s' are p'-element, this implies that $s = s'$, hence also $u = u'$. □

We call the decomposition $g = su$ as in this lemma the *p-Jordan decomposition* of $g \in G$.

For any positive integer $d \in \mathbb{Z}_{>0}$, we let $\mu_d \subset \mathbb{C}$ denote the set of $d-$th roots of unity. We write $\mathbb{Z}[\mu_d] \subset \mathbb{C}$ for the $\mathbb{Z}$-subalgebra of $\mathbb{C}$ generated by μ_d (this is a finitely generated free $\mathbb{Z}$-module).

Proposition 6.12. *Let G be a finite group of order d. Let $s \in G$ be a p'-element. Then there exists an element $\psi_s \in V_p \otimes_{\mathbb{Z}} \mathbb{Z}[\mu_d]$ such that*

1. *ψ_s is integer-valued,*
2. *$\psi_s(s) \not\equiv 0 \pmod{p}$, and*
3. *$\psi_s(s') = 0$ for any p'-element of G which is not conjugate to s.*

Proof. We let m be the order of s. We put $C := \langle s \rangle$. Let $\psi_C \colon C \to \mathbb{Z}$ be the function defined by

$$\psi_C(c) = \begin{cases} m & c = s, \\ 0 & c \neq s. \end{cases}$$

Then we can easily check that

$$\psi_C = \sum_{\chi \in \mathrm{Irr}(C)} \chi(s)^{-1} \cdot \chi.$$

As each $\chi(s)^{-1}$ belongs to $\mathbb{Z}[\mu_d]$, this implies that $\psi_C \in \mathbb{Z}[\mu_d][\mathrm{Irr}(C)]$.

We fix a p-Sylow subgroup P of the centralizer group $Z_G(s)$ of s in G (let p^r be the order of P). We put $H := C \times P = CP \subset G$. We let $\psi_H \colon H \to \mathbb{Z}$ be the composition of $\psi_C \colon C \to \mathbb{Z}$ and the projection $C \times P \twoheadrightarrow C$.

Now we let $\psi_s := \mathrm{Ind}_H^G \psi_H$ and prove that ψ_s satisfies the conditions as desired. By the Frobenius character formula, for any $g \in G$, we have

$$\psi_s(g) = \mathrm{Ind}_H^G \psi_H(g) = \sum_{\substack{x \in H\backslash G \\ xgx^{-1} \in H}} \psi_H(xgx^{-1}).$$

Hence the condition (1) is clearly satisfied.

To check (2) and (3), let us take g to be a p'-element $s' \in G$. Then

$$\psi_s(s') = \sum_{\substack{x \in H\backslash G \\ xs'x^{-1} \in H}} \psi_H(xs'x^{-1}).$$

If $xs'x^{-1} \in H$, then we must have $xs'x^{-1} \in C$ since $xs'x^{-1}$ is a p'-element. Moreover, by the definitions of ψ_H and ψ_C, $\psi_H(xs'x^{-1})$ is not zero only when $xs'x^{-1} = s$. Hence, if s' is not conjugate to s, we must have $\psi_s(s') = 0$. If $s' = s$, we get

$$\sum_{\substack{x \in H\backslash G \\ xsx^{-1} \in H}} \psi_H(xsx^{-1}) = \frac{1}{|H|} \sum_{\substack{x \in G \\ xsx^{-1} = s}} m = \frac{1}{mp^r} \cdot |Z_G(s)| \cdot m = \frac{|Z_G(s)|}{p^r}.$$

This is an integer which is not 0 modulo p since p^r is the largest power of p dividing $|Z_G(s)|$ (it is the order of a p-Sylow subgroup). □

Corollary 6.13. *There exists an element $\psi \in V_p \otimes_{\mathbb{Z}} \mathbb{Z}[\mu_d]$ such that*

(1) *ψ takes values in $\mathbb{Z}$ (viewed as a class function $\psi\colon G \to \mathbb{C}$), and*
(2) *$\psi(g) \not\equiv 0 \pmod{p}$ for any $g \in G$.*

Proof. Let $s_1, \ldots, s_n$ be a set of representatives of conjugacy classes of all p'-elements of G. For each s_i, we consider the function $\psi_{s_i}\colon G \to \mathbb{Z}$ as in Proposition 6.12. We put $\psi := \sum_{i=1}^n \psi_{s_i}\colon G \to \mathbb{Z}$; this is an element of $V_p \otimes_{\mathbb{Z}} \mathbb{Z}[\mu_d]$.

We check that this function satisfies the condition (2). If g is a p'-element, then (2) is clearly satisfied. Indeed, since there exists a unique s_j such that s_j is conjugate to g, we have

$$\psi(g) = \sum_{i=1}^n \psi_{s_i}(g) = \psi_{s_j}(g) \not\equiv 0 \pmod{p}$$

by the properties of ψ_{s_i}'s.

So let us consider the general case; let g be an element of G with p-Jordan decomposition $g = su$. In this case, it suffices to show that $\psi(g) \equiv \psi(s) \pmod{p}$. We put $C := \langle g \rangle$. As the restriction of ψ to C belongs to $\mathbb{Z}[\mu_d][\mathrm{Irr}(C)]$, we may write

$$\psi|_C = \sum_{k=1}^N a_k \chi_k,$$

where $a_k \in \mathbb{Z}[\mu_d]$ and $\chi_k \in \mathrm{Irr}(C)$. If we let p^r be the order of u, then $g^{p^r} = s^{p^r}$. Hence we get

$$\begin{aligned}\psi(g)^{p^r} = \Big(\sum_{k=1}^{N} a_k\chi_k(g)\Big)^{p^r} &\equiv \sum_{k=1}^{N} a_k^{p^r}\chi_k(g)^{p^r} \\ &= \sum_{k=1}^{N} a_k^{p^r}\chi_k(s)^{p^r} \\ &\equiv \Big(\sum_{k=1}^{N} a_k\chi_k(s)\Big)^{p^r} = \psi(s)^{p^r} \pmod{p\mathbb{Z}[\mu_d]}.\end{aligned}$$

As both $\psi(g)$ and $\psi(s)$ are integers, this implies that

$$\psi(g)^{p^r} \equiv \psi(s)^{p^r} \pmod{p}$$

(using that $p\mathbb{Z}[\mu_d] \cap \mathbb{Z} = p\mathbb{Z}$), which furthermore implies that $\psi(g) \equiv \psi(s) \pmod{p}$. □

To show the pre-Brauer theorem, we introduce a few more results.

For a cyclic group C, we define a function $\theta_C \colon C \to \mathbb{C}$ by

$$\theta_C(g) := \begin{cases} |C| & \text{if } \langle g\rangle = C, \\ 0 & \text{otherwise.}\end{cases}$$

Proposition 6.14. *Let G be a finite group. Then*

$$|G| \cdot \mathbb{1}_G = \sum_{C\subset G} \mathrm{Ind}_C^G \theta_C,$$

where the sum is over all cyclic subgroups $C \subset G$.

Proof. By the Frobenius character formula, for any $g \in G$, we have

$$\mathrm{Ind}_C^G \theta_C(g) = \sum_{\substack{x\in C\backslash G \\ xgx^{-1}\in C}} \theta_C(xgx^{-1}) = \sum_{\substack{x\in C\backslash G \\ xgx^{-1} \text{ generates } C}} |C| = \sum_{\substack{x\in G \\ xgx^{-1} \text{ generates } C}} 1.$$

Therefore, we have

$$\sum_{C\subset G} \mathrm{Ind}_C^G \theta_C(g) = \sum_{C\subset G} \sum_{\substack{x\in G \\ xgx^{-1} \text{ generates } C}} 1.$$

For any $g \in G$ and $x \in G$, there is exactly one cyclic subgroup $C \subset G$ such that xgx^{-1} generates C. Hence we get

$$\sum_{C\subset G} \sum_{\substack{x\in G \\ xgx^{-1} \text{ generates } C}} 1 = |G|.$$

This completes the proof. □

Lemma 6.15. *Let C be a cyclic group. Then $\theta_C \in \mathbb{Z}[\mathrm{Irr}(C)]$.*

Proof. We proceed with induction on the order of C. Since the statement is trivial for $|C| = 1$, we assume that $|C| > 1$. By Proposition 6.14, we have

$$|C| \cdot \mathbb{1}_C = \sum_{C'\subset C} \mathrm{Ind}_{C'}^C \theta_{C'} = \theta_C + \sum_{C'\subsetneq C} \mathrm{Ind}_{C'}^C \theta_{C'},$$

where the sum is over all subgroups of C. By the induction hypothesis, we have $\theta_{C'} \in \mathbb{Z}[\mathrm{Irr}(C')]$ for any $C' \subsetneq C$, hence $\mathrm{Ind}_{C'}^{C} \theta_{C'} \in \mathbb{Z}[\mathrm{Irr}(C)]$. As $|C| \cdot \mathbb{1}_C \in \mathbb{Z}[\mathrm{Irr}(C)]$, we get $\theta_C \in \mathbb{Z}[\mathrm{Irr}(C)]$. □

Corollary 6.16. *Let G be a finite group of order d. Then any class function $f\colon G \to d\mathbb{Z}$ (i.e., the values are integers divisible by d) can be written as*

$$f = \sum_{C \subset G} \sum_{\chi \in \mathrm{Irr}(C)} c_{C,\chi} \, \mathrm{Ind}_C^G \chi$$

where the sum is over all cyclic subgroups of G and $c_{C,\chi} \in \mathbb{Z}[\mu_d]$.

Proof. By assumption, there exists a class function $f'\colon G \to \mathbb{Z}$ such that $f = d \cdot f'$. By Proposition 6.14, we may write

$$d \cdot \mathbb{1}_G = \sum_{C \subset G} \mathrm{Ind}_C^G \theta_C.$$

Thus, by the projection formula, we have

$$f = d \cdot f' = \sum_{C \subset G} (\mathrm{Ind}_C^G \theta_C) \cdot f' = \sum_{C \subset G} \mathrm{Ind}_C^G (\theta_C \cdot f'|_C).$$

Here, $\theta_C \cdot f'|_C$ is written as a $\mathbb{Z}[\mu_d]$-linear combination of characters of C. Indeed, for any $\chi \in \mathrm{Irr}(C)$, we have

$$(\theta_C \cdot f'|_C, \chi)_C = \frac{1}{|C|} \sum_{c \in C} \theta_C(c) \cdot f'|_C(c) \cdot \overline{\chi(c)}.$$

Since $\theta_C(c)$ is divisible by $|C|$, $f'|_C(c)$ is an integer, and $\chi(c) \in \mu_d$, we must have that $(\theta_C \cdot f'|_C, \chi)_C \in \mathbb{Z}[\mu_d]$. This completes the proof. □

Now let us complete the proof of Theorem 6.9.

Proof of Theorem 6.9. Let $d := |G| = p^r \cdot m$. To show the assertion, it is enough to show that $m\mathbb{1}_G \in V_p$. Indeed, if we can show this, then the finite abelian group $\mathbb{Z}[\mathrm{Irr}(G)]/V_p$ is killed by m-multiplication. Hence its order must divide a power of m.

We consider the finite $\mathbb{Z}$-algebra $\mathbb{Z}[\mu_d] \subset \mathbb{C}$ generated by the d-th roots of unity μ_d over $\mathbb{Z}$. Since $V_p = (V_p \otimes_{\mathbb{Z}} \mathbb{Z}[\mu_d]) \cap \mathbb{Z}[\mathrm{Irr}(G)]$ (where the intersection and the equality are considered in $\mathbb{C}[\mathrm{Irr}(G)]$), in order to show that $m\mathbb{1}_G \in V_p$, it is enough to show that $m\mathbb{1}_G \in V_p \otimes_{\mathbb{Z}} \mathbb{Z}[\mu_d]$.

Let $\psi \in V_p \otimes_{\mathbb{Z}} \mathbb{Z}[\mu_d]$ be an element as in Corollary 6.13. If we let $N := |(\mathbb{Z}/p^r\mathbb{Z})^\times|$, then we have $\psi^N \equiv \mathbb{1}_G \pmod{p^r}$. Hence $m(\psi^N - \mathbb{1}_G)$ takes integer values divisible by $mp^r = d$. By Corollary 6.16, we can write

$$m(\psi^N - \mathbb{1}_G) = \sum_{C \subset G} c_C \, \mathrm{Ind}_C^G \pi$$

for some integers $c_C \in \mathbb{Z}[\mu_d]$ and $\pi \in \mathbb{Z}[\mathrm{Irr}(C)]$, where the sum is over all cyclic subgroups of G. As any cyclic group is p-elementary, we see that the right-hand side belongs to $V_p \otimes_{\mathbb{Z}} \mathbb{Z}[\mu_d]$. On the other hand, since $V_p \otimes_{\mathbb{Z}} \mathbb{Z}[\mu_d]$ is an ideal of the ring $\mathbb{Z}[\mu_d][\mathrm{Irr}(G)]$, we have $m\psi^N \in V_p \otimes_{\mathbb{Z}} \mathbb{Z}[\mu_d]$. Therefore, we also get

$$m\mathbb{1}_G = m\psi^N - m(\psi^N - \mathbb{1}_G) \in V_p \otimes_{\mathbb{Z}} \mathbb{Z}[\mu_d].$$

□

6.6. **Exercises.**

Exercise 6.1. Let G be a finite group. Let $f \in C(G)$ be a class function $G \to \mathbb{C}$. Prove that f is the character of a virtual representation of G (or equivalently, a $\mathbb{Z}$-linear combination of the characters of irreducible representations) if and only if $f|_H$ is the character of a virtual representation of H for any elementary subgroup $H \in \mathcal{E}$.

Exercise 6.2. Let G be a finite group. For $n \in \mathbb{Z}_{>0}$, we define a $\mathbb{C}$-linear map $\Psi_n \colon C(G) \to C(G)$ by

$$\Psi_n(f)(g) := f(g^n).$$

(This map is called the *Adams operator*.) Suppose that $(|G|, n) = 1$.

(1) For any subgroup $H \subset G$ and a 1-dimensional representation $\chi \in \mathrm{Irr}(H)$, prove that
$$\Psi_n(\Theta_{\mathrm{Ind}_H^G \chi}) = \Theta_{\mathrm{Ind}_H^G \chi^n}.$$
(2) Prove that Ψ_n maps a virtual representation to a virtual representation (i.e., if f is the character of a virtual representation, then so is $\Psi_n(f)$).
(3) Prove that Ψ_n maps an irreducible representation of G to an irreducible representation of G.

7. Lecture 7: Representations of topological groups

From now on, we will study representation theory of topological groups, especially, compact groups. As references, I would like to suggest [Rud87, Rud91] for basics of functional analysis, and [Fol16, Hal15] for basics of topological groups and their representation theory.

7.1. **Observation.** Now let us think about representations of infinite groups.

We start with the case where $G = \mathbb{R}$. Let us consider the following pathological example. We first regard $\mathbb{R}$ as a $\mathbb{Q}$-vector space and choose a basis containing 1 and $\sqrt{2}$. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a $\mathbb{Q}$-linear homomorphism determined by

$$f(1) = 0, \quad f(\sqrt{2}) = \pi = 3.1415\ldots,$$

and $f(v) = 0$ for any other basis vector $v \in \mathbb{R}$. If we define $\chi_f\colon \mathbb{R} \to \mathbb{C}^\times$ by

$$\chi_f(r) := e^{if(r)},$$

then this is a homomorphism. In other words, $(\chi_f, \mathbb{C})$ is a 1-dimensional representation of $\mathbb{R}$.

Note that, here I just randomly chose "$\sqrt{2}$" and "π". You can construct infinitely many examples of this kind easily by replacing $\{1, \sqrt{2}\}$ (and also $\{0, \pi\}$) with any set of $\mathbb{Q}$-linearly independent numbers. In particular, we see that there are uncontrollably many representations even in the case of a really basic group like $\mathbb{R}$.

The problem here is that, of course, χ_f is not continuous. So, the lesson from this example is that, when studying a group equipped with a topology ("topological group"), it is reasonable to restrict our attention to representations that are compatible with its topology. However, then we have to think about what "compatible with its topology" means. More precisely, if it simply means the "continuity", does it mean the continuity of the map

$$G \times V \to V,$$

or

$$G \to \mathrm{Aut}_{\mathbb{C}}(V)$$

(or something else)? In any case, if V is an infinite-dimensional $\mathbb{C}$-vector space, it is not clear what kind of topology we should give to V, and also $\mathrm{Aut}_{\mathbb{C}}(V)$.

Therefore, what we first have to do is to introduce a reasonable class of representations which reflects the nature of a topology of a given group G correctly.

7.2. **Topological groups.**

Definition 7.1. A group G is called a topological group if it is equipped with a topology such that

- G is Hausdorff,
- $G \times G \to G\colon (g, h) \mapsto gh$ is continuous, and
- $G \to G\colon g \mapsto g^{-1}$ is continuous.

Let us list several basic properties of a topological group.

Proposition 7.2. *Let G be a topological group.*

(1) (a) *For any $g \in G$, the left-translation map and the right-translation map*

$$G \to G\colon x \mapsto gx, \quad G \to G\colon x \mapsto xg$$

are homeomorphisms.

(b) *The inverse map*

$$G \to G \colon x \mapsto x^{-1}$$

is a homeomorphism.

(c) *For any $g \in G$, the conjugation map*

$$G \to G \colon x \mapsto gxg^{-1}$$

is a homeomorphism.

(2) *For any subgroup H of G, its closure $\overline{H}$ is again a subgroup of G.*

(3) *Any open subgroup of G is a closed subgroup of G.*

Example 7.3. (1) Any finite group is a topological group with respect to the discrete topology.

(2) The set of real numbers $\mathbb{R}$ is a topological group with respect to its addition and the Euclidean topology.

(3) The group $\mathrm{GL}_n(\mathbb{C})$ of n-by-n invertible matrices is a topological group with respect to its matrix multiplication and the induced topology from $\mathrm{GL}_n(\mathbb{C}) \subset M_n(\mathbb{C}) \cong \mathbb{C}^{\oplus n^2}$. (This is an example of a Lie group.)

(4) For any field F and its Galois extension E (possibly infinite), its Galois group $\mathrm{Gal}(E/F)$ is a topological group with respect to the Krull topology (profinite topology).

Remark 7.4. There might be several reasons why we restrict ourselves only to Hausdorff topological groups. For example, one reason is the following. Let G be a topological group which may not necessarily be Hausdorff. We consider the closure of the trivial subgroup $N := \overline{\{1\}} \subset G$. Then, this is a closed subgroup by Proposition 7.2 (2). Furthermore, it is normal. Indeed, for any $g \in G$, the conjugation map $G \to G \colon x \mapsto gxg^{-1}$ is a homeomorphism. In particular, it maps N to the smallest closed subgroup of G which contains 1, that is, N itself. Therefore, it makes sense to consider the quotient group G/N, which is again a topological group with respect to the quotient topology. Furthermore, by Exercise 7.2, this quotient group is Hausdorff. If we want to study continuous representations of G, then the action of N must be necessarily trivial (e.g., consider a continuous homomorphism $G \to \mathrm{GL}_n(\mathbb{C})$). Hence there is no difference between representation theories of G and G/N.

7.3. **Topological vector spaces.**

Definition 7.5. A $\mathbb{C}$-vector space V (possibly infinite-dimensional) is said to be a *topological $\mathbb{C}$-vector space* if it is Hausdorff and the maps

$$V \times V \to V \colon (v, w) \mapsto v + w, \quad \mathbb{C} \times V \to V \colon (a, v) \mapsto a \cdot v$$

are both continuous.

We remark that if V is finite-dimensional (say $\dim_{\mathbb{C}} V = n$), then V is in fact isomorphic to $\mathbb{C}^{\oplus n}$ as a topological $\mathbb{C}$-vector space (see [Rud91, Theorem 1.21]). However, if V is infinite-dimensional, there might be various different topologies on V so that V is a topological vector space.

Example 7.6. Let $C_c(\mathbb{N})$ be a $\mathbb{C}$-vector space consisting of sequences $a = (a_n)_{n \in \mathbb{N}}$ of complex numbers such that $a_n \neq 0$ only for finitely many natural numbers $n \in \mathbb{N}$.

If we put e_n to be the sequence with $a_m = \delta_{m,n}$ (Kronecker's delta), then we simply have

$$C_c(\mathbb{N}) = \bigoplus_{n=0}^{\infty} \mathbb{C} e_n.$$

We define the following two kinds of norms on $C_c(\mathbb{N})$:

$$\|a\|_1 := \sum_{n=0}^{\infty} |a_n|, \quad \|a\|_2 := \Big(\sum_{n=0}^{\infty} |a_n|^2 \Big)^{\frac{1}{2}}.$$

Then these norms are not equivalent; more precisely, the norm topology with respect to $\| - \|_1$ is finer. Indeed, for any $a \in C_c(\mathbb{N})$, we have

$$\|a\|_2^2 = \sum_{n=0}^{\infty} |a_n|^2 \leq \Big(\sum_{n=0}^{\infty} |a_n| \Big)^2 = \|a\|_1^2.$$

This means that if two points $a, a' \in C_c(\mathbb{N})$ are close to each other with respect to $\| - \|_1$, then they are also close with respect to $\| - \|_2$. However, for example, if we put

$$a^{(N)} := \frac{1}{N}(e_0 + \cdots + e_{N-1}) \in C_c(\mathbb{N}),$$

then we have

$$\|a^{(N)}\|_1 = N \cdot \frac{1}{N} = 1, \quad \|a^{(N)}\|_2 = \sqrt{N \cdot \frac{1}{N^2}} = \frac{1}{\sqrt{N}}.$$

This means that the sequence $\{a^{(N)}\}_{N \in \mathbb{N}}$ converges to 0 with respect to the norm $\| - \|_2$ but not with respect to $\| - \|_1$.

