THEORY OF ALGEBRAIC GROUPS

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Contents

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1. Week 1: Course overview

1.1. **Why algebraic groups?** If you have ever studied the theory of manifolds, you might have encountered the notion of a Lie group. A Lie group is a mathematical object equipped with two different kinds of mathematical structures in a consistent way; the one is a manifold structure, and the other is a group struture. An "algebraic group" is an algebraic version of the notion of a Lie group, where a "manifold structure" is replaced with an "algebraic variety structure".

The theory of algebraic groups is interesting in its own right, but it also plays a very important role in applications. For example, much of modern representation theory is founded on the theory of algebraic groups. Nowadays, theory of algebraic groups has became an indispensable "language" for developing representation theory.

The aim of this course is to learn basics of the theory of algebraic groups, mainly following the textbooks [Bor91, Spr09, Mil17].

1.2. **Algebraic varieties.** Before introducing the definition of an algebraic group, we briefly review the notion of schemes. See any textbook on algebraic geometry for more details, for example, Hartshorne, Liu, etc...

Definition 1.1. For a ring[1] R, we put $\operatorname{Spec} R$ to be the set of all prime ideals of R. We call $\operatorname{Spec} R$ the *spectrum* of R.

Let R be a ring. For any ideal $I \subset R$, we define a subset $V(I)$ of $\operatorname{Spec} R$ by

$$V(I) := \{\mathfrak{p} \in \operatorname{Spec} R \mid I \subset \mathfrak{p}\}.$$

When I is a principal ideal (f) generated by an element $f \in R$, we simply write $V(f)$ instead of $V((f))$. Also, we put $D(f) := \operatorname{Spec} R \smallsetminus V(f)$.

Lemma 1.2.
(1) For any ideals $I, J \subset R$, we have $V(I) \cup V(J) = V(I \cap J)$.
(2) For any family of ideals $\{I_\lambda\}_{\lambda\in\Lambda}$ of R, we have $\bigcap_{\lambda\in\Lambda} V(I_\lambda) = V(\sum_{\lambda\in\Lambda} I_\lambda)$.
(3) We have $V(R) = \varnothing$ and $V(0) = \operatorname{Spec} R$.

Exercise 1.3. Prove this lemma.

The above lemma shows that the family $\{V(I) \mid I \subset R\text{: ideal}\}$ defines a topology on $\operatorname{Spec} R$ such that the closed subsets are the sets of the form $V(I)$. We call the topology on $\operatorname{Spec} R$ defined in this way the *Zariski topology.*

Note that, from the above definition, the closed points of $\operatorname{Spec} R$ are nothing but the maximal ideals of R.

Example 1.4. Let k be an algebraically closed field. We put $\mathbb{A}^n_k := \operatorname{Spec} k[x_1, \ldots, x_n]$ (where $k[x_1, \ldots, x_n]$ is the polynomial ring with n variables over k). Then $\mathbb{A}^n_k$ is called the *n-dimensional affine space* over k.

(1) Let us first consider the subset of closed points of $\mathbb{A}^k_n$. By the Hilbert's Nullstellensatz, any maximal ideal of $k[x_1, \ldots, x_n]$ is of the form $(x_1 - a_1, \ldots, x_n - a_n)$ for some $a_1, \ldots, a_n \in k$ (note that, for this, it is needed that k is algebraically closed).
(2) Let us next consider a closed subset $V(I) \subset \mathbb{A}^n_k$ for an ideal $I = (f_1, \ldots, f_r)$ of R generated by $f_1, \ldots, f_r \in k[x_1, \ldots, x_n]$. Let $x \in \mathbb{A}^n_k$ be a closed point corresponding to a maximal ideal $\mathfrak{m} = (x_1 - a_1, \ldots, x_n - a_n)$. Then $x \in V(I)$ if and only if $\mathfrak{m} \supset I$, which is furthermore equivalent to $f_1(a_1, \ldots, a_n) = \cdots = f_r(a_1, \ldots, a_n) = 0$.