Now we define the notion of a continuous representation.

Definition 7.7. Let G be a topological group and (π, V) a representation of G. We say that (π, V) is a *continuous representation* if V is a topological $\mathbb{C}$-vector space such that the action map

$$G \times V \to V \colon (g, v) \mapsto g \cdot v$$

is continuous.

7.4. **Hilbert space representations.** We next consider when we can develop a satisfactory theory on semisimple representations of a given group G. Recall that every finite-dimensional representation of every finite group is semisimple. However, when a group G is infinite, the semisimplicity is no longer true without conditions.

For example, let us consider the following (from Exercises in Lecture 1):

Example 7.8. Let $G := \mathbb{Z}$. We consider the 2-dimensional representation (π, V) of G defined by $V = \mathbb{C}^{\oplus 2}$ and

$$\pi \colon \mathbb{Z} \to \mathrm{GL}_2(\mathbb{C}) \colon n \mapsto \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}.$$

Then (π, V) is not semisimple.

In this example, $\mathbb{Z}$ is thought of as a topological group with respect to the discrete topology. Also, since V is finite-dimensional, we can put on V a canonical structure of a topological $\mathbb{C}$-vector space. The representation (π, V) is continuous with respect to these topologies. So, what we should learn from this example is that the semisimplicity fails even in such a basic situation.

Here, let us go back to the proof of the semisimplicity of finite-dimensional representations of a finite group. Recall that the key is the unitarity of a representation (π, V), i.e., existence of a G-invariant inner product on V. If a representation (π, V) has such a nice inner product, then we can find a G-invariant complement of any G-subspace of V (Lemma 1.16). (So the point is that we can always find such an inner product in any representation in the case of finite groups; Proposition 1.18).

Keeping this in mind, we may understand that one of the natural classes of representations of a topological group G is continuous representations realized on a topological $\mathbb{C}$-vector space equipped with a G-invariant inner product.

However, here we also have the following observation:

Example 7.9. Let us again consider the $\mathbb{C}$-vector space $C_c(\mathbb{N})$ consisting of sequences $a = (a_n)_{n\in\mathbb{N}}$ with only finitely many nonzero entries. If we define an inner product on $C_c(\mathbb{N})$ by

$$((a_n)_n, (b_n)_n) := \sum_{n=0}^{\infty} a_n \cdot \overline{b_n},$$

then the norm $\| - \|_2$ is nothing but the norm determined by this inner product. Inside the space $C_c(\mathbb{N})$, we consider the following subspace:

$$W := \left\{ a = (a_n)_n \in C_c(\mathbb{N}) \,\middle|\, \sum_{n=0}^{\infty} \frac{a_n}{2^n} = 0 \right\} \subset C_c(\mathbb{N}).$$

Then we can check that W is a proper closed subspace of $C_c(\mathbb{N})$. (Closedness can be checked by noting that W is the kernel of a homomorphism $C_c(\mathbb{N}) \to \mathbb{C}$: $a \mapsto \sum_{n=0}^{\infty} \frac{a_n}{2^n}$, which is continuous.) However, we also see that its orthogonal complement is zero

$$W^{\perp} := \{a \in C_c(\mathbb{N}) \mid (a, b) = 0 \text{ for any } b \in W\} = 0.$$

Indeed, for any $n > 0$, we have

$$e_0 - 2^n e_n \in W.$$

Hence, if $a = (a_n)_n \in W^{\perp}$, then we have

$$(a, e_0 - 2^n e_n) = a_0 - 2^n a_n = 0.$$

This means that, for any $n > 0$, $a_0 = 0$ if and only if $a_n = 0$. If $a_0 \neq 0$, then $a_n \neq 0$ for all $n > 0$, which contradicts the fact that $a \in C_c(\mathbb{N})$. Hence $a = 0$.

As this example shows, in general (i.e., in the infinite-dimensional case), we cannot always take the orthogonal decomposition even if the norm is induced by an inner product. This observation naturally leads us to the notion of a Hilbert space:

Definition 7.10. Let V be a topological $\mathbb{C}$-vector space whose topology is given by the norm associated to an inner product $(-, -)$ on V. We call V a *Hilbert space* if V is complete.

Theorem 7.11 ([Rud91, Theorem 12.4])**.** *Let V be a Hilbert space. For any closed subspace W, its orthogonal complement $W^{\perp}$ satisfies $V = W \oplus W^{\perp}$.*

7.5. **Unitary representations.** Now, following the case of finite groups, let us introduce the notion of a unitary representation.

Definition 7.12. Let G be a topological group and V a Hilbert space. A *unitary representation* of G on V is a continuous representation (π, V) such that

$$(\pi(g)v, \pi(g)w) = (v, w)$$

for all $g \in G$ and $v, w \in V$.

Example 7.13. Note that the pathological example $\chi_f \colon \mathbb{R} \to \mathbb{C}$ takes values in $\mathbb{C}^1 := \{z \in \mathbb{C} \mid |z| = 1\}$. In other words, it satisfies the unitarity condition with respect to the standard norm of $\mathbb{C}$. Thus we can see that the continuity condition is independent of the unitarity condition for a representation of a topological group.

With this definition, we can prove the following by exactly the same argument as in the finite group case (Lemma 1.16).

Lemma 7.14. *Let G be a topological group and (π, V) be a unitary representation of G. Then, for any closed subrepresentation $W \subset V$, there exists a subrepresentation $W' \subset V$ such that $V = W \oplus W'$.*

Consequently, we can also show the following:

Corollary 7.15. *Any finite-dimensional unitary representation of a topological group G is semisimple.*

However, we have to note that here we crucially need the finiteness of the dimension. If the dimension of a unitary representation (π, V) is infinite, then the procedure of dividing V into smaller pieces might not stop at any stage. So at least we cannot prove the semisimplicity in general.

What we will learn starting the next week will be the case of compact topological groups, where all these problems can be resolved.

7.6. **Exercises.**

Exercise 7.1. Prove Proposition 7.2.

Exercise 7.2. Let G be a topological group which may not necessarily be Hausdorff. Show that the following are equivalent:

(1) G is Hausdorff,
(2) the trivial subgroup $\{1\} \subset G$ is closed.

8. Lecture 8: Representation theory of torus

8.1. **Irreducible representations of torus.** We consider the 1-dimensional real torus

$$\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z} \cong S^1 = \mathbb{C}^1 := \{z \in \mathbb{C} \mid |z| = 1\} \colon t \mapsto e^{it}$$

Note that this is an abelian topological group with respect to its natural topology (quotient topology of the Euclidean topology on $\mathbb{R}$).

We begin with classifying irreducible continuous finite-dimensional representations of $\mathbb{T}$.

Recall that any irreducible representation of a finite abelian group is necessarily 1-dimensional (Theorem 2.9 and Proposition 2.18). If we look at the proof of this fact, we can check that actually it is not used that the group is finite. Rather, what is important is that a given representation is finite-dimensional (necessary for finding an eigenvector). Therefore, also in the context of infinite groups, the same proof works to obtain the following (we do not even need to assume the continuity):

Proposition 8.1. *Any finite-dimensional irreducible representation of an abelian group is* 1*-dimensional.*

Let $(\chi, \mathbb{C})$ be a 1-dimensional continuous representation of $\mathbb{T}$, or equivalently, a continuous homomorphism

$$\chi \colon \mathbb{T} \to \mathbb{C}^\times.$$

Proposition 8.2. *Any continuous homomorphism $\chi \colon \mathbb{R} \to \mathbb{C}^\times$ must be of the form*

$$\chi(t) = e^{ct}$$

with some $c \in \mathbb{C}$.

Proof. Since $\chi(0) = 1 \neq 0$ and χ is continuous, we can find a sufficiently small positive real number $\varepsilon > 0$ such that

$$I := \int_0^\varepsilon \chi(s)\, ds \neq 0.$$

For any $t \in \mathbb{R}$, we have

$$I \cdot \chi(t) = \int_0^\varepsilon \chi(s+t)\, ds = \int_t^{t+\varepsilon} \chi(s)\, ds.$$

We can easily check that the right-hand side is differentiable with respect to t (use that χ is continuous), hence χ is also differentiable. To be more precise, the differentiation of this equality is

$$I \cdot \chi'(t) = \chi(t+\varepsilon) - \chi(t) = (\chi(\varepsilon) - 1) \cdot \chi(t).$$

In other words, if we let $c := \frac{\chi(\varepsilon)-1}{I}$, then

$$\chi'(t) = c \cdot \chi(t).$$

Combined with the initial condition $\chi(0) = 1$, the elementary theory of differential equations implies that

$$\chi(t) = e^{ct}.$$

□

Corollary 8.3. *Any unitary continuous homomorphism $\chi\colon \mathbb{R} \to \mathbb{C}^1$ must be of the form*

$$\chi(t) = e^{i\lambda t}$$

with some $\lambda \in \mathbb{R}$.

Proof. By Proposition 8.2, there exists $c \in \mathbb{C}$ such that $\chi(t) = e^{ct}$ for any $t \in \mathbb{R}$. Since $|\chi(t)| = 1$, c must be a purely imaginary number, hence written as $c = i\lambda$ with some $\lambda \in \mathbb{R}$. □

Corollary 8.4. *Any continuous homomorphism $\chi\colon \mathbb{T} \to \mathbb{C}^\times$ must be of the form*

$$\chi(t) = e^{i\lambda t}$$

with some $\lambda \in \mathbb{Z}$.

Proof. Since $\mathbb{T}$ is compact, χ factors through $\chi\colon \mathbb{T} \to \mathbb{C}^1 \subset \mathbb{C}^\times$ (see Exercise 8.1). Regarding χ as a unitary character of $\mathbb{R}$ by composing it with the natural projection $\mathbb{R} \twoheadrightarrow \mathbb{R}/2\pi\mathbb{Z} = \mathbb{T}$, Corollary 8.3 implies that there exists $\lambda \in \mathbb{R}$ such that $\chi(t) = e^{i\lambda t}$ for any $t \in \mathbb{R}$. Since χ must be periodic with respect to $2\pi\mathbb{Z}$-translation, i.e.,

$$\chi(t + 2\pi n) = e^{i\lambda(t+2\pi n)} = e^{i\lambda t} = \chi(t)$$

for any $n \in \mathbb{Z}$ and $t \in \mathbb{R}$, we see that $\lambda \in \mathbb{Z}$. (Conversely, if $\lambda \in \mathbb{Z}$, then $e^{i\lambda t}$ is periodic.) □

8.2. **Appearance of L^2-space.** Recall that, when G is a finite group, the regular representation of G on

$$\mathbb{C}[G] := \{f\colon G \to \mathbb{C}\}$$

has the following decomposition:

$$\mathbb{C}[G] \cong \bigoplus_{\pi \in \mathrm{Irr}(G)} \pi^{\oplus \dim \pi}.$$

Let us next consider developing an analogous result for $\mathbb{T}$. First, we obviously see that the space of all $\mathbb{C}$-valued functions on $\mathbb{T}$

$$\mathbb{C}[\mathbb{T}] := \{f\colon \mathbb{T} \to \mathbb{C}\}$$

should be too large to be studied; it contains tons of crazy functions. Then, how about focusing on continuous functions:

$$C(\mathbb{T}) := \{f\colon \mathbb{T} \to \mathbb{C} \mid \text{continuous}\}.$$

As explained in the last lecture, in order to talk about "continuous" representations of $\mathbb{T}$, we also have to think about which kind of topology we give on $C(\mathbb{T})$; ideally, we want $C(\mathbb{T})$ to be a Hilbert space.

Actually, we already have encountered a toy model of a candidate Hilbert space structure before in this course. Recall that, when G is a finite group, the following inner product played an important role:

$$(f_1, f_2)_G := \frac{1}{|G|} \sum_{g \in G} f_1(g) \cdot \overline{f_2(g)}.$$

So let us just imitate this definition by noting that the sum should be replaced with the integration over $\mathbb{T}$ since $\mathbb{T}$ is infinite:

$$(f_1, f_2)_{\mathbb{T}} := \int_{t \in \mathbb{T}} f_1(t) \cdot \overline{f_2(t)}\, dt,$$

where dt is the measure on $\mathbb{T}$ naturally induced from the Lebesgue measure on $\mathbb{R}$ (to be more precise, transfer the Lebesgue measure on the interval $[0, 2\pi]$ to $\mathbb{T}$ and normalize it so that the total volume is 1). At least this inner product on $C(\mathbb{T})$ defines a norm, hence gives a topological $\mathbb{C}$-vector space structure. We can check that the right-translation action of $\mathbb{T}$ on $C(\mathbb{T})$ is continuous with respect to this topology. Furthermore, we can also verify that the resulting "regular representation" $(R, C(\mathbb{T}))$ satisfies the unitarity condition. Indeed, for any $f_1, f_2 \in C(\mathbb{T})$ and $s \in \mathbb{T}$, we have

$$\begin{aligned}(R(s)f_1, R(s)f_2)_{\mathbb{T}} &= \int_{t\in\mathbb{T}} (R(s)f_1)(t) \cdot \overline{(R(s)f_2)(t)}\, dt \\ &= \int_{t\in\mathbb{T}} f_1(t+s) \cdot \overline{f_2(t+s)}\, dt \\ &= \int_{t\in\mathbb{T}} f_1(t) \cdot \overline{f_2(t)}\, dt = (f_1, f_2)_{\mathbb{T}}.\end{aligned}$$

This sounds great. However, it turns out that $C(\mathbb{T})$ is NOT complete with respect to this topology, i.e., $C(\mathbb{T})$ is not a Hilbert space! Based on this, we see that the next natural candidate is the completion of $C(\mathbb{T})$ with respect to the L^2-norm. Here, let us introduce the general notion of an L^2-space.

Definition 8.5 (L^2-space)**.** Let (X, μ) be a measure space. For any square-integrable functions f_1 and f_2 on X, we put

$$(f_1, f_2) := \int_X f_1(x) \cdot \overline{f_2(x)} d\mu.$$

(Here, with this notation, "square-integrable" means that (f_1, f_1) and (f_2, f_2) are finite; with this assumption, (f_1, f_2) is also finite.) We define the *L^2-space of X* by

$$L^2(X) = L^2(X, \mu) := \{f \colon X \to \mathbb{C} \mid f \text{ is measurable and } \|f\|_2 < \infty\}/\sim,$$

where $\|f\|_2 := (f, f)^{\frac{1}{2}}$ and the symbol $\sim$ denotes the equivalence relation such that $f_1 \sim f_2$ if and only if $\{x \in X \mid f_1(x) \neq f_2(x)\}$ has measure 0.

Theorem 8.6 (Riesz–Fischer's theorem, [Rud87, Theorem 3.11])**.** *The L^2-space $L^2(X)$ is a Hilbert space with respect to the inner product $(-,-)$.*

We now recall a general fact. Let X be a locally compact Hausdorff space equipped with a Radon measure μ. (Here, we do not explain the definition of a Radon measure; we just postpone it to the next lecture. The only important point here is that the natural measure on $\mathbb{T}$ is a Radon measure.) If we let

$$C_c(X) := \{f \colon X \to \mathbb{C} \mid f \text{ is continuous and has compact support}\},$$

then any element of $C_c(X)$ is L^2. Consequently, we get a map

$$C_c(X) \to L^2(X).$$

Furthermore, we have the following:

Theorem 8.7 ([Rud87, Theorem 3.14])**.** *The image of $C_c(X)$ is dense in $L^2(X)$. In other words, the completion of the image of $C_c(X)$ with respect to the L^2-norm is isomorphic to $L^2(X)$.*

In particular, since our group $\mathbb{T}$ is compact, we have $C(\mathbb{T}) = C_c(\mathbb{T}) \subset L^2(\mathbb{T})$. By this theorem, it turns out that $L^2(\mathbb{T})$ is nothing but the completion of $C(\mathbb{T})$.

With this, we are ready to proceed. In other words, for infinite groups (at least for a nice class of groups like our $\mathbb{T}$), the L^2-space is a natural choice of space on which to consider the "regular representation".

Example 8.8. Similarly to the last lecture, we consider the $\mathbb{C}$-vector space $C_c(\mathbb{Z})$ of sequences of complex numbers with only finitely many nonzero entries. This is nothing but the space of $\mathbb{C}$-valued functions on $\mathbb{Z}$ with compact support, where $\mathbb{Z}$ is regarded as a discrete topological group. Since $\mathbb{Z}$ with counting measure satisfies the assumption of the previous theorem, we have that $L^2(\mathbb{Z})$ is the completion of $C_c(\mathbb{Z})$ with respect to the L^2-norm.

8.3. **Fourier analysis on $\mathbb{T}$.** For $n \in \mathbb{Z}$, let us write $\chi_n \in L^2(\mathbb{T})$ for the function

$$\chi_n(t) = e^{int}.$$

Note that

$$\langle \chi_n, \chi_m \rangle_{\mathbb{T}} = \int_{\mathbb{T}} e^{int} \cdot \overline{e^{imt}}\, dt = \int_{\mathbb{T}} e^{i(n-m)t}\, dt = \delta_{nm}$$

(δ_{nm} denotes the Kronecker delta). In other words, $\{\chi_n \mid n \in \mathbb{Z}\}$ is an orthonormal system of $L^2(\mathbb{T})$. Classical theory of Fourier analysis asserts that this system is in fact an orthonormal basis in the sense of Hilbert space;

Theorem 8.9 (Fourier series expansion, [Fol16, Corollary 4.27])**.** *For any $f \in L^2(\mathbb{T})$, we have*

$$f = \sum_{n \in \mathbb{Z}} \hat{f}(n) \chi_n,$$

where $\hat{f}(n) := (f, \chi_n)_{\mathbb{T}}$. (Here, the infinite series on the right-hand side converges with respect to the L^2-norm.) In other words, we have the decomposition (as a Hilbert space)

$$L^2(\mathbb{T}) \cong \widehat{\bigoplus_{n \in \mathbb{Z}}} \mathbb{C} \cdot \chi_n.$$

Note that, each $\mathbb{C}\chi_n \subset L^2(\mathbb{T})$ is a 1-dimensional subrepresentation "χ_n" studied before. Indeed, for any $s \in \mathbb{T}$, we have

$$(R(s)\chi_n)(t) = \chi_n(t+s) = \chi_n(s) \cdot \chi_n(t).$$

This means that $\chi_n \in L^2(\mathbb{T})$ is an eigenvector with respect to the right-translation action of $\mathbb{T}$ with eigenvalue χ_n. Therefore, the above decomposition of $L^2(\mathbb{T})$ also gives the irreducible decomposition as a representation of $\mathbb{T}$. The only difference from the finite group situation is that we need to take the completion of the direct sum to get a Hilbert space.

Remark 8.10. By taking the square of the L^2-norm of the Fourier expansion (and using the orthonormality of $\{\chi_n\}_{n \in \mathbb{Z}}$), we obtain the formula

$$\|f\|_2^2 = \sum_{n \in \mathbb{Z}} |\hat{f}(n)|^2.$$

This is the identity known as Parseval's identity. Note that, if we regard $\hat{f}$ as a function $\mathbb{Z} \to \mathbb{C}\colon n \mapsto \hat{f}(n)$, then the right-hand side of Parseval's identity is nothing but the L^2-norm of $\hat{f}$. Therefore, the above identity says that

- $\hat{f}$ belongs to $L^2(\mathbb{Z})$, and
- the map $f \mapsto \hat{f}$ preserves the L^2-norm.

As the bijectivity of the map $f \mapsto \hat{f}$ can be checked easily, we can understand that the classical Fourier expansion says that we have the isomorphism of Hilbert spaces

$$L^2(\mathbb{T}) \xrightarrow{\cong} L^2(\mathbb{Z})\colon f \mapsto \hat{f}.$$

8.4. **Fourier analysis on $\mathbb{R}$.** We next consider the case of $\mathbb{R}$. At least we already have classified all unitary finite-dimensional irreducible representations of $\mathbb{R}$. (In fact, $\mathbb{R}$ has no infinite-dimensional unitary irreducible representation.) They are given by $\{\chi_\lambda\colon t \mapsto e^{i\lambda t}\}_{\lambda\in\mathbb{R}}$. Then, can we expect we have a decomposition of $L^2(\mathbb{R})$ similar to $L^2(\mathbb{T})$? The answer is NO.

To see this, let us pretend that we have a decomposition

$$L^2(\mathbb{R}) \cong \widehat{\bigoplus_{\lambda\in\mathbb{R}}} \mathbb{C}\cdot\chi_\lambda.$$

If we let $f \in L^2(\mathbb{R})$ be a function contained in $\mathbb{C}\cdot\chi_\lambda$, then $\mathbb{R}$ acts on f with eigenvalue χ_λ. In other words, for any $s \in \mathbb{R}$,

$$(R(s)f)(t) = f(t+s) = \chi_\lambda(s)\cdot f(t)$$

for almost all $t \in \mathbb{R}$ (recall that an element of $L^2(\mathbb{R})$ is an equivalence class). Since χ_λ is unitary, this implies that $|f(t+s)| = |f(t)|$ for any $t, s \in \mathbb{R}$. Combining this observation with the L^2-condition on f, i.e.,

$$\int_{\mathbb{R}} f(t)\cdot\overline{f(t)}\,dt < \infty,$$

we see that f necessarily satisfies $|f(t)| = 0$ for almost all $t \in \mathbb{R}$ (except for some measure zero set), which means that $f = 0$.