[1]In this lecture, the word "ring" always means a commutative ring with unit.

In other words, the subset of closed points of $V(I)$ is identified with the set of simultaneous solutions to polynomial equations $f_1 = \cdots = f_r = 0$ in k^n.

Definition 1.5. Let X be a topological space. A *presheaf* $\mathcal{F}$ of abelian groups (resp. rings) on X is a contravariant functor from the category of open sets of X to the category of abelian groups (resp. rings). More precisely, $\mathcal{F}$ associates an abelian group (resp. a ring) $\mathcal{F}(U)$ to each open set $U \subset X$ such that

(1) $\mathcal{F}(\varnothing) = 0$,
(2) for any open subsets $V \subset U \subset X$, we have a group homomoprhism (resp. ring homomorphism) $\rho_{U,V} \colon \mathcal{F}(U) \to \mathcal{F}(V)$ (called the *restriction* homomorphism) satisfying
   - $\rho_{U,U} = \mathrm{id}_U$ for any open subset $U \subset X$,
   - $\rho_{U,W} = \rho_{V,W} \circ \rho_{U,V}$ for any open subsets $W \subset V \subset U \subset X$.

For each open set $U \subset X$, we call an element $s \in \mathcal{F}(U)$ a *section* of $\mathcal{F}$ over U. We write $s|_V$ in short for $\rho_{V,U}(s)$.

Definition 1.6. We say that a presheaf $\mathcal{F}$ on X is a *sheaf* if it satisfies the following conditions:

(1) For any open subset $U \subset X$ and its open covering $\{U_i\}_{i\in I}$, if a section $s \in \mathcal{F}(U)$ satisfies $s|_{U_i} = 0$ for every $i \in I$, then $s = 0$.
(2) For any open subset $U \subset X$ and its open covering $\{U_i\}_{i\in I}$, if a family of sections $\{s_i \in \mathcal{F}(U_i)\}_{i\in I}$ satisfies $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$ for every $i, j \in I$, then there exists $s \in \mathcal{F}(U)$ satisfying $s|_{U_i} = s_i$.

Now we let $X = \operatorname{Spec} R$ for a ring R. Then we can construct a (unique) sheaf of rings $\mathcal{O}_X$ on X such that

- $\mathcal{O}_X(D(f)) = R_f$ for any $f \in R$ (R_f denotes the localization of R with respect to f), and
- for any $f, g \in R$ such that $D(g) \subset D(f)$, the restriction $\rho_{D(f),D(g)} \colon \mathcal{O}_X(D(f)) \to \mathcal{O}_X(D(g))$ is given by the natural homomorphism $R_f \to R_g$ (note that f is invertible in R_g when $D(g) \subset D(f)$).

We call the sheaf $\mathcal{O}_X$ the *structure sheaf* of X.

Definition 1.7. We call the pair $(X = \operatorname{Spec} R, \mathcal{O}_X)$ the *affine scheme* associated to the ring R. We refer to R as the *coordinate ring* of X.

In general, a topological space equipped with a sheaf of rings is called a "ringed space". For ringed spaces, we can define the notion of a morphism. When a ringed space $(X, \mathcal{O}_X)$ is locally isomorphic to affine schemes (more precisely, there exists an open covering $\{U_i\}_{i\in I}$ of X such that each $(X, \mathcal{O}_X)|_{U_i}$ is isomorphic to an affine scheme), we call $(X, \mathcal{O}_X)$ a *scheme*. (We often omit the symbol $\mathcal{O}_X$ of the structure sheaf and simply write "X" for a scheme $(X, \mathcal{O}_X)$.)

Note that, when we have a ring homomorphism $\varphi \colon R \to S$, we can naturally define a continuous map $\varphi^\sharp \colon \operatorname{Spec} S \to \operatorname{Spec} R$ by $\varphi^\sharp(\mathfrak{p}) := \varphi^{-1}(\mathfrak{p})$ for any $\mathfrak{p} \in \operatorname{Spec} S$. This map furthemore naturally induces a morphism between ringed spaces $(X := \operatorname{Spec} S, \mathcal{O}_X) \to (Y := \operatorname{Spec} R, \mathcal{O}_Y)$.

Fact 1.8. *The association $R \mapsto (X = \operatorname{Spec} R, \mathcal{O}_X)$ gives a contravariant equivalence between*

- *the category of rings and*
- *the category of affine schemes.*

The inverse is given by $(X, \mathcal{O}_X) \mapsto \mathcal{O}_X(X)$.

When a ring R is a k-algebra, we say that the affine scheme $\operatorname{Spec} R$ is "over k". When X is an affine scheme over k, its coordinate ring (i.e., the ring R when $X = \operatorname{Spec} R$) is often denoted by $k[X]$.

When a scheme is made from affine schemes over k (such that any restriction morphism is a k-algebra homomorphism), we say that the scheme is over k. Any scheme X over k is equipped with a moprhism $X \to \operatorname{Spec} k$; locally, this is a morphism of affine schemes corresponding to the structure morphism $k \to R$ of a k-algebra R. We call $X \to \operatorname{Spec} k$ the "structure morphism" of X.

Definition 1.9. Let k be an algebraically closed field.[2].

(1) When R is a reduced finitely generated k-algebra, we call $\operatorname{Spec} R$ an *affine algebraic variety* over k.
(2) When a scheme X over k has a finite open covering $\{U_i\}_{i \in I}$ such that each U_i is an affine algebraic variety, we call X an *algebraic variety* over k.

As long as k is fixed and there is no confusion, we often omit the word "over k".

1.3. **Definition and examples of algebraic groups.** For any schemes X and Y over k, there uniquely (up to a unique isomorphism) exists their "fibered product" $X \times_k Y$, which is a scheme over k equipped with morphisms $p_1 \colon X \times_k Y \to X$ and $p_2 \colon X \times_k Y \to Y$ over k satisfying the following "universal property":

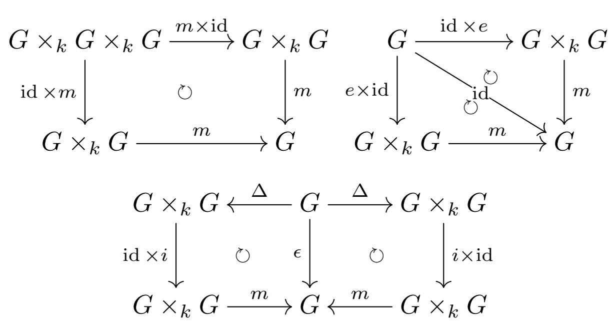
> for any scheme Z over k equipped with morphisms $q_1 \colon Z \to X$ and $q_2 \colon Z \to Y$ over k, there uniquely exists a morphism $f \colon Z \to X \times_k Y$ over k such that $q_1 = p_1 \circ f$ and $q_2 = p_2 \circ f$.

Note that, when $X = \operatorname{Spec} R$ and $Y = \operatorname{Spec} S$ for k-algebras R and S, their fibered product is simply given by $\operatorname{Spec}(R \otimes_k S)$ (the morphisms p_1 and p_2 are given by the natural k-algebra homomorphisms $R \to R \otimes_k S$ and $S \to R \otimes_k S$).

Definition 1.10 (algebraic group)**.** Let G be an algebraic variety over k. We say that G is an *algebraic group over* k if G is equipped with a group structure, i.e., morphisms of schemes over k

- $m \colon G \times_k G \to G$ ("multiplication morphism"),
- $i \colon G \to G$ ("inversion morphism"), and
- $e \colon \operatorname{Spec} k \to G$ ("unit element")

satisfying the axioms of groups. More precisely, the following diagrams are commutative:



Here, ϵ denotes the composition of the structure morphism $G \to \operatorname{Spec} k$ and $e \colon \operatorname{Spec} k \to G$.