So, this argument also shows that $L^2(\mathbb{R})$ does not contain any irreducible subrepresentation! (As mentioned above, we can show that $\mathbb{R}$ does not have an infinite-dimensional unitary irreducible representation. Therefore, we can say that $L^2(\mathbb{R})$ is a highly reducible representation which does not contain any irreducible subrepresentation.)

Then, why does such a big difference arise between $\mathbb{R}$ and $\mathbb{T}$? The point is that $\mathbb{T}$ is *compact* but $\mathbb{R}$ is not.

Remark 8.11. The classical Fourier inversion formula asserts that, for any $f \in L^2(\mathbb{R})$ satisfying enough nice conditions ("Schwartz functions"), we have

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \hat{f}(\lambda)\chi_\lambda(x)\,d\lambda,$$

where $\hat{f}$ denotes the Fourier transform

$$\hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(x)\chi_{-\lambda}(x)\,dx.$$

(See, e.g., [Fol16, Theorem 4.33]) This looks quite similar to the Fourier expansion theorem, but the infinite sum over $\mathbb{Z}$ in the Fourier expansion is replaced with the integration over $\mathbb{R}$ here. This suggests that the "irreducible decomposition" of $L^2(\mathbb{R})$ should be formulated in terms of something continuously generalizing the notion of the direct sum; it is called the "direct integral", but it is far out of reach of this course.

8.5. **Towards the Peter–Weyl theorem.** Based on the discussion so far, I hope that we are now sure that compact topological groups should be the natural next object to be studied after finite groups. Our aim starting the next week is the following:

Theorem 8.12 (Peter–Weyl's theorem)**.** *Let G be a compact topological group. Let* $\mathrm{Irr}(G)$ *denote the set of isomorphism classes of finite-dimensional irreducible unitary representations of G. Then we have a decomposition*

$$L^2(G) \cong \widehat{\bigoplus_{\pi \in \mathrm{Irr}(G)}} \pi^{\oplus \dim \pi}.$$

This theorem is beautiful and satisfactory from our viewpoint. Also, keeping the discussion above in mind, we may regard this theorem as a generalization of the classical Fourier theory ("non-abelian Fourier analysis").

8.6. **Exercises.**

Exercise 8.1. Let G be a compact group.

(1) Prove that the image of any continuous character $\chi\colon G \to \mathbb{C}^\times$ is contained in $\mathbb{C}^1$.
(2) Prove that the image of any continuous character $\chi\colon G \to \mathbb{R}^\times$ is contained in $\{\pm 1\}$.

Exercise 8.2. Determine all the continuous characters of $\mathrm{GL}_n(\mathbb{R})$.

Exercise 8.3. Determine all the continuous characters of the group

$$G := \prod_{n=0}^{\infty} \mathbb{Z}/2\mathbb{Z},$$

where G is equipped with the product topology.

9. Lecture 9: Haar measure and Schur's orthogonality

9.1. **Haar measure.** Our next aim is to prove Peter–Weyl's theorem. We first remark that the statement of Theorem 8.12 is actually not clear yet at present; this is because the choice of a measure on G has not been clarified (recall that the definition of L^2-space depends also on the choice of a measure). In the cases of $\mathbb{T}$ or $\mathbb{R}$, we just chose the standard Lebesgue measure. However, is there any "standard" choice of a measure for general compact groups? The point here is that actually we have secretly used some special feature of the Lebesgue measure on $\mathbb{T}$ in the above discussion. In the computation

$$(R(s)f_1, R(s)f_2)_{\mathbb{T}} = \int_{t\in\mathbb{T}} f_1(t+s)\cdot\overline{f_2(t+s)}\,dt = \int_{t\in\mathbb{T}} f_1(t)\cdot\overline{f_2(t)}\,dt = (f_1,f_2)_{\mathbb{T}},$$

we appealed to the translation-invariance of dt; this enabled us to do the change-of-variable argument $(t+s\mapsto t)$. Therefore, we must consider whether we can find such a nice measure also for general compact groups.

Let us begin with reviewing some basics of measure theory. Recall that, for a topological space X, a measure on the minimal σ-algebra $\mathcal{B}(X)$ generated by open subsets of X is called a *Borel measure*.

Definition 9.1. Let X be a locally compact Hausdorff topological space. We say that a Borel measure μ on X is a *Radon measure* if it satisfies the following conditions:

(1) (finiteness on compact sets) For any compact subset $K\subset X$, $\mu(K)<\infty$.
(2) (outer regularity) For any Borel subset $E\subset X$,
$$\mu(E) = \inf\{\mu(U)\mid E\subset U\colon \text{open}\}.$$
(3) (inner regularity) For any open subset $U\subset X$,
$$\mu(U) = \sup\{\mu(K)\mid K\subset U\colon \text{compact}\}.$$

Theorem 9.2 ([Fol16, Theorems 2.10, 2.20, Proposition 2.19])**.** *Let G be a locally compact topological group. Then there exists a nonzero Radon measure μ, which is right-invariant (resp. left-invariant) in the sense that*

$$\mu(Eg)=\mu(E)\quad (\textit{resp.}\ \mu(gE)=\mu(E))$$

for any Borel subset $E\subset G$. Moreover, such μ is unique up to $\mathbb{R}_{>0}$-multiplication. With respect to such a measure, the natural map $C_c(G)\to L^2(G)$ is injective.

Definition 9.3. For a locally compact topological group G, a measure as in the above theorem is called a *right (resp. left) Haar measure* on G.

Once a Radon measure μ on a locally compact topological space X is given, we can consider the integration of any continuous function with compact support. That is, we obtain a $\mathbb{C}$-linear form

$$C_c(X)\to\mathbb{C}\colon f\mapsto\int_X f(x)\,d\mu(x).$$

(The symbol $d\mu(x)$ is often simply written by dx. Or, a measure itself is written by dx.) In fact, it is known that any nonzero Radon measure is uniquely determined by this linear form. Therefore, a right Haar measure on a locally compact topological

group G is characterized as the unique (up to $\mathbb{R}_{>0}$-multiplication) nonzero Radon measure μ satisfying

$$\int_G f(g)\,d\mu(g) = \int_G f(gg_0)\,d\mu(g)$$

for any $f \in C_c(G)$ and $g_0 \in G$. (The same thing is true for a left Haar measure).

We note that a right Haar measure may not be a left Haar measure at the same time in general. When a right Haar measure of a locally compact group G is also a left Haar measure, we simply call it a "Haar measure" and say that G is *unimodular*.

Suppose that G is a locally compact topological group and μ a left Haar measure on G. For any $g \in G$, the Radon measure

$$\mathcal{B}(G) \to \mathbb{R}_{\geq 0} \sqcup \{\infty\} \colon E \mapsto \mu(Eg^{-1})$$

is again a left Haar measure. Since a left Haar measure is unique up to $\mathbb{R}_{>0}$-multiplication, there is a scalar $\delta_G(g) \in \mathbb{R}_{>0}$ such that $\mu(Eg^{-1}) = \delta_G(g)\mu(E)$ for any $E \in \mathcal{B}(G)$. We can easily check that the resulting function

$$\delta_G \colon G \to \mathbb{R}_{>0}$$

is a character (continuous homomorphism; see [Fol16, Proposition 2.24]). This character is called the *modulus character* of G. (Again noting the uniqueness of a left Haar measure up to $\mathbb{R}_{>0}$-multiplication, we easily see that δ_G is independent of the choice of μ.)

Note that a left Haar measure μ is right-invariant if and only if the modulus character δ_G is trivial. In particular, we obtain the following:

Proposition 9.4. *A locally compact topological group G is unimodular if and only if its modulus character δ_G is trivial.*

An important corollary of this proposition is as follows.

Corollary 9.5. *Any compact topological group is unimodular.*

Proof. For a compact topological group G, any continuous character $G \to \mathbb{R}_{>0}$ is necessarily trivial (Exercise 8.1). In particular, the modulus character is trivial. □

Example 9.6. Let G be a finite group, regarded as a discrete topological group. Then the counting measure

$$\mu(E) := |E| \quad \text{for any } E \subset G$$

(or its normalized version $\mu(E) := |E|/|G|$) is obviously a Haar measure.

Example 9.7. Let $G := \mathbb{R}$, regarded as a locally compact topological group with respect to the addition. Then the Lebesgue measure is a Haar measure. Also, the natural measure on $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$ induced by the Lebesgue measure of $\mathbb{R}$ is a Haar measure.

Example 9.8. Let $G := \mathrm{GL}_n(\mathbb{R})$. Then G is a locally compact topological group with respect to the induced topology from $\mathrm{GL}_n(\mathbb{R}) \subset M_n(\mathbb{R}) \cong \mathbb{R}^{\oplus n^2}$. Since $\mathrm{GL}_n(\mathbb{R})$ is an open subset of $M_n(\mathbb{R})$, we can restrict the Lebesgue measure $dX := \prod_{1 \leq i,j \leq n} dx_{ij}$ on $M_n(\mathbb{R})$ to $\mathrm{GL}_n(\mathbb{R})$. We define a measure dg on G by multiplying $\prod_{1 \leq i,j \leq n} dx_{ij}$ by the character $|\det|^{-n} \colon G \to \mathbb{R}_{>0}$. In other words, dg is the Radon measure on G characterized by

$$\int_{\mathrm{GL}_n(\mathbb{R})} f(g)\,dg = \int_{\mathrm{GL}_n(\mathbb{R})} f(g) \cdot |\det(g)|^{-n}\,dX$$

for every $f \in C_c(G)$. Then dg is a left Haar measure. Indeed, for any $g_0 \in G$, the automorphism $M_n(\mathbb{R}) \to M_n(\mathbb{R})\colon X \mapsto g_0 X$ has Jacobian $|\det(g_0)|^n$. In particular, for any $f \in C_c(M_n(\mathbb{R}))$, we have

$$\int_{M_n(\mathbb{R})} f(g_0 X)\, dX = \int_{M_n(\mathbb{R})} f(X) \cdot |\det(g_0)|^{-n}\, dX$$

(or, you can think of this identity as the definition of the "Jacobian"). Hence, for any $f \in C_c(\mathrm{GL}_n(\mathbb{R}))$, we have

$$\begin{aligned}\int_{\mathrm{GL}_n(\mathbb{R})} f(g_0 g)\, dg &= \int_{\mathrm{GL}_n(\mathbb{R})} f(g_0 g) \cdot |\det(g)|^{-n}\, dX \\ &= \int_{\mathrm{GL}_n(\mathbb{R})} (f(g_0 g) \cdot |\det(g_0 g)|^{-n}) \cdot |\det(g_0)|^n\, dX \\ &= \int_{\mathrm{GL}_n(\mathbb{R})} f(g) \cdot |\det(g)|^{-n}\, dX = \int_{\mathrm{GL}_n(\mathbb{R})} f(g)\, dg.\end{aligned}$$

In the same manner, we can also check that dg is a right Haar measure. Therefore, dg is a Haar measure and $\mathrm{GL}_n(\mathbb{R})$ is unimodular.

9.2. **Unitarity of continuous representations of compact groups.** What we will do in the following is to show that various basic properties of representations of finite groups (studied in Lectures 1 and 2) can be reproduced to compact groups, by replacing the sum over G with integration. In the following, we let G be a compact topological group. We fix the Haar measure dg on G normalized so that $dg(G) = 1$.

Proposition 9.9. *Let (π, V) be a finite-dimensional continuous representation of G. Then π is unitary, i.e., there exists a (positive-definite Hermitian) inner product on V which is G-invariant.*

Proof. The same idea as in the proof of Proposition 1.18 works as follows.

Let $(-,-)\colon V \times V \to \mathbb{C}$ be any inner product (which may not necessarily be G-invariant). We modify $(-,-)$ by "averaging" it with respect to the G-action:

$$(-,-)'\colon V \times V \to \mathbb{C}; \quad (v, v')' := \int_G (\pi(g)(v), \pi(g)(v'))\, dg$$

(note that $\frac{1}{|G|}\sum_{g\in G}$ in the proof of Proposition 1.18 is replaced with $\int_G dg$). Here, the integration is well-defined since the function

$$G \to \mathbb{C}\colon g \mapsto (\pi(g)(v), \pi(g)(v'))$$

is continuous by the continuity of (π, V) (and, of course, compactly supported since G is compact). This is obviously again an inner product of V. Moreover, it is G-invariant; indeed, for any $h \in G$ and $v, v' \in V$, we have

$$\begin{aligned}(\pi(h)(v), \pi(h)(v'))' &= \int_G (\pi(gh)(v), \pi(gh)(v'))\, dg \\ &= \int_G (\pi(g)(v), \pi(g)(v'))\, dg = (v, v')',\end{aligned}$$

where we used the right-invariance of dg in the middle equality. □

We remark that, in this proof, the compactness of G is crucially important. Indeed, if G is not compact, then the function $g \mapsto (\pi(g)(v), \pi(g)(v'))$ is not guaranteed to be compactly supported. Thus the integral in the definition $(v, v')'$ cannot be proved to be convergent.

Now we obtain the following:

Corollary 9.10. *Any finite-dimensional continuous representation of G is semisimple, i.e., decomposes into the direct sum of irreducible continuous representations of G.*

Remark 9.11. We say that a nonzero continuous representation (π, V) of a topological group G is *irreducible* if it does not contain any nontrivial *closed* subrepresentation. Note that, in contrast to the case of finite groups, we require being closed. If V is finite-dimensional (hence equipped with the canonical topological $\mathbb{C}$-vector space structure), any subspace is automatically closed, hence we do not have to care about this subtlety.

9.3. **Schur's orthogonality.** We first note that the proof of Schur's lemma Theorem 2.9 (and also Lemma 2.8) only uses that the dimension of a representation is finite. In particular, G does not need to be finite. Hence, the same proof implies the following (here, we do not even need that G is compact):

Theorem 9.12 (Schur's lemma)**.** *Let (π, V) and (π', V') be irreducible finite-dimensional continuous representations of a topological group G. Then we have*

$$\dim_{\mathbb{C}} \mathrm{Hom}_G(\pi, \pi') = \begin{cases} 1 & \text{if } \pi \cong \pi', \\ 0 & \text{if } \pi \ncong \pi'. \end{cases}$$

In the following, we again consider a compact topological group G, equipped with the normalized Haar measure dg. Note that, for any finite-dimensional $\mathbb{C}$-vector space W and continuous W-valued function $\varphi\colon G \to W$, we can define its integration

$$\int_G \varphi(g)\, dg \in W$$

by choosing a $\mathbb{C}$-basis of W and computing the integration entrywise with respect to the $\mathbb{C}$-basis (we can check that the resulting vector does not depend on the choice of a $\mathbb{C}$-basis).

Corollary 9.13. *Let (π, V) and (π', V') be irreducible finite-dimensional continuous representations of G. For any $f \in \mathrm{Hom}_{\mathbb{C}}(V, V')$, we define $f^0 \in \mathrm{Hom}_{\mathbb{C}}(V, V')$ by*

$$f^0 := \int_G \pi'(g)^{-1} \circ f \circ \pi(g)\, dg.$$

Here, since $\pi'(g)^{-1} \circ f \circ \pi(g)$ belongs to the finite-dimensional $\mathbb{C}$-vector space $\mathrm{Hom}_{\mathbb{C}}(V, V')$, the integration makes sense as explained above. Then we have

$$f^0 = \begin{cases} 0 & \text{if } \pi \ncong \pi', \\ \frac{\mathrm{Tr}(f)}{\dim(V)} \cdot \mathrm{id}_V & \text{if } \pi \cong \pi', \end{cases}$$

where, in the second case, we assume that $\pi = \pi'$.

Proof. We first check that f^0 is a G-homomorphism:

$$\begin{aligned} f^0 \circ \pi(h) &= \int_G \pi'(g)^{-1} \circ f \circ \pi(g) \circ \pi(h)\, dg \\ &= \int_G \pi'(g)^{-1} \circ f \circ \pi(gh)\, dg \\ &= \int_G \pi'(gh^{-1})^{-1} \circ f \circ \pi(g)\, dg \\ &= \int_G \pi'(h) \circ \pi'(g)^{-1} \circ f \circ \pi(g)\, dg = \pi'(h) \circ f^0, \end{aligned}$$

where, in the third equality, we replaced gh with g by using the right-invariance of dg. Therefore, by Schur's lemma, we get

$$f^0 = \begin{cases} 0 & \text{if } \pi \not\cong \pi', \\ \lambda \cdot \mathrm{id}_V & \text{if } \pi \cong \pi', \end{cases}$$

with some scalar $\lambda \in \mathbb{C}$. Thus our remaining task is to determine the scalar λ when $\pi \cong \pi'$. We may assume $\pi = \pi'$. Then

$$\mathrm{Tr}(f^0) = \int_G \mathrm{Tr}(\pi(g)^{-1} \circ f \circ \pi(g))\, dg = \int_G \mathrm{Tr}(f)\, dg = \mathrm{Tr}(f).$$

(Here we used that the trace can be interchanged with the integration, but this can be checked by hand.) As $f^0 = \lambda \cdot \mathrm{id}_V$, we also have $\mathrm{Tr}(f^0) = \lambda \cdot \dim(V)$. Hence we get $\lambda = \mathrm{Tr}(f)/\dim(V)$. □

We continue to proceed with the same argument as in Lecture 2.

Definition 9.14. Let (π, V) be a finite-dimensional continuous representation of G. By choosing a $\mathbb{C}$-basis of V, we define a continuous function π_{ij} on G by

$$\pi_{ij} \colon G \to \mathbb{C}; \quad g \mapsto \pi(g)_{ij},$$

where $\pi(g)_{ij}$ denotes the (i,j)-entry of the representation matrix of $\pi(g) \in \mathrm{Aut}_{\mathbb{C}}(V) \cong \mathrm{GL}_n(\mathbb{C})$ ($n := \dim(V)$). We call π_{ij} a *matrix coefficient* of π.

Corollary 9.15. *Let (π, V) and (π', V') be irreducible finite-dimensional continuous representations of G.*

(1) *If $\pi \not\cong \pi'$, then, for any i, j, p, q,*

$$\int_G \pi'_{ij}(g^{-1}) \cdot \pi_{pq}(g)\, dg = 0.$$

(2) *If $\pi \cong \pi'$ (assume $\pi = \pi'$), then, for any i, j, p, q,*

$$\int_G \pi_{ij}(g^{-1}) \cdot \pi_{pq}(g)\, dg = \begin{cases} \frac{1}{\dim(V)} & \text{if } i = q \text{ and } j = p, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let us take any $f \in \mathrm{Hom}_{\mathbb{C}}(V, V')$ and write it as $f = (f_{kl})_{kl}$ using the fixed $\mathbb{C}$-bases of V and V'. Then each entry of the representation matrix $(f^0_{iq})_{iq}$ of f^0 is given by:

$$f^0_{iq} = \int_G \sum_{j,p} \pi'_{ij}(g^{-1}) \cdot f_{jp} \cdot \pi_{pq}(g)\, dg.$$

We first consider the case where $\pi \not\cong \pi'$. In this case, by the above corollary, we have $(f^0_{iq})_{iq} = 0$ for any $f = (f_{kl})_{kl}$. Hence, the above equality implies that

$$\int_G \sum_{j,p} \pi'_{ij}(g^{-1}) \cdot f_{jp} \cdot \pi_{pq}(g)\, dg = 0$$

for each i, q (for any $f = (f_{kl})_{kl}$). Since f can vary arbitrarily, by looking at the coefficients of each f_{jp} on both sides of this equality, we get

$$\int_G \pi'_{ij}(g^{-1}) \cdot \pi_{pq}(g)\, dg = 0$$

for each i, j, p, q.

We next consider the case where $\pi = \pi'$. In this case, for any $f = (f_{kl})_{kl}$, f^0 is represented by the scalar matrix with diagonal entries $\mathrm{Tr}(f)/\dim(V)$. In other words, we have

$$\int_G \sum_{j,p} \pi'_{ij}(g^{-1}) \cdot f_{jp} \cdot \pi_{pq}(g)\, dg = \begin{cases} \frac{1}{\dim(V)}(f_{11} + \cdots + f_{nn}) & i = q, \\ 0 & i \neq q, \end{cases}$$

where $n = \dim(V)$. Therefore, we get

$$\int_G \pi'_{ij}(g^{-1}) \cdot \pi_{pq}(g)\, dg = \begin{cases} \frac{1}{\dim(V)} & i = q \quad \text{and} \quad j = p, \\ 0 & \text{otherwise.} \end{cases}$$

□

Now we define an inner product $(-,-)_G$ on $L^2(G)$ by

$$(f_1, f_2)_G := \int_G f_1(g) \cdot \overline{f_2(g)}\, dg.$$

Also, for any finite-dimensional continuous representation (π, V) of G, we define its *trace character* Θ_π in the same manner as before:

$$\Theta_\pi \colon G \to \mathbb{C} \colon g \mapsto \mathrm{tr}(\pi(g)).$$

Note that this is a continuous function. Furthermore, we have

$$\Theta_\pi(g^{-1}) = \overline{\Theta_\pi(g)}$$

for any $g \in G$. Indeed, since π is unitary, if we fix any G-invariant inner product on V and an orthonormal basis $\{e_1, \ldots, e_n\}$ of V, then we have

$$\Theta_\pi(g^{-1}) = \sum_{i=1}^n (\pi(g^{-1})e_i, e_i) = \sum_{i=1}^n (e_i, \pi(g)e_i) = \sum_{i=1}^n \overline{(\pi(g)e_i, e_i)} = \overline{\Theta_\pi(g)}.$$

Theorem 9.16. *Let (π, V) and (π', V') be finite-dimensional irreducible continuous representations of G. Then we have*

$$(\Theta_\pi, \Theta_{\pi'}) = \begin{cases} 0 & \text{if } \pi \not\cong \pi', \\ 1 & \text{if } \pi \cong \pi'. \end{cases}$$

Proof. By fixing $\mathbb{C}$-bases of V and V', we consider the matrix coefficients of π and π' as above. Then we have

$$\begin{aligned}(\Theta_\pi, \Theta_{\pi'}) &= \int_G \Theta_\pi(g) \cdot \overline{\Theta_{\pi'}(g)}\, dg = \int_G \Theta_\pi(g) \cdot \Theta_{\pi'}(g^{-1})\, dg \\ &= \int_G \Big(\sum_i \pi_{ii}(g)\Big) \cdot \Big(\sum_p \pi'_{pp}(g^{-1})\Big)\, dg\end{aligned}$$

If $\pi \not\cong \pi'$, then Corollary 9.15 (1) immediately implies that this is equal to 0. Thus let us suppose that $\pi = \pi'$. Again by Corollary 9.15 (2), we get

$$\int_G \Big(\sum_i \pi_{ii}(g)\Big) \cdot \Big(\sum_p \pi'_{pp}(g^{-1})\Big)\, dg = \sum_i \frac{1}{\dim(V)} = 1.$$

□

9.4. **Exercises.**

Exercise 9.1. Let μ be a Haar measure on $\mathrm{GL}_n(\mathbb{R})$. Let θ be the automorphism of $\mathrm{GL}_n(\mathbb{R})$ given by $\theta(g) := {}^t g^{-1}$. We let μ' be the pull-back of μ via θ, i.e., for any measurable set E, $\mu'(E) := \mu(\theta(E))$. Show that $\mu' = \mu$.

Exercise 9.2. Let G be a compact topological group equipped with Haar measure dg such that $dg(G) = 1$. Let (π, V) be an irreducible finite-dimensional continuous representation of G with $d = \dim(V)$.

(1) Show that, for every $h \in G$,

$$\int_G \Theta_\pi(ghg^{-1}h^{-1})\, dg = \frac{|\Theta_\pi(h)|^2}{d}.$$

(2) Show that

$$\int_G \int_G \Theta_\pi(ghg^{-1}h^{-1})\, dg\, dh = \frac{1}{d}.$$

10. Lecture 10: Peter–Weyl's theorem

Recall that we have stated the following theorem in a previous lecture:

Theorem 10.1 (Peter–Weyl's theorem)**.** *Let G be a compact topological group. Let $\mathrm{Irr}(G)$ denote the set of isomorphism classes of finite-dimensional irreducible unitary representations of G. Then we have a decomposition*

$$L^2(G) \cong \widehat{\bigoplus_{\pi \in \mathrm{Irr}(G)}} \pi^{\oplus \dim \pi}.$$

Today's aim is to explain the proof of this theorem. (For the main reference, I would like to suggest [Fol16, Chapter 5] although the proof given here is not exactly following that.)

10.1. **Matrix coefficients: revisited.** Let $(\pi, V) \in \mathrm{Irr}(G)$, i.e., (π, V) is a finite-dimensional irreducible unitary representation of G. Recall that matrix-entry functions of representations $\pi_{ij} \colon G \to \mathbb{C}$ are called "matrix coefficients" of π. We introduce a more sophisticated version of this notion as follows:

Definition 10.2. For any $v \in V$ and $v^\vee \in V^\vee$, we define a function $f_{v,v^\vee} \colon G \to \mathbb{C}$ by

$$f_{v,v^\vee}(g) := \langle \pi^\vee(g) v^\vee, v \rangle.$$

We call $f_{v,v^\vee}$ the matrix coefficient (associated to $v, v^\vee$).

You may easily realize that the function $g \mapsto \pi_{ij}(g^{-1})$ is obtained by choosing v and $v^\vee$ to be any basis vector of V and its dual basis vector.

The association $v \otimes v^\vee \mapsto f_{v,v^\vee}$ induces a well-defined $\mathbb{C}$-linear map

$$V \otimes V^\vee \to C(G) \subset L^2(G) \colon v \otimes v^\vee \mapsto f_{v,v^\vee}.$$

Here, note that the left-hand side can be viewed as a representation of $G \times G$. Also, the right-hand side can be viewed as a representation of $G \times G$; let us explain this more precisely. So far, we have only considered the right-translation action R of G on $L^2(G)$, which is given by

$$R(g)(f)(x) := f(xg)$$

for any $g \in G$ and $f \in L^2(G)$. However, we can also consider the left-translation action L of G on $L^2(G)$, which is given by

$$L(g)(f)(x) := f(g^{-1}x).$$

Obviously these actions commute, hence we have an action $L \times R$ of $G \times G$ on $L^2(G)$, which is given by

$$(L \times R)(g_1, g_2)(f)(x) := f(g_1^{-1} x g_2).$$

Lemma 10.3. *The $\mathbb{C}$-linear map*

$$V \otimes V^\vee \to C(G) \subset L^2(G) \colon v \otimes v^\vee \mapsto f_{v,v^\vee}$$

preserves $G \times G$-actions, i.e., is a homomorphism as representations of $G \times G$.

Proof. This easily follows from the definition:

$$\begin{aligned} f_{\pi(g_1)v,\pi^\vee(g_2)v^\vee}(x) &:= \langle \pi^\vee(x)\pi^\vee(g_2)v^\vee, \pi(g_1)v\rangle \\ &= \langle \pi^\vee(g_1^{-1}xg_2)v^\vee, v\rangle \\ &= f_{v,v^\vee}(g_1^{-1}xg_2) = (L\times R)(g_1,g_2)(f_{v,v^\vee})(x). \end{aligned}$$

□

We modify the matrix coefficient map slightly as follows:

$$\Phi_\pi \colon V\otimes V^\vee \to C(G)\subset L^2(G)\colon v\otimes v^\vee \mapsto (\dim V)^{\frac{1}{2}}\cdot f_{v,v^\vee}.$$

Proposition 10.4. *This modified map is an injective homomorphism as unitary representations of $G\times G$. Furthermore, for any nonisomorphic (π_1,V_1), $(\pi_2,V_2)\in \mathrm{Irr}(G)$, the images of Φ_{π_1} and Φ_{π_2} are orthogonal to each other in $L^2(G)$.*

Proof. This is a consequence of Schur's orthogonality relation (Corollary 9.15). □

Consequently, we have seen that the injective homomorphisms $\Phi_\pi\colon V\otimes V^\vee \hookrightarrow L^2(G)$ induce an injective homomorphism of unitary representations of $G\times G$

$$\widehat{\bigoplus_{(\pi,V)\in\mathrm{Irr}(G)}} V\otimes V^\vee \hookrightarrow L^2(G).$$

(To be more precise, we may first view the algebraic direct sum $\bigoplus_{(\pi,V)\in\mathrm{Irr}(G)} V\otimes V^\vee$ as a subspace of $L^2(G)$. Then we consider its closure in $L^2(G)$, which is identified with the completion of the algebraic direct sum.)

In fact, what we will show in the following is a stronger version of Theorem 10.1 as follows:

Theorem 10.5. *The homomorphism*

$$\widehat{\bigoplus_{(\pi,V)\in\mathrm{Irr}(G)}} V\otimes V^\vee \hookrightarrow L^2(G)$$

is in fact an isomorphism as unitary representations of $G\times G$.

(Note that Theorem 10.1 follows from this theorem just by forgetting the action L of the first G of $G\times G$.)

So the main part of the proof of Peter–Weyl's theorem is how to show the surjectivity of the map.

10.2. **Key Proposition.** The key to the proof of Peter–Weyl's theorem is the following:

Proposition 10.6. *Let G be a compact topological group. For any nonzero unitary continuous representation (π,V) of G, there exists a finite-dimensional irreducible subrepresentation $W\subset V$.*

Note that, as an immediate consequence of this proposition, we also have the following corollary:

Corollary 10.7. *Any irreducible unitary continuous representation of a compact group G is finite-dimensional.*

Let us first see that we can complete the proof of Peter–Weyl's theorem by admitting Proposition 10.6.

Proof of Theorem 10.5. Our task is to show the surjectivity of the map

$$\overbrace{\bigoplus_{(\pi,V)\in \mathrm{Irr}(G)} V\otimes V^\vee} \hookrightarrow L^2(G).$$

Let $W \subset L^2(G)$ be the orthogonal complement of $\bigoplus_{(\pi,V)\in \mathrm{Irr}(G)} V\otimes V^\vee$. It suffices to show that $W = \{0\}$. For the sake of contradiction, we assume that $W \neq \{0\}$.

Since $\bigoplus_{(\pi,V)\in \mathrm{Irr}(G)} V\otimes V^\vee$ is invariant under $G\times G$-action, W is a $G\times G$-subrepresentation of $L^2(G)$. By forgetting the left-translation, let us view W as a G-subrepresentation of $L^2(G)$. By using Proposition 10.6, we may find a nonzero finite-dimensional irreducible unitary G-subrepresentation W_f of W. We write (τ, W_f) for this representation of G.

If we fix an orthonormal basis $\{w_1,\dots,w_n\}$ of W_f, then we get matrix coefficients $\{\tau_{ij}\}_{ij}$. In other words, for any $g\in G$ and $1\le j\le n$, we have

$$\tau(g)w_j = \sum_{i=1}^n \tau_{ij}(g)w_i.$$

Here, recall that w_i is an element of $W_f \subset W \subset L^2(G)$, hence it is a function on G. As τ is the restriction of the right-translation action of G on $L^2(G)$ to W_f, evaluating the above identity at $1\in G$ gives

$$w_j(g) = (\tau(g)w_j)(1) = \sum_{i=1}^n \tau_{ij}(g)w_i(1).$$

(Precisely speaking, when we view the above identity in $L^2(G)$, it asserts that two functions on G coincide outside some measure zero set. Hence, here we need to take w_i to be a continuous function on G so that we can evaluate the identity at $1\in G$. In this situation, this is possible by the assumption that G is compact; see [Fol16, Propositions 2.41, 2.44].) In other words, the function w_j can be written as a $\mathbb{C}$-linear combination of matrix coefficients $\{\tau_{ij}\}$. This means that w_j belongs to $\bigoplus_{(\pi,V)\in \mathrm{Irr}(G)} V\otimes V^\vee$. Consequently, we obtain $W_f \subset \bigoplus_{(\pi,V)\in \mathrm{Irr}(G)} V\otimes V^\vee$, which is a contradiction. □

10.3. **Idea of the proof of Key Proposition.** Here let us again go back to the proof of Schur's lemma for finite-dimensional irreducible representation. To be more precise, let G be any group and (π, V) be its finite-dimensional irreducible representation. Then Schur's lemma says that $\mathrm{End}_G(V) = \mathbb{C}\cdot \mathrm{id}_V$. To show this, let us take any $f\in \mathrm{End}_G(V)\smallsetminus\{0\}$. The point is that, since V is finite-dimensional, we can always find a nonzero eigenvalue λ of f. Then the kernel $\mathrm{Ker}(f-\lambda\cdot \mathrm{id}_V)$ (i.e., the eigenspace of f with eigenvalue λ) is a nonzero subrepresentation of V. Hence the irreducibility of V implies that $V = \mathrm{Ker}(f-\lambda\cdot\mathrm{id}_V)$, which means that $f = \lambda\cdot \mathrm{id}_V$.

So, what is important here is the following elementary observation:

> For any representation (π, V) of G, if we can find a G-endomorphism $f\in \mathrm{End}_G(V)$ and a nonzero eigenvalue λ, then its eigenspace $\mathrm{Ker}(f-\lambda\cdot\mathrm{id}_V)$ gives a nonzero subrepresentation of V.

The main idea of the proof of Proposition 10.6 is to realize a finite-dimensional subrepresentation as an eigenspace of a certain G-endomorphism. However, in general (when we think about infinite-dimensional $\mathbb{C}$-vector spaces), it is not always guaranteed that an endomorphism has a nonzero eigenvalue. Therefore, so that our

"main idea" works, we must consider a nice class of endomorphisms of a Hilbert space so that we can always find an eigenvalue.

Before we introduce various notions related to Hilbert spaces (from functional analysis), let us also go back to some basic linear algebra to get any sense. Recall the following:

Proposition 10.8. *Let $A \in M_n(\mathbb{C})$ be a Hermitian matrix, i.e., ${}^t\bar{A} = A$. Then A can be diagonalized to a real diagonal matrix via a unitary matrix. That is, there exists a matrix $Q \in \mathrm{GL}_n(\mathbb{C})$ such that ${}^t\bar{Q}^{-1} = Q$ and $Q^{-1}AQ$ is a real diagonal matrix.*

Let us interpret this statement without appealing to matrices. We first set $V := \mathbb{C}^{\oplus n}$ to be the n-dimensional $\mathbb{C}$-vector space equipped with the standard inner product, i.e., for any $x = {}^t(x_1, \ldots, x_n) \in V$ and $y = {}^t(y_1, \ldots, y_n) \in V$, we have

$$(x, y) := ({}^t x)\bar{y} = \sum_{i=1}^{n} x_i \overline{y_i}.$$

Note that, for an invertible matrix $Q \in \mathrm{GL}_n(\mathbb{C}) = \mathrm{Aut}_{\mathbb{C}}(V)$, we have

$$(Qx, Qy) := {}^t(Qx)\bar{Q}\bar{y} = {}^t x({}^t Q\bar{Q})\bar{y}.$$

Thus, for Q, being unitary is exactly preserving the inner product (i.e., unitary endomorphism with respect to the Hilbert space structure of V). In particular, if we replace the standard basis of V using Q as the change-of-basis matrix, then the resulting basis is again an orthonormal basis of V.

On the other hand, for $A \in M_n(\mathbb{C})$, we have

$$(Ax, y) = {}^t(Ax)\bar{y} = {}^t x\, {}^t A\bar{y} = (x, {}^t\overline{A}y).$$

Therefore, for $A \in M_n(\mathbb{C})$, being Hermitian is equivalent to satisfying the following identity for any $x, y \in V$:

$$(Ax, y) = (x, Ay).$$

Keeping these observations in mind, let us rewrite the above proposition:

Proposition 10.9. *Let $A \in \mathrm{End}_C(V)$ be an endomorphism satisfying the identity $(Ax, y) = (x, Ay)$ for any $x, y \in V$. Then we can find a finite set of nonzero real numbers $\{\lambda_i\}_{i \in I}$ such that*

$$V = V_0 \oplus \bigoplus_{i \in I} V_{\lambda_i},$$

where $V_\lambda := \mathrm{Ker}(A - \lambda \cdot \mathrm{id}_V)$ for any $\lambda \in \mathbb{R}$.

So this proposition tells us that, once an additional structure on a $\mathbb{C}$-vector space (i.e., a Hilbert space structure) is given, and also the assumption that $A \in \mathrm{End}_{\mathbb{C}}(V)$ satisfies a nice property (i.e., $(Ax, y) = (x, Ay)$), we can find a particularly nice eigenspace decomposition of V.

What we next see is how these phenomena can be generalized to Hilbert spaces.

10.4. **Spectral decomposition theorem.** In the following, we let V be a Hilbert space over $\mathbb{C}$ equipped with an inner product $(-,-)_V$ and a norm $\|\cdot\|_V$.

Definition 10.10. Let $T \in \mathrm{End}_{\mathbb{C}}(V)$ be any $\mathbb{C}$-linear endomorphism of V.

(1) We define the *operator norm* $\|T\| \in \mathbb{R} \cup \{\infty\}$ by

$$\|T\| := \sup_{\substack{v \in V \\ \|v\|_V = 1}} \|T(v)\|_V.$$

We call T a *bounded operator* if $\|T\| < \infty$.

(2) We call T a *finite-rank operator* if T is bounded and $\mathrm{Im}(T)$ is a finite-dimensional subspace of V.

(3) We call T a *compact operator* if T is bounded and the image of

$$B := \{v \in V \mid \|v\|_V < 1\}$$

under T is relatively compact in V.

We write $\mathcal{B}(V)_f$, $\mathcal{B}(V)_c$, $\mathcal{B}(V)$ for the set of finite-rank, compact, bounded operators, respectively.

Fact 10.11 ([Rud91, Theorem 4.18 (c)])**.** *We have*

$$\mathcal{B}(V)_f \subset \mathcal{B}(V)_c \subset \mathcal{B}(V) \subset \mathrm{End}_{\mathbb{C}}(V).$$

Moreover, $\mathcal{B}(V)_c$ is complete with respect to the operator norm provided that V is separable.

Fact 10.12 ([Rud91, §12.9])**.** *Let $T \in \mathcal{B}(V)$ be a bounded operator on V. There exists a unique bounded operator $T^* \in \mathcal{B}(V)$ satisfying*

$$(T(v), v')_V = (v, T^*(v'))_V$$

for any $v, v' \in V$.

Definition 10.13. The operator T^* as in the above fact is called the *adjoint operator* of T. If $T = T^*$, we say that T is a *self-adjoint* operator.

Theorem 10.14 (Spectral decomposition theorem, [Rud91, Theorems 12.29, 12.30])**.** *Let $T \in \mathcal{B}(V)_c$ be a compact self-adjoint operator. For $\lambda \in \mathbb{C}$, we put $V_\lambda := \mathrm{Ker}(T - \lambda \cdot \mathrm{id}_V : V \to V)$. Then we have a countable family of nonzero real numbers $\{\lambda_i \in \mathbb{R}^\times\}_{i \in I}$ such that*

$$V = V_0 \oplus \widehat{\bigoplus_{i \in I}} V_{\lambda_i}.$$

Moreover,

(1) *each V_{λ_i} is finite-dimensional,*

(2) *for any $\lambda \in \mathbb{C}^\times$, $V_\lambda \neq 0$ if and only if $\lambda = \lambda_i$ for some $i \in I$,*

(3) *0 is only a possible accumulation point of $\{\lambda_i\}_{i \in I}$.*

10.5. **Proof of Key Proposition.** Now let us prove Proposition 10.6:

Proof of Proposition 10.6. We take any $v \in V$ such that $\|v\|_V = 1$. We define a $\mathbb{C}$-linear endomorphism $T \in \mathrm{End}_{\mathbb{C}}(V)$ by

$$T(u) := (u, v)_V \cdot v$$

for $u \in V$. In other words, T is the projector of V onto $\mathbb{C} \cdot v$; hence T is an idempotent ($T \circ T = T$) and finite-rank. Also, T is a self-adjoint operator since

$$\begin{aligned}(T(u), u')_V = ((u, v)_V \cdot v, u')_V &= (u, v)_V \cdot (v, u')_V \\ &= (u, v)_V \cdot \overline{(u', v)_V} \\ &= (u, (u', v)_V \cdot v)_V = (u, T(u'))_V\end{aligned}$$

for any $u, u' \in V$.

Here, following the manner of Corollary 9.13, we define an endomorphism $T^0 \in \mathrm{End}_{\mathbb{C}}(V)$ by

$$T^0 := \int_G \pi(g)^{-1} \circ T \circ \pi(g)\, dg.$$

To be more precise, this is an endomorphism which maps any $v \in V$ to

$$T^0(v) := \int_G \pi(g)^{-1} \circ T \circ \pi(g)(v)\, dg$$

(Here, we do not go into details of how this integration makes sense; note that now V is an infinite-dimensional vector space.) Then, by the same argument as in the proof of Corollary 9.13, we can check that this is a G-homomorphism, i.e., $T^0 \in \mathrm{End}_G(V)$. Moreover, T^0 is a compact self-adjoint operator of V. Indeed, as the endomorphism $\pi(g)^{-1} \circ T \circ \pi(g)$ in the integrand is a finite-rank operator, hence compact, its integration is also a compact operator (note that $\mathcal{B}(V)_c$ is complete). Also, for any $u, u' \in V$, we have

$$(T^0)^* = \int_G \pi(g)^* \circ T^* \circ \pi^{-1}(g)^*\, dg = \int_G \pi(g) \circ T \circ \pi^{-1}(g)\, dg = T^0$$

using that π is unitary.

Therefore, we can apply Theorem 10.14 to T^0 to get the decomposition

$$V = V_0 \oplus \widehat{\bigoplus_{i \in I}} V_{\lambda_i}$$

with a countable family of nonzero real numbers $\{\lambda_i\}_{i \in I}$. Here, recall that each V_{λ_i} is defined by

$$V_{\lambda_i} := \mathrm{Ker}(T^0 - \lambda_i \cdot \mathrm{id} \colon V \to V).$$

We note that I is not empty. Indeed, if I is empty, then it implies that $V = V_0$. In other words, $T^0 = 0$. However, then

$$\begin{aligned} 0 = (T^0(v), v)_V &= \int_G (\pi(g)^{-1} \circ T \circ \pi(g)(v), v)_V\, dg \\ &= \int_G (T \circ \pi(g)(v), \pi(g)(v))_V\, dg \end{aligned}$$

(used that π is unitary, hence $(-,-)_V$ is G-invariant). Using that $T \circ T = T$ and the self-adjointness of T, we furthermore have

$$\begin{aligned} \int_G (T \circ \pi(g)(v), \pi(g)(v))_V\, dg &= \int_G (T \circ T \circ \pi(g)(v), \pi(g)(v))_V\, dg \\ &= \int_G (T \circ \pi(g)(v), T \circ \pi(g)(v))_V\, dg. \end{aligned}$$

The integrand is a continuous function $G \to \mathbb{R}_{\geq 0}$ which is not identically zero (at least the value at $g = 1$ is 1). Thus the integral cannot be zero, which leads to a contradiction.