[2]In this lecture, for the definition of an algebraic variety, we always assume that k is an algebraically closed field. Also, please be careful that the definition of the word "algebraic variety" heavily depends on textbooks. The definition given here may not be very universal.

Definition 1.11. Let G and H be algebraic group over k. We say that a morphism $f\colon G \to H$ over k is a *homomorphism* of algebraic groups if the following diagram is commutative:

$$\begin{array}{ccc} G \times_k G & \xrightarrow{f \times f} & H \times_k H \\ {\scriptstyle m}\downarrow & \circlearrowleft & \downarrow{\scriptstyle f} \\ G & \xrightarrow{\quad m \quad} & H \end{array}$$

Here, the left vertical arrow denotes the multiplication morphism for G and the right one denotes that for H.

Remark 1.12. Suppose that G is an affine algebraic variety with coordinate ring $k[G]$ (i.e., $G = \operatorname{Spec} k[G]$). Recall that the category of affine schemes is equivalent to the category of rings. Thus giving G an algebraic group structure is equivalent to defining k-algebra homomorphisms

- $m\colon k[G] \to k[G] \otimes_k k[G]$,
- $i\colon k[G] \to k[G]$,
- $e\colon k[G] \to k$.

In general, a commutative ring equipped with such an additional structure is called a *Hopf algebra*.

Example 1.13. (1) We put $\mathbb{G}_{\mathrm{a}} := \operatorname{Spec} k[x]$ and define m, i, and e at the level of rings as follows:
- $m\colon k[x] \to k[x] \otimes_k k[x]; \quad x \mapsto x \otimes 1 + 1 \otimes x$,
- $i\colon k[x] \to k[x]; \quad x \mapsto -x$,
- $e\colon k[x] \to k; \quad x \mapsto 0$.

Then $\mathbb{G}_{\mathrm{a}}$ is an algebraic group over k with respect to the corresponding morphisms. We call $\mathbb{G}_{\mathrm{a}}$ the *additive group* over k.

(2) We put $\mathbb{G}_{\mathrm{m}} := \operatorname{Spec} k[x, x^{-1}]$ and define m, i, and e at the level of rings as follows:
- $m\colon k[x, x^{-1}] \to k[x, x^{-1}] \otimes_k k[x, x^{-1}]; \quad x \mapsto x \otimes x$,
- $i\colon k[x, x^{-1}] \to k[x, x^{-1}]; \quad x \mapsto x^{-1}$,
- $e\colon k[x, x^{-1}] \to k; \quad x \mapsto 1$.

Then $\mathbb{G}_{\mathrm{m}}$ is an algebraic group over k with respect to the corresponding morphisms. We call $\mathbb{G}_{\mathrm{m}}$ the *multiplicative group* over k.

(3) We put $\mathrm{GL}_n := \operatorname{Spec} k[x_{ij}, D^{-1} \mid 1 \le i, j \le n]$, where $D := \det(x_{ij})_{1 \le i,j \le n}$. We define m, i, and e at the level of rings as follows:
- $m(x_{ij}) := \sum_{k=1}^{n} x_{ik} \otimes x_{kj}$,
- $i(x_{ij}) :=$ the (i, j)-entry of the inverse of the matrix $(x_{ij})_{1 \le i,j \le n}$,
- $e(x_{ij}) := \delta_{ij}$ (Kronecker's delta).

Then GL_n is an algebraic group over k with respect to the corresponding morphisms. We call GL_n the *general linear group (of rank n)* over k. (Note that $\mathrm{GL}_1 \cong \mathbb{G}_{\mathrm{m}}$.)

Now we explain a "functorial" viewpoint of algebraic groups, which is more practical.