Now we consider V_{λ_i} for any $i \in I$. Since $T^0 - \lambda_i \cdot \mathrm{id}$ is a G-homomorphism, each V_{λ_i} is a subrepresentation of V. Moreover, by Theorem 10.14, each V_{λ_i} is finite-dimensional. In particular, it contains a finite-dimensional irreducible subrepresentation. □

10.6. **Exercise.**

Exercise 10.1. Let G be a compact topological group.

(1) Let $g_1, \ldots, g_n$ be distinct elements of G. Show that there exist functions $f_1, \ldots, f_n \in C(G)$ such that
$$f_i(g_j) = \delta_{ij}.$$

(2) Prove that G is finite if and only if it has only finitely many isomorphism classes of irreducible continuous unitary representations.

11. Lecture 11: Lie groups and abstract Lie algebras

For the remainder of this course, we will study the representation theory of a very important class of topological groups called "Lie groups". Lie groups and their representation theory are important for at least two reasons. First, they are useful in many applications, hence practically important. Second, they admit systematic structure and classification theories, hence they are beautiful and important mathematical subjects in their own right.

The algebraic objects obtained by linearly approximating Lie groups, which are generally noncommutative, are called "Lie algebras". The structure theory and representation theory of Lie algebras are often relatively simpler than those of Lie groups.

What we aim to understand in the following lectures are some basics of the theoretical framework to study these: what Lie groups and Lie algebras are, what their structure and representation theories look like, and how these theories are related to each other.

(Main reference for today's lecture: [Hal15, Chapter 3])

11.1. Definition and Examples of Lie groups.

Recall that a smooth manifold is a Hausdorff and second countable topological space that is locally homeomorphic to Euclidean spaces such that the transition maps are smooth (C^∞).

Definition 11.1. A *Lie group* is a smooth manifold G equipped with a group structure on G such that the multiplication map

$$G \times G \to G \colon (g, h) \mapsto gh$$

and the inversion map

$$G \to G \colon g \mapsto g^{-1}$$

are smooth.

We define the notion of a homomorphism between Lie groups in the obvious way: we require that the map be both a group homomorphism and a smooth map between manifolds.

Note that, in particular, a Lie group is a topological group in our sense. Furthermore, a Lie group is locally compact since it is locally homeomorphic to Euclidean spaces. Therefore, by Theorem 9.2, any Lie group admits a right/left Haar measure.

Example 11.2.
(1) Any finite group is a Lie group with respect to the discrete topology.
(2) The additive groups $\mathbb{R}^n$ and $\mathbb{C}^n$ are Lie groups with respect to the Euclidean topology.
(3) The 1-dimensional torus $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$ is a Lie group with respect to the quotient topology.

Example 11.3. The general linear groups $\mathrm{GL}_n(\mathbb{R})$ and $\mathrm{GL}_n(\mathbb{C})$ are Lie groups with respect to the topology induced from $M_n(\mathbb{R}) \cong \mathbb{R}^{\oplus n^2}$ or $M_n(\mathbb{C}) \cong \mathbb{C}^{\oplus n^2}$. Indeed, the multiplication and inversion maps are expressed in terms of rational functions (i.e., quotients of polynomial functions) with respect to the coordinates of matrices. For example, for $\mathrm{GL}_2(\mathbb{R})$, the multiplication map is given by

$$\mathrm{GL}_2(\mathbb{R}) \times \mathrm{GL}_2(\mathbb{R}) \to \mathrm{GL}_2(\mathbb{R}) \colon \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} x & y \\ z & w \end{pmatrix} \right) \mapsto \begin{pmatrix} ax + bz & ay + bw \\ cx + dz & cy + dw \end{pmatrix},$$

and the inversion map is given by

$$\mathrm{GL}_2(\mathbb{R}) \to \mathrm{GL}_2(\mathbb{R})\colon \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \mapsto \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Hence, in particular, they are smooth.

To obtain more examples of Lie groups, the following theorem is useful:

Theorem 11.4 (von Neumann–Cartan's theorem, [Hal15, §3.8])**.** *Every closed subgroup of* $\mathrm{GL}_n(\mathbb{C})$ *admits a canonical structure of a real Lie group. More precisely, the inclusion map of a closed subgroup* $H \subset \mathrm{GL}_n(\mathbb{C})$ *is a smooth embedding with respect to this structure.*

Example 11.5. The *special linear groups*

$$\mathrm{SL}_n(\mathbb{R}) := \{g \in \mathrm{GL}_n(\mathbb{R}) \mid \det(g) = 1\},$$
$$\mathrm{SL}_n(\mathbb{C}) := \{g \in \mathrm{GL}_n(\mathbb{C}) \mid \det(g) = 1\}$$

are Lie groups. Indeed, these are the preimages of the closed subsets $\{1\} \subset \mathbb{R}$ or $\{1\} \subset \mathbb{C}$ under the determinant map, which is continuous:

$$\det\colon \mathrm{GL}_n(\mathbb{C}) \to \mathbb{C}^\times\colon (g_{ij})_{ij} \mapsto \sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} g_{i,\sigma(i)}.$$

In particular, $\mathrm{SL}_n(\mathbb{R})$ and $\mathrm{SL}_n(\mathbb{C})$ are closed subgroups of $\mathrm{GL}_n(\mathbb{R})$ and $\mathrm{GL}_n(\mathbb{C})$, respectively.

Example 11.6. The (special) orthogonal group

$$\mathrm{O}(n) := \{g \in \mathrm{GL}_n(\mathbb{R}) \mid {}^t g g = I_n\},$$
$$\mathrm{SO}(n) := \{g \in \mathrm{GL}_n(\mathbb{R}) \mid {}^t g g = I_n,\ \det(g) = 1\},$$

is a Lie group (I_n denotes the identity matrix of size n). This can be checked by noting that each entry of the matrix identity ${}^t g g = I_n$ can be expressed as a polynomial function of the entries of g.

Here, let us explain the interpretation of the defining equation ${}^t g g = I_n$. We put $V := \mathbb{R}^{\oplus n}$ and consider its standard inner product

$$(-,-)\colon V \times V \to \mathbb{R}; \quad (x,y) = \left(\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \right) \mapsto \sum_{i=1}^{n} x_i y_i = {}^t x y.$$

For any $g \in \mathrm{GL}_n(\mathbb{R})$, we naturally regard it as an automorphism of V, i.e., $V \to V\colon x \mapsto gx$. Then we have

$$(gx, gy) = {}^t(gx)(gy) = {}^t x({}^t g g) y.$$

From this, we see that g preserves the inner product $(-,-)$ (in the sense that $(gx, gy) = (x, y)$ for any $x, y \in V$) if and only if ${}^t g g = I_n$. Thus the orthogonal group $\mathrm{O}(n)$ can be thought of as the group of automorphisms of V preserving the standard inner product.

With this viewpoint, we notice that it also makes sense to consider similar groups for any nondegenerate quadratic form (or, equivalently, nondegenerate symmetric bilinear form) on V (not necessarily a positive definite inner product). For any nondegenerate quadratic form $Q\colon V \to \mathbb{R}$, if we write J for its representation matrix (with respect to the standard basis of V), then the condition $Q(gx) = Q(x)$

is equivalent to ${}^tgJg = J$. Thus it is natural to define the following generalization of the orthogonal group:

$$\mathrm{O}(V,Q) := \{g \in \mathrm{GL}_n(\mathbb{R}) \mid {}^tgJg = J\},$$
$$\mathrm{SO}(V,Q) := \{g \in \mathrm{GL}_n(\mathbb{R}) \mid {}^tgJg = J \ \det(g) = 1\}.$$

Particularly important examples of quadratic forms are the *Minkowski forms* $Q_{p,q}$ on $\mathbb{R}^{\oplus n}$ defined by

$$Q_{p,q}(x_1, \ldots, x_n) := x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_n^2$$

for any $p, q \in \mathbb{Z}_{\geq 0}$ with $p + q = n$. We write $\mathrm{O}(p,q) := \mathrm{O}(V,Q)$ and $\mathrm{SO}(p,q) := \mathrm{SO}(V,Q)$ for the associated (special) orthogonal groups.

Note that, when two quadratic forms are equivalent, the associated orthogonal groups are isomorphic. It is known that any nondegenerate quadratic form over $\mathbb{R}$ is equivalent to a Minkowski form $Q_{p,q}$ for some $p, q \in \mathbb{Z}_{\geq 0}$ with $p + q = n$. In this sense, $\mathrm{O}(p,q)$ and $\mathrm{SO}(p,q)$ exhaust all possible orthogonal groups over $\mathbb{R}$.

Example 11.7. The (special) unitary group

$$\mathrm{U}(n) := \{g \in \mathrm{GL}_n(\mathbb{C}) \mid {}^tg\overline{g} = I_n\},$$
$$\mathrm{SU}(n) := \{g \in \mathrm{GL}_n(\mathbb{C}) \mid {}^tg\overline{g} = I_n, \det(g) = 1\},$$

is a Lie group. This can be checked by noting that each entry of the matrix identity ${}^tg\overline{g} = I_n$ can be expressed as the combination of polynomial functions and the complex conjugates of the entries of g.

Also, we can perform a similar construction to the previous example for the complex vector space $V := \mathbb{C}^{\oplus n}$ equipped with the standard Hermitian inner product:

$$(-,-)\colon V \times V \to \mathbb{C}; \quad (x,y) = \left(\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \right) \mapsto \sum_{i=1}^{n} x_i \overline{y_i} = {}^tx\overline{y}.$$

Then we see that g preserves the inner product $(-,-)$ (in the sense that $(gx, gy) = (x,y)$ for any $x, y \in V$) if and only if ${}^tg\overline{g} = I_n$. Thus the unitary group $\mathrm{U}(n)$ can be thought of as the group of automorphisms of V preserving the standard inner product.

Hence, again, we can naturally consider the following generalization of the unitary group:

$$\mathrm{U}(V,Q) := \{g \in \mathrm{GL}_n(\mathbb{C}) \mid {}^tgJ\overline{g} = J\},$$
$$\mathrm{SU}(V,Q) := \{g \in \mathrm{GL}_n(\mathbb{C}) \mid {}^tgJ\overline{g} = J, \det(g) = 1\},$$

where Q is any nondegenerate Hermitian form on V and J is its representation matrix.

Particularly important examples of Hermitian forms are the *Minkowski forms* $Q_{p,q}$ on $\mathbb{C}^{\oplus n}$ defined by

$$Q_{p,q}(x_1, \ldots, x_n) := |x_1|^2 + \cdots + |x_p|^2 - |x_{p+1}|^2 - \cdots - |x_n|^2$$

for any $p, q \in \mathbb{Z}_{\geq 0}$ with $p + q = n$. We write $\mathrm{U}(p,q) := \mathrm{U}(V,Q)$ and $\mathrm{SU}(p,q) := \mathrm{SU}(V,Q)$ for the associated (special) unitary groups.

Note that, when two Hermitian forms are equivalent, the associated unitary groups are isomorphic. It is known that any nondegenerate Hermitian form over $\mathbb{C}$ is equivalent to a Minkowski form $Q_{p,q}$ for some $p, q \in \mathbb{Z}_{\geq 0}$ with $p + q = n$. In this sense, $\mathrm{U}(p,q)$ and $\mathrm{SU}(p,q)$ exhaust all possible unitary groups over $\mathbb{C}$.

Example 11.8. The symplectic group

$$\mathrm{Sp}_{2n}(\mathbb{R}) := \{g \in \mathrm{GL}_{2n}(\mathbb{R}) \mid {}^t g J_{2n} g = J_{2n}\},$$

is a Lie group, where J_{2n} is the $2n \times 2n$ anti-diagonal matrix whose anti-diagonal entries are $1, -1, 1, -1, \ldots, 1, -1$. This can be checked by a similar argument to above.

Similarly to above, this can be thought of as the group of automorphisms of $V = \mathbb{R}^{\oplus 2n}$ preserving the symplectic form

$$(-,-)\colon V \times V \to \mathbb{R}; \quad (x,y) = \left(\begin{pmatrix} x_1 \\ \vdots \\ x_{2n} \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ y_{2n} \end{pmatrix} \right) \mapsto \sum_{i=1}^{2n} (-1)^{i+1} x_i y_{2n+1-i}.$$

With this viewpoint, one can consider the symplectic group associated to any symplectic form on V. However, unlike in the preceding cases, there is only one equivalence class of nondegenerate symplectic forms on V (and also its dimension must be even). In other words, any "symplectic group" is isomorphic to Sp_{2n} for some $2n$.

The groups introduced here are examples of a particular class of Lie groups called "classical groups".

Example 11.9. We consider the following subgroups of $\mathrm{GL}_n(\mathbb{R})$:

$$B_n = \{g \in \mathrm{GL}_n(\mathbb{R}) \mid g \text{ is upper triangular}\},$$

$$N_n = \{g \in \mathrm{GL}_n(\mathbb{R}) \mid g \text{ is upper triangular and unipotent}\}.$$

Then these groups are closed subgroups of $\mathrm{GL}_n(\mathbb{R})$ since the condition "upper triangular" is expressed via equations "$g_{ij} = 0$ for $i > j$" and the condition "unipotent" is expressed via equations "$g_{ii} = 1$ for all i". Hence these are Lie groups.

11.2. **Abstract Lie algebra.** We first introduce the following purely algebraic notion:

Definition 11.10. Let k be a field. Let $\mathfrak{g}$ be a k-vector space equipped with a k-bilinear map $[-,-]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$. We say that $(\mathfrak{g}, [-,-])$ is a *Lie algebra* if the following are satisfied:

(1) $[x,x] = 0$ for any $x \in \mathfrak{g}$,
(2) $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$ for any $x, y, z \in \mathfrak{g}$.

Example 11.11. For any k-vector space V, we can define a Lie algebra structure on the vector space V by

$$[X,Y] = 0.$$

In general, a Lie algebra with the trivial bracket is called an *abelian* Lie algebra.

Example 11.12. For any k-algebra A (i.e., an associative k-algebra with unit, which may not be commutative), we can define a Lie algebra structure on the vector space A by

$$[X,Y] = XY - YX.$$

It is routine to check that this bracket satisfies the Jacobi identity.

For example you can choose A to be the set of n-by-n matrices $M_n(k)$. Then the resulting Lie algebra $(M_n(k), [-,-])$ is denoted by $\mathfrak{gl}_n(k)$.

Let us introduce some basic notions related to Lie algebras.

Definition 11.13. Let $\mathfrak{g}$ be a Lie algebra over a field k. A k-subspace $\mathfrak{h} \subset \mathfrak{g}$ is called a *Lie subalgebra* if it is closed under the Lie bracket, i.e.,

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}.$$

Definition 11.14. Let $\mathfrak{g}$ and $\mathfrak{g}'$ be Lie algebras over k. A homomorphism from $\mathfrak{g}$ to $\mathfrak{g}'$ as Lie algebras is a k-linear map

$$f \colon \mathfrak{g} \to \mathfrak{g}'$$

preserving the Lie brackets, i.e., satisfying

$$f([X, Y]) = [f(X), f(Y)]$$

for every $X, Y \in \mathfrak{g}$. When f is bijective, we say that f is a Lie algebra isomorphism (and say that $\mathfrak{g}$ and $\mathfrak{g}'$ are isomorphic).

Definition 11.15. Let $\mathfrak{g}$ be a Lie algebra over k. A k-subspace $\mathfrak{h} \subset \mathfrak{g}$ is called an *ideal* if it is closed under the Lie bracket with any element of $\mathfrak{g}$, i.e.,

$$[\mathfrak{g}, \mathfrak{h}] \subset \mathfrak{h}.$$

When $\mathfrak{h}$ is an ideal of $\mathfrak{g}$, the Lie bracket on $\mathfrak{g}$ naturally induces a Lie bracket on the quotient vector space $\mathfrak{g}/\mathfrak{h}$, which makes it a Lie algebra.

Remark 11.16. Note that, by definition, any ideal is a Lie subalgebra. If you are familiar with commutative ring theory, you might feel that this sounds somehow strange; in commutative ring theory, any ideal I of a commutative ring R is not a subring unless $I = R$. So, please keep in mind that the notion of ideal in Lie algebra theory is NOT analogous to that in commutative ring theory. Rather, it is analogous to the notion of normal subgroup in group theory.

Lemma 11.17. *Let $f \colon \mathfrak{g} \to \mathfrak{h}$ be a Lie algebra homomorphism. Then the kernel $\operatorname{Ker}(f)$ is an ideal of $\mathfrak{g}$ and the image $\operatorname{Im}(f)$ is a Lie subalgebra of $\mathfrak{h}$. Furthermore, f induces a Lie algebra isomorphism $\mathfrak{g}/\operatorname{Ker}(f) \cong \operatorname{Im}(f)$.*

Example 11.18. We define a subspace $\mathfrak{sl}_n(k)$ of $\mathfrak{gl}_n(k)$ by

$$\mathfrak{sl}_n(k) = \{X \in \mathfrak{gl}_n(k) \mid \operatorname{Tr}(X) = 0\}.$$

Then this is a Lie subalgebra of $\mathfrak{gl}_n(k)$.

To get an intuition, let us look at the case $n = 2$ and $k = \mathbb{C}$ in more detail. We have

$$\mathfrak{sl}_2(k) = kH \oplus kE \oplus kF,$$

where

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then the bracket structure is described by

$$[H, E] = 2E, \qquad [H, F] = -2F, \qquad [E, F] = H.$$

Example 11.19. We define a subspace $\mathfrak{so}_n(k)$ of $\mathfrak{sl}_n(k)$ by

$$\mathfrak{so}_n(k) = \{X \in \mathfrak{sl}_n(k) \mid {}^tX + X = 0\}.$$

Then this is a Lie subalgebra of $\mathfrak{sl}_n(k)$. More generally, for any nondegenerate symmetric matrix $J \in M_n(k)$, we can define the Lie subalgebra $\mathfrak{so}(J)$ of $\mathfrak{sl}_n(k)$ by

$$\mathfrak{so}(J) = \{X \in \mathfrak{sl}_n(k) \mid {}^tXJ + JX = 0\}.$$

Example 11.20. Here we consider the case where $k = \mathbb{R}$. We define a subspace $\mathfrak{u}(n)$ of $\mathfrak{gl}_n(\mathbb{C})$ by

$$\mathfrak{u}(n) = \{X \in \mathfrak{gl}_n(\mathbb{C}) \mid {}^tX + \overline{X} = 0\},$$
$$\mathfrak{su}(n) = \{X \in \mathfrak{gl}_n(\mathbb{C}) \mid {}^tX + \overline{X} = 0,\ \mathrm{Tr}(X) = 0\},$$

Then these are real Lie subalgebras of $\mathfrak{gl}_n(\mathbb{C})$. More generally, for any Hermitian matrix $J \in M_n(\mathbb{C})$, we can define the $\mathbb{R}$-Lie subalgebra $\mathfrak{u}(J)$ of $\mathfrak{gl}_n(\mathbb{C})$ by

$$\mathfrak{u}(J) = \{X \in \mathfrak{gl}_n(\mathbb{C}) \mid {}^tXJ + J\overline{X} = 0\}.$$

Example 11.21. We define a subspace $\mathfrak{sp}_{2n}(k)$ of $\mathfrak{gl}_{2n}(k)$ by

$$\mathfrak{sp}_{2n}(k) = \{X \in \mathfrak{gl}_{2n}(k) \mid {}^tXJ_{2n} + J_{2n}X = 0\}.$$

Then this is a Lie subalgebra of $\mathfrak{gl}_{2n}(k)$.

Example 11.22. We define subspaces of $\mathfrak{gl}_n(k)$ by

$$\mathfrak{b}_n(k) = \{X \in \mathfrak{gl}_n(k) \mid X \text{ is upper triangular}\},$$
$$\mathfrak{n}_n(k) = \{X \in \mathfrak{gl}_n(k) \mid X \text{ is upper triangular and nilpotent}\}.$$

Then these are Lie subalgebras of $\mathfrak{gl}_n(k)$.

Note that the definitions of these Lie algebras are quite similar to those of the classical groups. In fact, as we will see in the next lecture, these Lie algebras are obtained by linearly approximating the corresponding classical groups; note that, if you take the derivative of the defining equations of the classical groups (such as ${}^tgJg = J$ by formally regarding g as a variable and J as a constant), you get the defining equations of the corresponding Lie algebras. In general, we can associate a Lie algebra to any Lie group in a natural way. (Note that the differentiation is the operation unique to Lie groups, which is not available for general topological groups.)

11.3. **Exercises.**

Exercise 11.1. We consider the 2-dimensional k-vector space with basis X, Y:

$$\mathfrak{g} := kX \oplus kY.$$

(1) Prove that the identity $[X, Y] = Y$ gives a unique well-defined Lie algebra structure on $\mathfrak{g}$.
(2) Show that any 2-dimensional non-abelian Lie algebra is isomorphic to $\mathfrak{g}$.

Exercise 11.2. (1) Determine the dimension of the Lie algebra $\mathfrak{so}_{p,q}(\mathbb{R})$ as an $\mathbb{R}$-vector space.
(2) Determine when the Lie group $\mathrm{SO}_{p,q}(\mathbb{R})$ is compact.

12. Lecture 12: From Lie groups to Lie algebras

(Main reference for today's lecture: [Hal15, Chapter 3])

12.1. **From Lie groups to Lie algebras.** We encounter the notion of a Lie algebra naturally in the context of manifold theory.