Let $X = \operatorname{Spec} k[X]$ be an affine scheme over k. We consider a functor $X(-)$ from the category of k-algebras to the category of sets given by

$$X(R) := \operatorname{Hom}_k(\operatorname{Spec} R, X)$$

for any k-algebra R, where $\operatorname{Hom}_k(-,-)$ denotes the set of morphisms of affine schemes over k. Since the category of affine schemes is equivalent to the category of rings, we have

$$\operatorname{Hom}_k(\operatorname{Spec} R, X) \cong \operatorname{Hom}_k(k[X], R),$$

where the latter $\operatorname{Hom}_k(-,-)$ denotes the set of k-algebra homomorphisms. In fact, the affine scheme X is determined by the functor $X(-)$. Therefore, we may regard the affine scheme X as a "machine" which associate to each k-algebra R a set $X(R)$ in a functorial way. (More precisely, the association $X \mapsto X(-)$ gives a fully faithful functor from the category of affine schemes over k to the category of functors from the category of affine schemes over k to the category of sets; this is so-called "Yoneda's lemma".)

We call an element of $X(R)$ an *R-valued point* or an *R-rational point* of X.

Example 1.14. Let $X = \operatorname{Spec} k[x,y]/(y^2 - x^3)$. Then, for any k-algebra R, we have

$$X(R) \cong \operatorname{Hom}_k(k[x,y]/(y^2 - x^3), R).$$

Note that, any k-algebra homomorphism f from $k[x,y]/(y^2-x^3)$ to R is uniquely detemined by the images $f(x), f(y) \in R$ of x, y. Since x and y satisfies the equation $y^2 - x^3 = 0$ in the coordinate ring $k[x,y]/(y^2 - x^3)$, their images must satisfy $f(y)^2 - f(x)^3 = 0$. Conversely, for any elements $(a,b) \in R^2$ satisfying the equation $b^2 - a^3 = 0$, we can define a k-algebra homomorphism $f \colon k[x,y]/(y^2 - x^3) \to R$ by $f(x) = a$ and $f(y) = b$. Therefore, we get

$$X(R) \cong \operatorname{Hom}_k(k[x,y]/(y^2 - x^3), R) \cong \{(a,b) \in R^2 \mid b^2 - a^3 = 0\}.$$

In other words, we can think of X as a machine which associates to each R the set of solutions to the equation $y^2 - x^3 = 0$ in R^2.

Now let G be an algebraic group over k. Then the multiplication morphism $m \colon G \times_k G \to G$ induces a map $m_R \colon G(R) \times G(R) \to G(R)$ for each k-algebra R. Indeed, let $g_1, g_2 \in G(R) = \operatorname{Hom}_k(\operatorname{Spec} R, G)$. Then we can define an element $m_R(g_1, g_2) \in G(R)$ to by

$$m_R(g_1, g_2) \colon \operatorname{Spec} R \xrightarrow{(g_1,g_2)} G \times_k G \xrightarrow{m} G.$$

(Here, (g_1, g_2) denotes the morphism induced from g_1 and g_2 by the universal property of the fibered product $G \times_k G$.) Similarly, we also have a map $i_R \colon G(R) \to G(R)$ induced by i. Furthermore, the unit morphism $e \colon \operatorname{Spec} k \to G$ induces an element $e_R \in G(R)$ given by $e_R \colon \operatorname{Spec} R \to \operatorname{Spec} k \xrightarrow{e} G$, where the first arrow is the structure morphism for $\operatorname{Spec} R$. Then, it can be easily checked that the axiom of an algebraic group implies that $G(R)$ is a group in the usual sense with respect to the map m_R with inversion map i_R and unit element e_R. As a result, $G(-)$ gives a functor from the category of k-algebras to the category of groups.

Example 1.15. (1) For a k-algebra R, we have $\mathbb{G}_{\mathrm{a}}(R) \cong R$, where the group structure on R is given by the additive structure of R. Indeed, we have

$$\mathbb{G}_{\mathrm{a}}(R) = \operatorname{Hom}_k(\operatorname{Spec} R, \mathbb{G}_{\mathrm{a}}) \cong \operatorname{Hom}_k(k[x], R) \cong R,$$

where the last map is given by $f \mapsto f(x)$. The multiplication map m_R induced on $\mathbb{G}_{\mathrm{a}}(R)$ corresponds to the addition on R. Indeed, let us take any elements $g_1, g_2 \in \mathbb{G}_{\mathrm{a}}(R)$, hence $m_R(-,-)$ is given by the composition

$$m_R(g_1, g_2) \colon \operatorname{Spec} R \xrightarrow{(g_1,g_2)} G \times_k G \xrightarrow{m} G.$$

At the level of rings, this amounts to the composition

$$k[x] \xrightarrow{m} k[x] \otimes_k k[x] \xrightarrow{g_1 \otimes g_2} R.$$

Since $m(x) = x \otimes 1 + 1 + \otimes x$ by definition, we get

$$(g_1 \otimes g_2) \circ m(x) = (g_1 \otimes g_2)(x \otimes 1 + 1 + \otimes x) = g_1(x) + g_2(x).$$

This is why $\mathbb{G}_{\mathrm{a}}$ is called the "additive group".

(2) For a k-algebra R, we have $\mathbb{G}_{\mathrm{m}}(R) \cong R^\times$, where $R^\times$ denotes the unit group of R with respect to the multiplicative structure of R. Indeed, we have

$$\mathbb{G}_{\mathrm{m}}(R) = \mathrm{Hom}_k(\mathrm{Spec}\, R, \mathbb{G}_{\mathrm{m}}) \cong \mathrm{Hom}_k(k[x, x^{-1}], R) \cong R^\times,$$

where the last map is given by $f \mapsto f(x)$. In a similar manner to above, we can check that the multiplication map m_R on $\mathbb{G}_{\mathrm{m}}(R)$ corresponds to the multiplication on $R^\times$. This is why $\mathbb{G}_{\mathrm{m}}$ is called the "multiplicative group".

(3) For a k-algebra R, we have

$$\mathrm{GL}_n(R) \cong \{g = (g_{ij})_{i,j} \in M_n(R) \mid \det(g) \in R^\times\}.$$

Indeed, by definition, we have

$$\mathrm{GL}_n(R) = \mathrm{Hom}_k(\mathrm{Spec}\, R, \mathrm{GL}_n) \cong \mathrm{Hom}_k(k[x_{ij}, D^{-1} \mid 1 \leq i, j \leq n], R).$$

The right-hand side is isomorphic to (at least as sets) $\{g = (g_{ij})_{i,j} \in M_n(R) \mid \det(g) \in R^\times\}$ by the map $f \mapsto (f(x_{ij}))_{i,j}$. It is a routine work to check that this bijection is indeed a group isomorphism.

(4) The *symplectic group* Sp_{2n} is an affine algebraic group such that the group of its R-valued points is given as follows:

$$\mathrm{Sp}_{2n}(R) \cong \{g = (g_{ij})_{i,j} \in \mathrm{GL}_{2n}(R) \mid {}^t g J_{2n} g = J_{2n}\},$$

where J_{2n} denotes the antidiagonal matrix whose antidiagonal entries are given by 1 and -1 alternatively:

$$J_{2n} := \begin{pmatrix} & & & 1 \\ & & -1 & \\ & 1 & & \\ \iddots & & & \end{pmatrix}.$$

Here, we don't explain how to define the coordinate ring of Sp_{2n} and also how to introduce the group structure at the level of the coordinate ring. Only the important viewpoint here is what kind of groups are associated as the groups of R-valued points! So, in this course, let us just believe that the functor Sp_{2n} is indeed *representable*, i.e., realized as the functor of points of some affine algebraic groups. This remark is always applied to any affine algebraic group which we will encounter in the future.