Definition 12.1. Let k be any field and R be a commutative k-algebra. Let M be an R-module. A *derivation* from R to M is a k-linear homomorphism $D\colon R \to M$ satisfying the "Leibniz rule"

$$D(fg) = f \cdot D(g) + g \cdot D(f)$$

for any $f, g \in R$. We write $D_k(R, M)$ for the k-vector space consisting of derivations from R to M.

Recall that, for any smooth manifold X and its point $p \in X$, the *tangent space* T_pX of X at p is defined by

$$T_pX := D_{\mathbb{R}}(C^\infty(X), \mathbb{R}_p),$$

where $\mathbb{R}_p$ denotes $\mathbb{R}$ regarded as a $C^\infty(X)$-module through the map $\mathrm{ev}_p\colon C^\infty(X) \twoheadrightarrow \mathbb{R}\colon f \mapsto f(p)$ (i.e., $f \cdot r := f(p)r$). Elements of T_pX are called tangent vectors of X at p.

Remark 12.2. Note that the tangent space has a nice description in terms of local coordinates. If we let $x_1, \ldots, x_n$ be the local coordinates of a real manifold X around a point $p \in X$, then T_pX is the n-dimensional $\mathbb{R}$-vector space spanned by

$$\left(\frac{\partial}{\partial x_1}\right)(p), \ldots, \left(\frac{\partial}{\partial x_n}\right)(p),$$

where $(\frac{\partial}{\partial x_i})(p) : f(x) \mapsto (\frac{\partial f}{\partial x_i})(p)$. For this reason, T_pX is often defined as $\oplus_{i=1}^n \mathbb{R}(\frac{\partial}{\partial x_i})(p)$.

Also recall that, a *vector field* on X is an association $X \ni p \mapsto v_p \in T_pX$ such that v_p "varies smoothly in $p \in X$". Let $\mathfrak{X}(X)$ be the set of vector fields on X. For any $v \in \mathfrak{X}(X)$, we define an $\mathbb{R}$-linear homomorphism $v\colon C^\infty(X) \to C^\infty(X)$ by $v(f)(p) := v_p(f)$. Then we can check that $v(-)$ satisfies the Leibniz rule, i.e., $v \in D_{\mathbb{R}}(C^\infty(X), C^\infty(X))$. In fact, this procedure gives an identification between $\mathfrak{X}(X)$ and $D_{\mathbb{R}}(C^\infty(X), C^\infty(X))$:

$$\begin{array}{ccc} \mathfrak{X}(X) & \xrightarrow{\cong} & D_{\mathbb{R}}(C^\infty(X), C^\infty(X)) \\ {\scriptstyle v \mapsto v_p}\downarrow & & \downarrow{\scriptstyle D \mapsto \mathrm{ev}_p \circ D} \\ T_pX & = & D_{\mathbb{R}}(C^\infty(X), \mathbb{R}_p) \end{array}$$

For any $u, v \in \mathfrak{X}(X)$, we define their *bracket product* $[u, v] \in \mathfrak{X}(X)$ by $[u, v] := u \circ v - v \circ u$, where u and v are regarded as elements of $D_{\mathbb{R}}(C^\infty(X), C^\infty(X))$.

Lemma 12.3. *The space of vector fields $\mathfrak{X}(X)$ forms a Lie algebra over $\mathbb{R}$ with respect to $[-,-]$.*

Proof. Recall that we can introduce a Lie algebra structure on any associative algebra. In particular, $\mathrm{End}_{\mathbb{R}}(C^\infty(X))$ forms a Lie algebra with respect to the bracket product given by $[u, v] := u \circ v - v \circ u$. The bracket product on $\mathfrak{X}(X)$ is nothing but

the restriction of this bracket product from $\mathrm{End}_{\mathbb{R}}(C^\infty(X))$ to $D_{\mathbb{R}}(C^\infty(X))$. Hence, the only thing we have to check is that $D_{\mathbb{R}}(C^\infty(X))$ is closed under $[-,-]$ (i.e., the well-definedness of $[-,-]$); we omit the proof since it is routine. □

Now let us suppose that G is a Lie group (a manifold equipped with a group structure which is compatible with the manifold structure). In this case, G acts on G itself via left and right translations, hence $\mathfrak{X}(G)$ also has the left and right translations by G. Let us explain the meaning of the translation-invariance more explicitly (only in the case of left translations). For any $g \in G$, we consider the left translation action L_g of G on $C^\infty(G)$ defined by $L_g(f)(x) := f(g^{-1}x)$ for any $f \in C^\infty(G)$ and $x \in G$. We say that $v \in \mathfrak{X}(G)$ is left-invariant if it satisfies $v \circ L_g = L_g \circ v$ for any $g \in G$, where v is viewed as an element of $D_{\mathbb{R}}(C^\infty(G), C^\infty(G))$.

Let $\mathfrak{X}_l(G)$ (resp. $\mathfrak{X}_r(G)$) denote the subset of left-invariant (resp. right-invariant) vector fields on G.

Lemma 12.4. *The association $v \mapsto v_e$ gives an isomorphism of $\mathbb{R}$-vector spaces*

$$\mathfrak{X}_l(G) \xrightarrow{\cong} T_eG \quad \textit{and} \quad \mathfrak{X}_r(G) \xrightarrow{\cong} T_eG.$$

Proof. Let us first show the injectivity. Since $D \in \mathfrak{X}(G) \cong D_{\mathbb{R}}(C^\infty(G), C^\infty(G))$ is uniquely determined by $\mathrm{ev}_g \circ D$ for all $g \in G$, it is enough to show that $\mathrm{ev}_g \circ D$ is specified by $\mathrm{ev}_e \circ D$ by assuming that D is left-invariant. For any $f \in C^\infty(G)$, we have $\mathrm{ev}_g \circ D(f) = D(f)(g)$. On the other hand, by the left-invariance of D, we have $L_g \circ D \circ L_g^{-1} = D$. Hence, in particular,

$$D(f)(g) = (L_g \circ D \circ L_g^{-1}(f))(g) = D \circ L_g^{-1}(f)(e).$$

The right-hand side is nothing but $(\mathrm{ev}_e \circ D)(L_g^{-1}(f))$, which shows that the value of $D(f)$ at g is determined by the value of $D(L_g^{-1}(f))$ at e.

The surjectivity can be checked by reversing this argument. More precisely, for any $D_e \in T_eG \cong D_{\mathbb{R}}(C^\infty(G), \mathbb{R}_e)$, we can define a left-invariant vector field $D \in D_{\mathbb{R}}(C^\infty(G), C^\infty(G))$ by $D(f)(g) := D_e \circ L_g^{-1}(f)$. □

Note that, $\mathfrak{X}_l(G)$ and $\mathfrak{X}_r(G)$ are obviously closed under the bracket product, hence are Lie subalgebras of $\mathfrak{X}(G)$. In particular, the tangent space at the unit element T_eG also gets a Lie algebra structure through the identification $\mathfrak{X}_l(G) \cong T_eG$. We write $\mathrm{Lie}\, G := T_eG$ and call it the *Lie algebra* of the Lie group G.

12.2. **Exponential map and Lie algebra.** We next discuss how to compute the Lie algebra of a Lie group G in the case where G is a closed subgroup of $\mathrm{GL}_n(\mathbb{R})$.

We first note the following lemma:

Lemma 12.5. *Let $f\colon G \to H$ be a homomorphism of Lie groups. Then the differential $df\colon T_eG \to T_eH$ is a homomorphism of Lie algebras.*

This lemma can be proved by tracing all the definitions, so let us omit the proof.

Lemma 12.6. *The Lie algebra of $\mathrm{GL}_n(\mathbb{R})$ is $\mathfrak{gl}_n(\mathbb{R})$.*

Proof. Since $\mathrm{GL}_n(\mathbb{R})$ is embedded into $M_n(\mathbb{R})$ as an open submanifold, the tangent space at $e = I_n$ is identified with that of $M_n(\mathbb{R})$ at 0, which is given by $M_n(\mathbb{R})$. To be more explicit, each $A = (a_{ij})_{ij} \in M_n(\mathbb{R})$ is identified with the following tangent vector v_A:

$$v_A\colon f \mapsto \sum_{i,j=1}^{n} a_{ij} \cdot \frac{\partial f}{\partial x_{ij}}(1)$$

The element D_A of $D_{\mathbb{R}}(C^\infty(\mathrm{GL}_n(\mathbb{R})), C^\infty(\mathrm{GL}_n(\mathbb{R})))$ corresponding to A is given by

$$(D_A(f))(g) := v_A(L_g^{-1}(f))$$

for any $f \in C^\infty(G)$ and $g \in \mathrm{GL}_n(\mathbb{R})$.

Recall that $(L_g^{-1}f)(x) = f(gx)$. If we choose f to be the coordinate function $x \mapsto x_{ij}$, then we obtain

$$(D_A x_{ij})(g) = (gA)_{ij},$$

i.e.,

$$D_A x_{ij} = \sum_{k=1}^{n} x_{ik} a_{kj}.$$

Thus, for any $A = (a_{ij})_{ij}, B = (b_{ij})_{ij} \in M_n(\mathbb{R})$,

$$\begin{aligned} D_A(D_B x_{ij}) &= D_A\left(\sum_{k=1}^{n} x_{ik} b_{kj}\right) \\ &= \sum_{k=1}^{n}\left(\sum_{l=1}^{n} x_{il} a_{lk}\right) b_{kj} \\ &= \sum_{l=1}^{n} x_{il}\left(\sum_{k=1}^{n} a_{lk} b_{kj}\right) = D_{AB} x_{ij}. \end{aligned}$$

Hence

$$[D_A, D_B](x_{ij}) = D_{[A,B]} x_{ij}.$$

Since a tangent vector at each point is determined by its values on the coordinate functions, this implies that

$$[D_A, D_B] = D_{[A,B]}.$$

Consequently, the identification $M_n(\mathbb{R}) \cong T_{I_n}\mathrm{GL}_n(\mathbb{R}) \cong A \mapsto D_A$ preserves the Lie bracket structures. □

In particular, if G is a closed subgroup of $\mathrm{GL}_n(\mathbb{R})$, then its Lie algebra $\mathfrak{g}$ is identified with a Lie subalgebra of $\mathfrak{gl}_n(\mathbb{R})$. Let us consider how this can be explicitly computed.

Definition 12.7. For any $X \in \mathfrak{gl}_n(\mathbb{C})$, we define its *exponential* by

$$\exp(X) := \sum_{n=0}^{\infty} \frac{X^n}{n!}.$$

(The infinite series on the right-hand side can be checked to converge entry-wise.)

Lemma 12.8 ([Hal15, Proposition 2.16])**.** *The association $X \mapsto \exp(X)$ defines a smooth map*

$$\exp\colon \mathfrak{gl}_n(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{R}).$$

The same is true when $\mathbb{R}$ is replaced with $\mathbb{C}$. Moreover, the differential of this map at $0 \in \mathfrak{gl}_n(\mathbb{R})$ is identified with the identity map of $\mathfrak{gl}_n(\mathbb{R})$.

Note that this lemma implies that, by the inverse function theorem, a local neighborhood of $I_n \in \mathrm{GL}_n(\mathbb{R})$ can be identified with a local neighborhood of $0 \in \mathfrak{gl}_n(\mathbb{R})$.

Lemma 12.9. *We have the following:*

(1) *For any* $X, Y \in \mathfrak{gl}_n(\mathbb{C})$ *satisfying* $[X, Y] = 0$*, we have* $\exp(X + Y) = \exp(X)\exp(Y)$.
(2) *For any* $X \in \mathfrak{gl}_n(\mathbb{C})$*, we have* $\det(\exp(X)) = \exp(\mathrm{Tr}(X))$.

Proof. The first assertion can be immediately checked by looking at the definition of the exponential map.

For the second assertion, we note that the exponential map is conjugation-equivariant, i.e., for any $g \in \mathrm{GL}_n(\mathbb{C})$ and $X \in M_n(\mathbb{C})$, we have

$$\exp(gXg^{-1}) = g\exp(X)g^{-1}.$$

In particular, we get

$$\det(\exp(gXg^{-1})) = \det(g\exp(X)g^{-1}) = \det(\exp(X)).$$

On the other hand, we also have

$$\exp(\mathrm{Tr}(gXg^{-1})) = \exp(\mathrm{Tr}(X)).$$

Hence, to show the claim, we may assume that X is a triangular matrix. If we write $X = (x_{ij})_{ij}$, then the diagonal entries of $\exp(X)$ are given by $\exp(x_{ii})$, hence

$$\det(\exp(X)) = \prod_{i=1}^{n} \exp(x_{ii}) = \exp\Big(\sum_{i=1}^{n} x_{ii}\Big) = \exp(\mathrm{Tr}(X)).$$

□

Remark 12.10. (1) For any fixed $X \in M_n(\mathbb{R})$, the map

$$\mathbb{R} \to \mathrm{GL}_n(\mathbb{R})\colon t \mapsto \exp(tX)$$

is a Lie group homomorphism. This homomorphism is referred to as a *one-parameter subgroup* of $\mathrm{GL}_n(\mathbb{R})$.

(2) The exponential map $\exp\colon \mathfrak{gl}_n(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{R})$ is not surjective. This can be seen obviously by looking at connected components. More precisely, we consider the continuous surjective map

$$\mathrm{GL}_n(\mathbb{R}) \xrightarrow{\det} \mathbb{R}^\times \twoheadrightarrow \{\pm 1\},$$

where the last map is the signature character $x \mapsto x/|x|$. Then, as the exponential map is continuous and $\mathfrak{gl}_n(\mathbb{R}) \cong \mathbb{R}^{\oplus n^2}$ is connected, the image of the exponential map must be trivial in $\{\pm 1\}$ after being sent under the above map. In other words, at least the image of the exponential map must be contained in the following subgroup of $\mathrm{GL}_n(\mathbb{R})$:

$$\mathrm{GL}_n(\mathbb{R})^+ := \{g \in \mathrm{GL}_n(\mathbb{R}) \mid \det(g) > 0\}.$$

Now we introduce a proposition, which is quite convenient for determining the Lie algebra explicitly for a Lie group realized as a closed subgroup of $\mathrm{GL}_n(\mathbb{R})$:

Proposition 12.11 ([Hal15, Corollary 3.46])**.** *Let G be a closed subgroup of $\mathrm{GL}_n(\mathbb{R})$. Then its Lie algebra $\mathfrak{g}$ is identified with a Lie subalgebra of $\mathfrak{gl}_n(\mathbb{R})$ which is given by*

$$\{X \in \mathfrak{gl}_n(\mathbb{R}) \mid \exp(tX) \in G \text{ for all } t \in \mathbb{R}\}.$$

Example 12.12. Let us check that the Lie algebra of $\mathrm{SL}_n(\mathbb{R})$ is $\mathfrak{sl}_n(\mathbb{R})$. Since we have $\det(\exp(X)) = \exp(\mathrm{Tr}(X))$ for any $\mathfrak{gl}_n(\mathbb{R})$, we get $\det(\exp(tX)) = \exp(\mathrm{Tr}(tX)) = \exp(t\,\mathrm{Tr}(X))$. Hence, for $X \in \mathfrak{gl}_n(\mathbb{R})$, we have

$$\begin{aligned} \exp(tX) \in \mathrm{SL}_n(\mathbb{R}) \text{ for all } t \in \mathbb{R} &\iff \exp(t\,\mathrm{Tr}(X)) = 1 \text{ for all } t \in \mathbb{R} \\ &\iff \mathrm{Tr}(X) = 0 \\ &\iff X \in \mathfrak{sl}_n(\mathbb{R}). \end{aligned}$$

Example 12.13. Let us check that the Lie algebra of $\mathrm{SO}(J)$ is $\mathfrak{so}(J)$, where J is a nondegenerate symmetric matrix in $M_n(\mathbb{R})$. Recall that the defining equations of $\mathrm{SO}(J)$ and $\mathfrak{so}(J)$ are

$$^t gJg = J, \quad \det(g) = 1$$

and

$$^t XJ + JX = 0,$$

respectively. For any $X \in \mathfrak{gl}_n(\mathbb{R})$, we consider the condition

$$^t \exp(tX) J \exp(tX) = J.$$

By differentiating with respect to the parameter $t \in \mathbb{R}$, we see that this implies that condition $^t XJ + JX = 0$. (To be more precise, regard $^t \exp(tX)J\exp(tX) = J$ as the n^2-entrywise simultaneous equations, and consider the differentiation of each entry.) Conversely, if we have $^t XJ + JX = 0$, then

$$\begin{aligned} ^t \exp(tX) J \exp(tX) &= \left(\sum_{n=0}^{\infty} \frac{t^n \cdot {}^t X^n}{n!} \right) \cdot J \cdot \left(\sum_{n=0}^{\infty} \frac{t^n \cdot X^n}{n!} \right) \\ &= J \cdot \left(\sum_{n=0}^{\infty} \frac{t^n \cdot (-X)^n}{n!} \right) \cdot \left(\sum_{n=0}^{\infty} \frac{t^n \cdot X^n}{n!} \right) \\ &= J \cdot \exp(-tX) \cdot \exp(tX) = J. \end{aligned}$$

(Note that the condition $\det(g) = 1$ is automatically satisfied as remarked before.)

Example 12.14. We can check that the Lie algebra of $\mathrm{U}(J)$ is $\mathfrak{u}(J)$, where J is a nondegenerate Hermitian matrix in $M_n(\mathbb{C})$ by the same argument as above.

Example 12.15. We can check that the Lie algebra of Sp_{2n} is $\mathfrak{sp}_{2n}$ by the same argument as above.

12.3. **Exercises.**

Exercise 12.1. Prove that the image of the exponential map $\exp\colon \mathfrak{gl}_n(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{R})$ is even smaller than $\mathrm{GL}_n(\mathbb{R})^+$ when $n > 1$.

Exercise 12.2. (1) Prove that the exponential map for $\mathfrak{gl}_2(\mathbb{R})$ induces a surjective homomorphism

$$\exp\colon \mathfrak{so}_2(\mathbb{R}) \twoheadrightarrow \mathrm{SO}(2).$$

(2) Determine the preimage of I_2 under the map $\exp\colon \mathfrak{so}_2(\mathbb{R}) \twoheadrightarrow \mathrm{SO}(2)$.

13. Lecture 13: Representations of Lie groups and Lie algebras

Let G be a Lie group. From now on, we consider a general strategy for classifying all finite-dimensional irreducible continuous representations of G.

Recall that, when G is compact, any finite-dimensional irreducible continuous representation is automatically unitary (Proposition 9.9). In fact, this fact can be strengthened even more; e.g., if an irreducible continuous representation is realized on a Hilbert space, then it can be unitarized. Conversely, any irreducible unitary continuous representation is automatically finite-dimensional (Corollary 10.7). Therefore, when G is compact, we can say that it is enough to understand all finite-dimensional irreducible continuous representations of G.

On the other hand, when G is non-compact, there could be a lot of non-unitary irreducible continuous representations. Moreover, even if an irreducible continuous representation is unitary, it is not necessarily the case that it is finite-dimensional. (Being unitary is not equivalent to being finite-dimensional.) In general, we can find plenty of infinite-dimensional irreducible continuous representations.

In fact, generally speaking, classifying infinite-dimensional irreducible continuous (unitary) representations of a non-compact Lie group G is a very difficult problem. Even in the case where G belongs to a class of Lie groups called "reductive Lie groups", where a very sophisticated structure theory is available, this is a very deep matter related to so-called "the Langlands program".

Therefore, in the following, we focus only on finite-dimensional irreducible continuous representations of G (but, instead, we allow G to be non-compact).

Before proceeding, let us present several concrete examples:

Proposition 13.1. *Let G be one of the Lie groups $\mathrm{SL}_2(\mathbb{R})$, $\mathrm{SU}(2)$, and $\mathrm{SO}(3)$. Then, for each positive integer $n \in \mathbb{Z}_{>0}$, there exists at most one n-dimensional irreducible continuous representation of G (up to isomorphism). Moreover, the set of possible dimensions is given by*

$$\begin{cases} \mathbb{Z}_{>0} & \text{if } G = \mathrm{SL}_2(\mathbb{R}), \mathrm{SU}(2), \\ 2\mathbb{Z}_{\geq 0} + 1 & \text{if } G = \mathrm{SO}(3). \end{cases}$$

The aim of this lecture and the next lecture is to prove this proposition.

13.1. Representations of Lie algebras.

Definition 13.2. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{R}$. A *representation* of $\mathfrak{g}$ is a pair (ρ, V) of a $\mathbb{C}$-vector space V and a homomorphism

$$\rho \colon \mathfrak{g} \to \mathrm{End}_{\mathbb{C}}(V)$$

of Lie algebras over $\mathbb{R}$. (Here, we consider the standard Lie algebra structure on $\mathrm{End}_{\mathbb{C}}(V)$, i.e., $[X, Y] := X \circ Y - Y \circ X$ for any $X, Y \in \mathrm{End}_{\mathbb{C}}(V)$.)

We can define the notions of a subrepresentation (or "$\mathfrak{g}$-invariant subspace") and an irreducible representation in the natural way; for example, a $\mathbb{C}$-vector subspace $W \subset V$ is called a subrepresentation if $\rho(X)(W) \subset W$ for any $X \in \mathfrak{g}$.

Note that, for a Lie algebra $\mathfrak{g}$ over $\mathbb{R}$, we can define its *complexification* $\mathfrak{g}_{\mathbb{C}}$ by $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ with the naturally extended Lie bracket structure, i.e., for any $X, Y \in \mathfrak{g}$ and $a, b \in \mathbb{C}$, we define $[aX, bY] := ab[X, Y]$. Then, any representation (ρ, V) of $\mathfrak{g}$ can be also regarded as a representation of $\mathfrak{g}_{\mathbb{C}}$ by

$$\rho(aX) := a \cdot \rho(X)$$

for any $X \in \mathfrak{g}$ and $a \in \mathbb{C}$. Conversely, if a representation (ρ, V) of the complexification $\mathfrak{g}_{\mathbb{C}}$ is given, then we may regard it as a representation of $\mathfrak{g}$ just by restriction.

It is not difficult to see that the notions of subrepresentations or homomorphisms of representations do not change under the base change $\mathfrak{g} \mapsto \mathfrak{g}_{\mathbb{C}}$. In particular, we can easily check that any representation (ρ, V) of $\mathfrak{g}$ is irreducible if and only if it is irreducible as a representation of $\mathfrak{g}_{\mathbb{C}}$. Consequently, we get the following:

Proposition 13.3. *We have a natural bijection between the set of isomorphism classes of irreducible representations of $\mathfrak{g}$ and that of $\mathfrak{g}_{\mathbb{C}}$.*

Now recall that, if a Lie group homomorphism $f\colon G \to H$ is given, then its differential at the unit $df\colon T_eG \to T_eH$ gives a Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{h}$ (Lemma 12.5). We apply this fact to a finite-dimensional continuous representation of a Lie group. The key is the following:

Lemma 13.4. *Let G be a Lie group. Let (π, V) be a finite-dimensional continuous representation of G (say $\dim_{\mathbb{C}}(V) = n$). Then the continuous homomorphism $\pi\colon G \to \mathrm{Aut}_{\mathbb{C}}(V)(\cong \mathrm{GL}_n(\mathbb{C}))$ is automatically smooth, hence a homomorphism as Lie groups.*

Proof. We consider the "graph" Γ_π of π, i.e., Γ_π is the subgroup of $G \times H$ defined by

$$\Gamma_\pi := \{(g, \pi(g)) \mid g \in G\}.$$

We write $i\colon \Gamma_\pi \hookrightarrow G \times H$ for the inclusion, $p_1\colon G \times H \twoheadrightarrow G$ for the first projection, and $p_2\colon G \times H \twoheadrightarrow H$ for the second projection. By the von Neumann–Cartan theorem (Theorem 11.4), Γ_π admits a structure of a Lie group such that i is a smooth embedding of Lie groups and also that

$$(\mathrm{id}, \pi)\colon G \to \Gamma_\pi\colon g \mapsto (g, \pi(g))$$

is a Lie group isomorphism (the inverse is given by $p_1 \circ i$; the smoothness of this map can be checked by looking at the differential). Since the projection p_2 is smooth, we have that $p_2 \circ i \circ (\mathrm{id}, \pi)$ is also smooth. As $p_2 \circ i \circ (\mathrm{id}, \pi) = \pi$, we conclude that π is smooth. □

Let G be a Lie group and (π, V) a finite-dimensional continuous representation of G. Then, by the above lemma, we can apply Lemma 12.5 to π to get a Lie algebra representation

$$d\pi\colon \mathfrak{g} \to \mathrm{End}_{\mathbb{C}}(V).$$

Let us investigate the relationship between π and $d\pi$.

Proposition 13.5. *Suppose that G is connected.*

1. *A subspace $W \subset V$ is G-invariant if and only if it is $\mathfrak{g}$-invariant. In particular, π is irreducible if and only if $d\pi$ is irreducible.*
2. *For any other irreducible finite-dimensional continuous representation (π', V') of G, we have*
$$\mathrm{Hom}_G(V, V') = \mathrm{Hom}_{\mathfrak{g}}(V, V').$$

Proof. Here, for simplicity, we only consider the case where G is a closed subgroup of a general linear group. Recall that, in this case, the matrix exponential map $\mathfrak{g} \to G$ is available.

We only show (1); (2) can be proved by a similar discussion. If a subspace $W \subset V$ is stable under G-action, then for $X \in \mathfrak{g}$ and $w \in W$, the image of the 1-parameter

subgroup $[t \mapsto \pi(\exp(tX))w]$ is contained in W. By differentiating at $t = 0$, this implies that $d\pi(X)w \in W$. Let us consider the converse. For this, we note that the image of exp contains an open neighborhood of the identity, the subgroup of G generate by the image of exp is open. Thus, by Proposition 7.2 (3), it is also a closed subgroup; in particular, connected. Therefore, as G is connected, it coincides with the whole G. Hence, if a subspace W is stable under $\mathfrak{g}$-action, then it is also stable under $\exp(\mathfrak{g})$-action, which implies that it is stable under G-action. □

By this proposition, in order to see if given finite-dimensional irreducible continuous representatios of G are isomorphic or not, it is enough to consider the same problem for their differential representations on the Lie algebras. However, from this, we cannot immediately say that "classfiying finite-dimensional irreducible continuous representations of G is reduced to classfiying finite-dimensional irreducible representations of $\mathfrak{g}$" because we do not know yet whether any Lie algebra representation can be lifted to a Lie group representation.

13.2. **Universal covering group.** See [Hal15, Section 13.1] for the details of the following.

Definition 13.6. Let X be a topological space. A *covering* of X is a continuous surjection $p\colon \widetilde{X} \to X$ such that every $x \in X$ has an open neighborhood U for which $p^{-1}(U)$ is a disjoint union of open sets, each mapped homeomorphically onto U by p. We call p a *universal covering* of X if $\widetilde{X}$ is simply connected.

Theorem 13.7 (Unique existence of a universal covering)**.** *Let X be a connected topological manifold. There exists a universal covering $p\colon \tilde{X} \to X$. Furthermore, such a universal covering is unique in the following sense. We fix a point $x \in X$ and also $\tilde{x} \in p^{-1}(x)$. If $p'\colon \tilde{X}' \to X$ is another universal covering with a point $\tilde{x}' \in p'^{-1}(x)$, then there is a unique isomorphism $F\colon \tilde{X} \to \tilde{X}'$ such that $p' \circ F = p$ and $F(\tilde{x}) = \tilde{x}'$.*

If X is a connected topological manifold, then its universal covering $\tilde{X}$ also admits a structure of a topological manifold. Moreover, if X is smooth, then $\tilde{X}$ also admits a structure of a smooth manifold such that $p\colon \tilde{X} \twoheadrightarrow X$ is a map of smooth manifolds.

Theorem 13.8. *Let G be a connected Lie group and $p\colon \tilde{G} \twoheadrightarrow G$ its universal covering. Then, for any $\tilde{e} \in p^{-1}(e)$ (where $e \in G$ is the unit), $\tilde{G}$ admits a Lie group structure such that $\tilde{e}$ is the unit and p is a Lie group homomorphism such that $dp_{\tilde{e}}$ is a Lie algebra isomorphism.*

We call $\tilde{G}$ as in the above theorem the *simply-connected covering group* of G.

Theorem 13.9. *Let G be a connected and simply connected Lie group and H a Lie group. Every Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{h}$ is the differential of a unique Lie group homomorphism $G \to H$.*

Corollary 13.10. *Let G be a connected and simply-connected Lie group. For any finite-dimensional representation $\Pi\colon \mathfrak{g} \to \mathrm{End}_{\mathbb{C}}(V)$, there exists a unique continuous homomorphism $\pi\colon G \to \mathrm{Aut}_{\mathbb{C}}(V)$ whose differential $d\pi$ equals Π.*

Note that, for a connected Lie group G with universal covering $p\colon \tilde{G} \twoheadrightarrow G$, a finite-dimensional continuous representation $\pi\colon G \to \mathrm{Aut}_{\mathbb{C}}(V)$ obviously induces a

finite-dimensional continuous representation $\tilde{\pi} := \pi \circ p \colon \tilde{G} \to \mathrm{Aut}_{\mathbb{C}}(V)$. Conversely, a finite-dimensional continuous representation $\tilde{\pi} \colon \tilde{G} \to \mathrm{Aut}_{\mathbb{C}}(V)$ factors through $G \to \mathrm{Aut}_{\mathbb{C}}(V)$ if and only if $\mathrm{Ker}(p) \subset \mathrm{Ker}(\tilde{\pi})$.

Let us summarize our discussion so far. Let G be a connected Lie group with universal covering $p \colon \tilde{G} \to G$.

(1) The isomorphism classes of finite-dimensional irreducible representation of G are identified with those of $\tilde{G}$ which are trivial on $\mathrm{Ker}(p)$.
(2) Therefore, to classify the isomorphism classes of finite-dimensional irreducible representations of G, it is enough to classify those of $\tilde{G}$.
(3) Any finite-dimensional irreducible representation of $\tilde{G}$ gives a finite-dimensional irreducible representation of $\mathfrak{g}$ by differentiation (Proposition 13.5 (1)).
(4) The differential representations of $\mathfrak{g}$ are isomorphic if and only if the original representations of $\tilde{G}$ are isomorphic (Proposition 13.5 (2)).
(5) Conversely, any finite-dimensional irreducible representation of $\mathfrak{g}$ can be "lifted" uniquely to a finite-dimensional irreducible representation of $\tilde{G}$ (Corollary 13.10).
(6) Therefore, the isomorphism classes of finite-dimensional irreducible continuous representations of $\tilde{G}$ bijectively correspond to those of $\mathfrak{g}$ by differentiation.
(7) By Proposition 13.3, we are furthermore reduced to classifying the isomorphism classes of finite-dimensional irreducible continuous representations of $\mathfrak{g}_{\mathbb{C}}$.

13.3. **Example: $\mathrm{SO}(3)$.** Recall from Lecture 11 the definitions of $\mathrm{SU}(2)$ and $\mathrm{SO}(3)$:

$$\mathrm{SU}(2) := \{g \in \mathrm{SL}_2(\mathbb{C}) \mid {}^t\bar{g} \cdot g = I_2\},$$

$$\mathrm{SO}(3) := \{g \in \mathrm{SL}_3(\mathbb{R}) \mid {}^t g \cdot g = I_3\}.$$

Also, recall that the Lie algebras of these Lie groups are given as follows:

$$\mathfrak{su}(2) := \{g \in \mathfrak{sl}_2(\mathbb{C}) \mid {}^t\bar{g} + g = 0\},$$

$$\mathfrak{so}(3) := \{g \in \mathfrak{sl}_3(\mathbb{R}) \mid {}^t g + g = 0\}.$$

We consider the conjguate action of $\mathrm{SU}(2)$ on $\mathfrak{su}(2)$, i.e.,

$$g \cdot X := gXg^{-1}$$

for any $g \in \mathrm{SU}(2)$ and $\mathfrak{su}(2)$. It is not difficult to check that gXg^{-1} again belongs to $\mathfrak{su}(2)$. Furthermore, note that $\mathfrak{su}(2)$ is a 3-dimensional $\mathbb{R}$-vector space equipped with a symmetric bilinear form

$$(X, Y) \mapsto -\frac{1}{2}\,\mathrm{tr}(XY).$$

In fact, we can also check that this form is positive-definite and preserved by the conjugate action of $\mathrm{SU}(2)$. Consequently, the conjugate action induces a map

$$\mathrm{SU}(2) \to \mathrm{O}(\mathfrak{su}(2)) \cong \mathrm{O}(3).$$

In fact, the image of this map is contained in $\mathrm{SO}(3)$.

Fact 13.11 ([Hal15, Example 5.15])**.** *This map is the universal covering with kernel* $\{\pm I_2\}$.

13.4. **Exercises.**

Exercise 13.1. Complete the discussion of Section 13.3. More precisely,

(1) Check that the bilinear form on $\mathfrak{su}(2)$ is indeed positive-definite.
(2) Show that the image of the map $\mathrm{SU}(2) \to \mathrm{O}(3)$ is contained in $\mathrm{SO}(3)$.

14. Lecture 14: Representaions of $\mathfrak{sl}_2$

Today we discuss representation theory of $\mathfrak{sl}_2(\mathbb{C})$:

$$\mathfrak{sl}_2(\mathbb{C}) := \{X \in M_2(\mathbb{C}) \mid \mathrm{tr}(X) = 0\}.$$

Recall that the Lie bracket structure on $\mathfrak{sl}_2(\mathbb{C})$ is defined by

$$[X, Y] := XY - YX$$

for $X, Y \in M_2(\mathbb{C})$. Also recall that we have

$$\mathfrak{sl}_2(\mathbb{C}) = \mathbb{C}H \oplus \mathbb{C}E \oplus \mathbb{C}F,$$

where

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then the bracket structure is described by

$$[H, E] = 2E, \qquad [H, F] = -2F, \qquad [E, F] = H.$$

The aim of this week is to classify the irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$ as a Lie algebra over $\mathbb{C}$.

14.1. **Standard representation of $\mathfrak{sl}_2(\mathbb{C})$.** We consider the 2-dimensional representation (std, V_2) of $\mathfrak{sl}_2(\mathbb{C})$ naturally realized on $V_2 := \mathbb{C}^{\oplus 2}$:

$$\mathrm{std}\colon \mathfrak{sl}_2(\mathbb{C}) \hookrightarrow \mathfrak{gl}_2(\mathbb{C}) \cong \mathrm{End}_{\mathbb{C}}(V_2).$$

More precisely, for any $X = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathfrak{sl}_2(\mathbb{C})$ and $v = \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in V_2$, the action of X on v is given by the usual matrix multiplication:

$$\mathrm{std}(X)(v) := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}.$$

We put $e_1 := \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ and $e_2 := \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ so that $V_2 = \mathbb{C}e_1 \oplus \mathbb{C}e_2$.

Recall that, in the case of representations of a group G, we can consider the tensor product representation $(\pi \otimes \pi', V \otimes V')$ for any representations (π, V) and (π', V') of G. Also, for any representation (π, V) of G, we can consider its *n-th symmetric power representation* $(\mathrm{Sym}^n(\pi), \mathrm{Sym}^n(V))$ in a natural way.

We can consider a similar construction for representations of Lie algebras, but we have to be a bit careful. For representations (ρ, V) and (ρ', V') of a Lie algebra $\mathfrak{g}$, we define their tensor product representation $(\rho \otimes \rho', V \otimes V')$ by

$$(\rho \otimes \rho')(X)(v \otimes v') := (\rho(X)v) \otimes v' + v \otimes (\rho'(X)v')$$

for $X \in \mathfrak{g}$, $v \in V$, $v' \in V'$. (The necessary observations for understanding the validity of this definition are as follows: (1) If we adopt the exactly same definition as in the group case, then we cannot obtain a representation of a Lie algebra, in other words, $\rho \otimes \rho'\colon \mathfrak{g} \to \mathrm{End}_{\mathbb{C}}(V)$ is not a Lie algebra homomorphism in general. (2) The above defining formula naturally appear when we compute the differential representation of the tensor representation of a Lie group.) We define the symmetric power representations $(\mathrm{Sym}^n(\rho), \mathrm{Sym}^n(V))$ of a representation (ρ, V) of a Lie algebra $\mathfrak{g}$ in a similar way. More precisely, for any $X \in \mathfrak{g}$, we define

$$\mathrm{Sym}^n(\rho)(X)(v_1 \otimes \cdots \otimes v_n) := \sum_{i=1}^{n} v_1 \otimes \cdots \otimes v_{i-1} \otimes \rho(v_i) \otimes v_{i+1} \otimes \cdots \otimes v_n.$$

We also consider the n-th symmetric power representation

$$(s_{n+1}, V_{n+1}) := (\mathrm{Sym}^n \mathrm{std}, \mathrm{Sym}^n V_2)$$

of the standard representation for any $n \geq 1$. For convenience, we put (s_1, V_1) to be the trivial representation of $\mathfrak{sl}_2(\mathbb{C})$, i.e., $V_0 = \mathbb{C}$ and any element of $\mathfrak{sl}_2(\mathbb{C})$ acts on $\mathbb{C}$ via zero. Here, please be careful about the numbering; we put $(+1)$-shift so that $\dim_{\mathbb{C}} V_{n+1} = n + 1$. By definition of the symmetric power, we can find the following standard basis of V_{n+1}:

$$\{v_i \mid 0 \leq i \leq n\}, \quad v_i := e_1^{\otimes i} \otimes e_2^{\otimes (n-i)}.$$

In the following, as long as there is no risk of confusion, we simply omit the symbol for a representation when we do any computation, e.g., $X \cdot v$ denotes $s_{n+1}(X)(v)$ for any $X \in \mathfrak{sl}_2(\mathbb{C})$ and $v \in V_{n+1}$.

Proposition 14.1. *The representation (s_{n+1}, V_{n+1}) is an irreducible $(n+1)$-dimensional representaion of $\mathfrak{sl}_2(\mathbb{C})$.*

Proof. We first note that, for any $0 \leq i \leq n$,

$$H \cdot v_i = (-n + 2i)v_i.$$

Indeed, noting that $H \cdot e_1 = e_1$ and $H \cdot e_2 = -e_2$ we have

$$\begin{aligned} H \cdot v_i &= H \cdot e_1^{\otimes i} \otimes e_2^{\otimes (n-i)} \\ &= i \cdot e_1^{\otimes i} \otimes e_2^{\otimes (n-i)} - (n-i) \cdot e_1^{\otimes i} \otimes e_2^{\otimes (n-i)} \\ &= (-n + 2i) \cdot e_1^{\otimes i} \otimes e_2^{\otimes (n-i)} = (-n + 2i)v_i. \end{aligned}$$

In other words, v_i is an eigenvector of H with eigenvalue $-n + 2i$. Since $\{v_i \mid 0 \leq i \leq n\}$ forms a $\mathbb{C}$-basis of V_{n+1}, this means that the action of H on V_{n+1} is diagonalized with respect to $\{v_0, \ldots, v_n\}$ with eigenvalues $\{-n, -n+2, \ldots, n-2, n\}$ (the multiplicity of each eigenvalue is one).

We also consider how E and F acts on v_i's. We first compute $E \cdot v_i$. Since $E \cdot e_1 = 0$ and $E \cdot e_2 = e_1$, we have

$$E \cdot v_i = E \cdot e_1^{\otimes i} \otimes e_2^{\otimes (n-i)} = (n-i) \cdot e_1^{\otimes (i+1)} \otimes e_2^{\otimes (n-i-1)} = (n-i)v_{i+1}.$$

We similarly compute $F \cdot v_i$. Since $F \cdot e_1 = e_2$ and $F \cdot e_2 = 0$, we have

$$F \cdot v_i = F \cdot e_1^{\otimes i} \otimes e_2^{\otimes (n-i)} = i \cdot e_1^{\otimes (i-1)} \otimes e_2^{\otimes (n-i+1)} = iv_{i-1}.$$

Now let us show the irreducibility of (s_{n+1}, V_{n+1}). For this, suppose that we have a nonzero subrepresentation $W \subset V_{n+1}$. Since W is a nonzero subrepresentation (in particular, stable under the action of H), we can find an eigenvector of H belonging to W. As $v_0, \ldots, v_n$ are only the possible eigenvectors of H (up to scalar), we must have $v_j \in W$ for some $0 \leq j \leq n$. As we saw above, we can obtain all v_i's by applying E and F repeatedly to v_j. Thus we necessarily have $W = V_{n+1}$. □

By this proposition implies that, for each $n \in \mathbb{Z}_{>0}$, there exists at least one n-dimensional irreducible representaion, i.e., (s_n, V_n). So the question is; whether these exhaust all irreducible representaions of $\mathfrak{sl}_2(\mathbb{C})$? Our next aim is to show that this is indeed true.

14.2. **Irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$.**

Proposition 14.2. *For any finite-dimensional irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$ is isomorphic to (s_{n+1}, V_{n+1}) for some $n \in \mathbb{Z}_{\geq 0}$.*

Proof. Let (ρ, V) be a finite-dimensional irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$. Similarly to the proof of the previous proposition, we note the action of H first. Since V is finite-dimensional, there exists an eigenvector $v \in V$ of with respect to the action of H. Let $\lambda \in \mathbb{C}$ be the eigenvalue of v.

We first consider $E \cdot v$. By using the formula $[H, E] = 2E$, we get[3]

$$H(E \cdot v) = ([H, E] + EH) \cdot v = 2E \cdot v + E \cdot (\lambda \cdot v) = (\lambda + 2)E \cdot v.$$

In other words, $E \cdot v$ is an eigenvector of H with eigenvalue $\lambda + 2$ provided that $E \cdot v \neq 0$. Here, note that there are only finitely many eigenvalues of H. In particular, we may choose $v \in V$ and $\lambda \in \mathbb{C}$ such that $\lambda + 2$ is not an eigenvalue of H. If we choose v and λ in this way, then we necessarily see that $E \cdot v = 0$.

We next consider $F \cdot v$. By using the formula $[H, F] = -2F$ similarly, we get

$$H(F \cdot v) = ([H, F] + FH) \cdot v = -2F \cdot v + F \cdot (\lambda \cdot v) = (\lambda - 2)F \cdot v.$$

In other words, $F \cdot v$ is an eigenvector of H with eigenvalue $\lambda - 2$ provided that $F \cdot v \neq 0$. By repeating this computation, we see that $F^k \cdot v := F \cdots F \cdot v$ (F is applied k times)[4] is an eigenvector of H with eigenvalue $\lambda - 2k$ provided that $F \cdots F \cdot v \neq 0$.

Again by noting that there are only finitely many eigenvalues of H, we necessarily have $F^n \cdot v = 0$ for some $n \in \mathbb{Z}_{>0}$. We let $n \in \mathbb{Z}_{\geq 0}$ be the unique non-negative integer such that $F^n \cdot v \neq 0$ and $F^{n+1} \cdot v = 0$.

Now, we note that

$$E \cdot F^k \cdot v = k(\lambda - k + 1) \cdot F^{k-1} \cdot v.$$

Indeed, we can verify this via induction as follows. When $k = 1$, we have

$$E \cdot F \cdot v = ([E, F] + F \cdot E) \cdot v = (H + F \cdot E) \cdot v = \lambda \cdot v.$$

If the formula is true for k, then we have

$$\begin{aligned}
E \cdot F^{k+1} \cdot v &= (E \cdot F) \cdot F^k \cdot v \\
&= ([E, F] + F \cdot E) \cdot F^k \cdot v \\
&= (H + F \cdot E) \cdot F^k \cdot v \\
&= (\lambda - 2k) \cdot F^k \cdot v + F \cdot (E \cdot F^k \cdot v) \\
&= (\lambda - 2k) \cdot F^k \cdot v + F \cdot (k(\lambda - k + 1) \cdot F^{k-1} \cdot v) \\
&= ((\lambda - 2k) + k(\lambda - k + 1)) \cdot (F^k \cdot v) \\
&= (k + 1)(\lambda - k) \cdot F^k \cdot v.
\end{aligned}$$

[3] Please be careful about the abbreviation of the notation here. The symbol $H(E \cdot v)$ means $\rho(H)\rho(E)(v)$. This is not equal to $\rho(HE)(v)$ since ρ is a Lie algebra homomorphism (does not preserve the matrix multiplication!). If we write the computation here without this abbreviation, it is like this: $\rho(H)\rho(E)(v) = (\rho([H, E]) + \rho(E)\rho(H))(v)$.

[4] Similarly to the previous remark, please be careful that this is $\rho(F) \cdots \rho(F)(v)$, which is not equal to $\rho(F^k)(v)$, where F^k in $\rho(-)$ is taken as a matrix product.

We apply this formula to $k = n+1$. Since $F^{n+1}v = 0$ and $F^n v \neq 0$, the formula

$$E \cdot F^{n+1} \cdot v = (n+1)(\lambda - n) \cdot F^n \cdot v$$

implies that $\lambda = n$. Now we consider the subspace W of V as follows:

$$W = \bigoplus_{i=0}^{n} \mathbb{C}F^i v.$$

Each $F^i v$ is an eigenvector of H with eigenvalue $n - 2i$. Obviously F maps each $\mathbb{C}F^i v$ to $\mathbb{C}F^{i+1}v$. Also, as we saw above, E maps each $\mathbb{C}F^i v$ to $\mathbb{C}F^{i-1}v$. Therefore, W is closed under the actino of $\mathfrak{sl}_2(\mathbb{C})$, hence a subrepresentation of V. Since V is irreducible, we necessarily have $V = W$.

Now let us construct an isomorphism between (ρ, V) and (s_{n+1}, V_{n+1}). Recall that we have a basis $\{v_0, \dots, v_n\}$ of V_{n+1} such that

- $H \cdot v_i = (-n + 2i)v_i$,
- $E \cdot v_i = (n - i)v_{i+1}$,
- $F \cdot v_i = i v_{i-1}$.

On the other hand, the basis $\{v, F \cdot v, \cdots, F^n \cdot v\}$ satisfies

- $H \cdot (F^j \cdot v) = (n - 2j) \cdot F^j \cdot v$,
- $E \cdot F^j \cdot v = j(n - j + 1) \cdot F^{j-1} \cdot v$,
- $F \cdot F^j \cdot v = F^{j+1} \cdot v$.

Therefore, if we put $\varphi(F^j \cdot v) := \frac{n!}{(n-j)!} v_{n-j}$, we see that the resulting map

$$\varphi \colon V = \bigoplus_{j=0}^{n} \mathbb{C}F^j \cdot v \to \bigoplus_{i=0}^{n} \mathbb{C}v_i$$

is an isomoprhism as representaions of $\mathfrak{sl}_2(\mathbb{C})$. □

Remark 14.3. The above proposition is important, but note that its proof is also important. From this proof, we can see that any semisimple representation of $\mathfrak{sl}_2(\mathbb{C})$ can be decomposed into irreducible representations by choosing a vector "v" as in the above proof (generally called the *highest weight vector*) and producing an irreducible subrepresentation by applying F repeatedly to v.

14.3. **How to compute differential representations.** Let G be a Lie group. We take a finite-dimensional representation $\pi \colon G \to \mathrm{GL}_n(\mathbb{C})$ and its differential representation $d\pi \colon \mathfrak{g} \to \mathfrak{gl}_n(\mathbb{C})$.

Let us suppose that G a linear group (i.e., contained in a general linear group as its closed subgroup), so that we have the exponential map $\exp \colon \mathfrak{g} \to G$. To be more precise, if we let $G \hookrightarrow \mathrm{GL}_N(\mathbb{C})$, then we can define the exponential for G to be the restriction of that for $\mathrm{GL}_N(C)$;

$$\begin{array}{ccc} G & \hookrightarrow & \mathrm{GL}_N(\mathbb{C}) \\ {\scriptstyle \exp}\uparrow & & \uparrow{\scriptstyle \exp} \\ \mathfrak{g} & \hookrightarrow & \mathfrak{gl}_N(\mathbb{C}) \end{array}$$

In this case, we can compute $d\pi$ using the exponential map as follows.

Let $X \in \mathfrak{g}$ (so we want to determine $d\pi(X)$). We consider the 1-parameter subgroup

$$\gamma_X \colon \mathbb{R} \to G; \quad t \mapsto \exp(tX)$$

and compose it with π:

$$\pi \circ \gamma_X \colon \mathbb{R} \xrightarrow{\gamma_X} G \xrightarrow{\pi} \mathrm{GL}_n(\mathbb{C}).$$

Then note that

$$d(\pi \circ \gamma_X)_0 = d\pi_e \circ d\gamma_{X,0}. \tag{1}$$

The point here is that $\pi \circ \gamma_X$ is a smooth map from $\mathbb{R}$ to $\mathrm{GL}_n(\mathbb{C})$, which is open in $M_N(\mathbb{C}) \cong \mathbb{C}^{N^2}$, we may compute its differential in a standard way (i.e., take the high-school differentiation of the $\mathbb{C}^{N^2}$-valued function $\pi \circ \gamma_X$ entry-wise). To be more precise, if we let $(\frac{\partial}{\partial t})(0)$ be the standard basis of $T_0\mathbb{R}$ (i.e., taking the differential coefficient at $t = 0$), we have

$$d(\pi \circ \gamma_X)_0((\tfrac{\partial}{\partial t})(0)) = \frac{\partial}{\partial t}(\pi \circ \gamma_X)|_{t=0} = \frac{\partial}{\partial t}\pi(\exp(tX))|_{t=0}.$$

On the other hand, we can apply the same discussion to compute $d\gamma_{X,0}$. Since the exponential map for G is induced from that of $\mathrm{GL}_N(\mathbb{C})$ and the embedding $G \hookrightarrow \mathrm{GL}_N(\mathbb{C})$ is smooth, we can compute

$$d\gamma_{X,0}((\tfrac{\partial}{\partial t})(0)) = \frac{\partial}{\partial t}\exp(tX)|_{t=0} = \frac{\partial}{\partial t}(1 + tX + \frac{1}{2!}(tX)^2 + \cdots)|_{t=0} = X$$

(here, the target of $d\gamma_{X,0}$ is T_eG, which is identified with $\mathfrak{g} \subset \mathfrak{gl}_N(\mathbb{C})$.)

Hence, by combining these computations with (1), we obtain

$$d\pi_e(X) = \frac{\partial}{\partial t}\pi(\exp(tX))|_{t=0}.$$

Example 14.4. Let $G := \mathrm{SL}_2(\mathbb{R})$ or $\mathrm{SL}_2(\mathbb{C})$ and consider its standard representation $(\mathrm{Std}, V_2 := \mathbb{C}^{\oplus 2})$, i.e., $\mathrm{Std}\colon G \hookrightarrow \mathrm{GL}_2(\mathbb{C})$ is the natural injection. In this case, we have

$$d\mathrm{Std}_e(X) = \frac{\partial}{\partial t}\mathrm{Std}(\exp(tX))|_{t=0} = \frac{\partial}{\partial t}(1 + tX + \frac{1}{2!}(tX)^2 + \cdots)|_{t=0} = X.$$

Hence the differential representation $d\mathrm{Std}$ is nothing but the inclusion $\mathfrak{g} \hookrightarrow \mathfrak{gl}_2(\mathbb{C})$, which was introduced as the standard representation of $\mathfrak{sl}_2(\mathbb{C})$ earlier (in the case where $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$).

Example 14.5. Let G be a linear group and consider its finite-dimensional representations (π_1, V_1) and (π_2, V_2). We consider the differential representation of $(\pi_1 \otimes \pi_2, V_1 \otimes V_2)$. By the product rule for differentiation, we get

$$\begin{aligned} d(\pi_1 \otimes \pi_2)_e(X) &= \frac{\partial}{\partial t}\big(\pi_1(\exp(tX)) \otimes \pi_2(\exp(tX))\big)|_{t=0} \\ &= \Big(\frac{\partial}{\partial t}\pi_1(\exp(tX))|_{t=0} \otimes \pi_2(e) + \pi_1(e) \otimes \frac{\partial}{\partial t}\pi_2(\exp(tX))|_{t=0}\Big) \\ &= d\pi_1(X) \otimes \mathrm{id}_{V_2} + \mathrm{id}_{V_1} \otimes d\pi_2(X). \end{aligned}$$

This is nothing but the tensor product of representations of a Lie algebra introduced before. In particular, if we consider the n-th symmetric power representation $(S_{n+1}, V_{n+1}) := (\mathrm{Sym}^n \mathrm{Std}, \mathrm{Sym}^n V_2)$ of the standard representation of $\mathrm{SL}_2(\mathbb{C})$, then its differential representation is nothing but $(s_{n+1}, V_{n+1}) := (\mathrm{Sym}^n \mathrm{std}, \mathrm{Sym}^n V_2)$, i.e., that of $\mathfrak{sl}_2(\mathbb{C})$ introduced earlier.

14.4. **Irreducible representations of $\mathrm{SL}_2(\mathbb{R})$, $\mathrm{SU}(2)$, $\mathrm{SO}(3)$.** Recall that we have a surjective map

$$\mathrm{SU}(2) \twoheadrightarrow \mathrm{SO}(3)$$

which is a universal cover with kernel $\{\pm I_2\}$. In particular, this map induces an isomorphism on the Lie algebras.

Lemma 14.6. *The complexification of $\mathfrak{su}(2)$ is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$.*

Proof. Note that, by definition, we have a natural injective homomorphism

$$\mathfrak{su}(2) \hookrightarrow \mathfrak{sl}_2(\mathbb{C})$$

as Lie algebras over $\mathbb{R}$. We extend this to a homomorphism as Lie algebras over $\mathbb{C}$ naturally:

$$\mathfrak{su}(2)_{\mathbb{C}} \to \mathfrak{sl}_2(\mathbb{C})\colon X \otimes z \mapsto zX.$$

Let us show that this is an isomorphism. Since the bracket structure on $\mathfrak{su}(2)_{\mathbb{C}}$ is also extended from that of $\mathfrak{su}(2)$ naturally, this homomorphism preserves the bracket products. For any $A \in \mathfrak{sl}_2(\mathbb{C})$, we put

$$X = \frac{A - {}^t\bar{A}}{2}, \qquad Y = \frac{A + {}^t\bar{A}}{2\sqrt{-1}}.$$

Then X and Y belongs to $\mathfrak{su}(2)$ and satisfy $A = X + \sqrt{-1}Y$. This shows that $\mathfrak{su}(2)_{\mathbb{C}} \hookrightarrow \mathfrak{sl}_2(\mathbb{C})$ is surjective. Since these have the same dimension, this must be bijective. □

So the relations of Lie groups of our interest are summarized in the following diagram:

$$\mathrm{SL}_2(\mathbb{R}) \hookrightarrow \mathrm{SL}_2(\mathbb{C}) \hookleftarrow \mathrm{SU}(2) \twoheadrightarrow \mathrm{SO}(3)$$

$$\mathfrak{sl}_2(\mathbb{R}) \hookrightarrow \mathfrak{sl}_2(\mathbb{C}) \hookleftarrow \mathfrak{su}(2) \xrightarrow{\cong} \mathfrak{so}(3)$$

Here, note that all $\mathfrak{sl}_2(\mathbb{R})$, $\mathfrak{su}(2)$, and $\mathfrak{so}(3)$ have the same complexifications $\mathfrak{sl}_2(\mathbb{C})$.

Now we complete the classification of irreducible representations of $\mathrm{SL}_2(\mathbb{R})$, $\mathrm{SU}(2)$, and $\mathrm{SO}(3)$.

14.4.1. $\mathrm{SL}_2(\mathbb{R})$. We define the standard representation (Std, V_2) and its n-th symmetric power representation (S_{n+1}, V_{n+1}) of $\mathrm{SL}_2(\mathbb{R})$ in the same manner as before.

Proposition 14.7. *Each (S_{n+1}, V_{n+1}) is an irreducible $(n+1)$-dimensional representaion of $\mathrm{SL}_2(\mathbb{R})$. Conversely, for $n \in \mathbb{Z}_{\geq 0}$, any irreducible $(n+1)$-dimensional representaion of $\mathrm{SL}_2(\mathbb{R})$ is isomorphic to (S_{n+1}, V_{n+1}).*

Proof. The differential representation of (S_{n+1}, V_{n+1}) is equal to (s_{n+1}, V_{n+1}) (viewed as a representation of the complexification $\mathfrak{sl}_2(\mathbb{C})$). Since this is irreducible, so is (S_{n+1}, V_{n+1}) (Proposition 13.5 (1); note that $\mathrm{SL}_2(\mathbb{R})$ is connected). Suppose that there is another $(n+1)$-dimensional irreducible representation (π, V) of $\mathrm{SL}_2(\mathbb{R})$. Then its differential representation $(d\pi, V)$ also defines an irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$ again by Proposition 13.5 (1). Since the irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$ are classfied only by the dimensions, we see that $(d\pi, V)$ is isomorphic to (s_{n+1}, V_{n+1}). THis implies that $(\pi, V) \cong (S_{n+1}, V_{n+1})$ by Proposition 13.5 (2). □

14.4.2. $\mathrm{SU}_2(\mathbb{R})$. We consider the standard Std representation of SU(2)

$$\mathrm{Std}\colon \mathrm{SU}(2) \hookrightarrow \mathrm{SL}_2(\mathbb{C}).$$

Similarly to the case of $\mathrm{SL}_2(\mathbb{R})$, we consider its n-th symmetric power representation (S_{n+1}, V_{n+1}) of SU(2). Then we can check that its differential representation exactly induces the representation of (s_{n+1}, V_{n+1}) of $\mathfrak{sl}_2(\mathbb{C})$ after identifying $\mathfrak{su}(2)_{\mathbb{C}} \cong \mathfrak{sl}_2(\mathbb{C})$. Hence, the same proof as in the case of $\mathrm{SL}_2(\mathbb{R})$ implies the following:

Proposition 14.8. *Each (S_{n+1}, V_{n+1}) is an irreducible $(n+1)$-dimensional representaion of* SU(2)*. Conversely, for $n \in \mathbb{Z}_{\geq 0}$, any irreducible $(n+1)$-dimensional representaion of* SU(2) *is isomorphic to (S_{n+1}, V_{n+1}).*

(If you want only know that the isomoprhism classes of the irreducible representaions of SU(2) are determined by dimensions, you can appeal to the simply-connectedness of SU(2). So the point of the discussion is that each irreducible representation can be constructed in a concrete way as above.)

14.4.3. SO(3).

Proposition 14.9. *The dimension of any irreducible continuous representation of* SO(3) *is necessarily odd. Conversely, for each odd integer $2m+1$, there exists a unique (up to isomorphism) irreducible continuous $(2m+1)$-dimensional representation of $SO(3)$.*

Proof. Since $\mathrm{SU}(2) \twoheadrightarrow \mathrm{SO}(3)$ is the universal covering with kernel $\{\pm I_2\}$, it is enough to determine when the $(n+1)$-dimensional irreducible representation (S_{n+1}, V_{n+1}) of SU(2) is trivial on $-I_2$. Since (S_{n+1}, V_{n+1}) is the n-th symmetric power of the standard representation of SU(2), the action of $-I_2$ is given by $(-1)^n$. Hence it is trivial if and only if n is even. □

14.5. **Exercises.**

Exercise 14.1. For $n \geq 1$, we consider a $\mathbb{C}$-linear homomorphism

$$m\colon V_2 \otimes V_{n+1} \to V_{n+2}; \quad v \otimes v' \mapsto v \otimes v'.$$

(1) Show that m is a surjective homomorphism as representations of$\mathfrak{sl}_2(\mathbb{C})$.
(2) Construct an isomorphism $\mathrm{Ker}(m) \cong V_n$ as representations of$\mathfrak{sl}_2(\mathbb{C})$ to conclude that $V_2 \otimes V_{n+1} \cong V_n \oplus V_{n+2}$.

Appendix A. Regulations for exercise sessions and grading

This course has Exercise Sessions once every four lectures. Thus it will be like:

$$\text{2h lecture} \to \text{2h lecture} \to \text{2h lecture} \to \text{2h Exercise Session}$$
$$\to \text{2h lecture} \to \text{2h lecture} \to \text{2h lecture} \to \text{2h Exercise Session} \to \cdots$$

The grade for this course will be determined by

(1) written reports and
(2) oral presentations.

Lecture notes will be uploaded for each lecture, and exercise problems will be included in the notes. Solve these problems and submit your solutions. Basically, you can receive up to **4 points** for each problem. Your final score will be the total number of points you earn. Please collect as many points as you want!

However, I also would like to introduce the following special rules:

- You can receive up to **80 points** in total by submitting written reports.
- If you give an oral presentation of your solution to any problem at the blackboard during an Exercise Session, you can receive up to **20 points**.
- To pass the course, you must give an oral presentation **at least once**.

Example A.1. (1) [oral $\varnothing$] + [written 80 pts] $\rightsquigarrow$ [No credit].
(2) [oral 5 pts] + [written 80 pts] $\rightsquigarrow$ [85 pts].
(3) [oral 15 pts] + [oral 20 pts] + [written 70 pts] $\rightsquigarrow$ [100 pts].
(4) [oral 20 pts]$\times$5 +[written $\varnothing$] $\rightsquigarrow$ [100 pts].

The detailed regulations are explained below.

A.1. **Written reports.**

(1) Please submit your written solution through NTU COOL or by email to me.
(2) You may submit your reports at any time during the semester.
(3) After each Exercise Session, the reports submitted up to that point will be returned to you with your score and brief feedback.
(4) After getting a feedback, if your score is less than 4, you can modify your solution following the feedback and **re-submit up to once**. If your re-submitted solution is improved, you may get more points.

A.2. **Oral presentations.**

(1) During an Exercise Session, students will present solutions to problems they chose at the blackboard.
(2) Each presentation must be given **within 15 minutes**, and then a 5 minutes question/comment time will be given. (So, at maximum, 6 students can give oral presentations in one Exercise Session.)
(3) Before each Exercise Session, students can express which problem they want to present (through the Google spreadsheet for this course).
    - The list of students and their preferred problems will be visible to everyone.
    - Each student can select more than one problems.
    - If two or more students wish to present the same problem, priority will be given to the student who has given fewer oral presentations. (If they

have given the same number of oral presentations, the presenter will be selected randomly.)
    - Within the day before an Exercise Session, the list of presenters will be announced.
    - I would like to do my best so that presentation opportunities will be allocated to every student equally.
(4) If you are selected as a presenter, please prepare a 15-minute presentation of your solution. You should try to explain your argument so that the other students, TA (Hao-An Wu), and I can follow it. Please regard this as an opportunity to develop your presentation skills.
(5) In addition to the oral presentation, **the presenter must submit a written solution** to the problem by the Exercise Session.
(6) You will receive feedback on your written solution and may revise and resubmit it afterward. An improved written solution may result in a higher score as well. However, the quality of your oral presentation has most weights. For example;
    - Suppose that you have prepared your presentation seriously, but during the presentation a mistake in your argument is discovered and the proof breaks down. This is not necessarily a serious problem. If you understand the issue through the subsequent discussion and successfully correct it in your revised written solution, you may still receive a high score.
    - On the other hand, suppose that your written solution is perfect, but it is clear that you have not prepared for the presentation. Similarly, if you are unable to explain your own argument or answer basic questions about your solution, this may result in a low score even if the written solution itself is excellent.

A.3. **Q & A.**

Question 1. Can I give an oral presentation of a problem which were already presented by someone at the blackboard?

Answer 1. No. Each problem can be presented at the blackboard at most once.

Question 2. Can I submit a written solution to a problem which were already presented by someone at the blackboard?

Answer 2. Yes. In this case, probably you already got to know how to solve the problem. However, what is important in writing a report is not only its correctness, but also how it is organized/explained. Please try to write a report in your words with your understanding even if you already have listened to other student's oral presentation on the problem.

Question 3. Can I submit all written solutions together at the end of the course?

Answer 3. Yes, in principle. However, please keep in mind that it will be more difficult for you to know how many points you can get precisely in advance.

Question 4. Is there any other necessary condition to get the course credit (e.g., attendance)?

Answer 4. No, but please do not forget to give an oral presentation at least once.

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