2. Week 2: Very basic properties of general algebraic groups

Recall that, in general, a *scheme* X is a topological space equipped with a sheaf of rings $\mathcal{O}_X$ ("structure sheaf") which is locally isomorphic to affine schemes ("$\operatorname{Spec} A$" for a commutative ring A).

In the following, we let k be an algebraically closed field. Also, when we say "an algebraic variety", it always means "an algebraic variety over k". Here, recall that we say that a scheme X is an algebraic variety over k if it is locally isomoprhic to $\operatorname{Spec} A$ for a finitely generated reduced k-algebra (hence, in particular, A is of the form $k[x_1, \dots, x_n]/I$ for an ideal I of $k[x_1, \dots, x_n]$).

For any algebraic variety X over k, the subset of closed points of X can be identified with the set $X(k)$ of k-rational points of X; for any k-rational point $\operatorname{Spec} k \to X$, the image of the unique point of $\operatorname{Spec} k$ is a closed point of X, and vice versa. From now on, we freely identify the set of closed points of X with $X(k)$. Moreover, the subset of closed points of X is dense in X because k is algebraically closed. (Both these facts are consequences of Hilbert's "nullstellensatz", which asserts that any maximal ideal of $k[x_1, \dots, x_n]$ is of the form $(x_1 - a_1, \dots, x_n - a_n)$ for some $a_1, \dots, a_n \in k$; this fact assumes that k is algebraically closed.)

2.1. Identity component subgroup.

Let G be an algebraic group over k. Recall that, in particular, G is equipped with a unit element $e \in G(k)$. Let G° denote the connected component of G containing the closed point e.

Proposition 2.1. *The subset G° is a subgroup of G. Moreover, G° is normal of finite index in G.*

Proof. We have to show that G° is closed under the multiplication morphism $m\colon G \times G \to G$ and the inversion morphism $i\colon G \to G$. More precisely, our task is to check that $m(G^\circ, G^\circ) \subset G^\circ$ and $i(G^\circ) \subset G^\circ$. But both statements follow by combining a general fact that the image of a connected set under a continuous map is again connected with that $m(e, e) = e$ and $i(e) = e$.

To show the second assertion, let us take $g \in G(k)$. (By definition, being normal means that $gG^\circ g^{-1} \subset G^\circ$ for any $g \in G(k)$.) Then it can be easily checked that $gG^\circ g^{-1}$ is a subgroup of G which is connected and contains the unit element. Hence we get $gG^\circ g^{-1} \subset G^\circ$. The finite-index property follows from that the set of connected components of an algebraic variety is finite. □

Definition 2.2. We call the algebraic subgroup G° of G the *identity component* of G.

2.2. Smoothness of algebraic groups.

Let us first look at the following example: we consider an affine algebraic variety $X := \operatorname{Spec} k[x, y]/(y^2 - x^3)$, i.e., X is the spectrum of the quotient ring of $k[x, y]$ by the ideal generated by $(y^2 - x^3)$. Recall that, X represents the space of solutions to the equation $y^2 - x^3 = 0$. More precisely, for any k-algebra R, the set $X(R)$ of R-rational points of X is equal to the set of solutions to $y^2 - x^3 = 0$ in R. If we try to draw a picture of the set $X(\mathbb{R}) \subset \mathbb{R}^2$, then we can immediately notice that the resulting curve is "smooth" except for the origin $(0, 0)$; at the origin, the curve has a "singular point"[3].

In fact, the difference between the point $(0, 0)$ and the other points in this example can be explained in terms of ring-theoretic properties of the coordinate ring $k[x, y]/(y^2 - x^3)$.

[3]Because we assume k is algebraically closed in this lecture, it's not actually allowed to take R to be $\mathbb{R}$. If you want to be rigorous please take the coefficient k to be any smaller field, for example, $\mathbb{Q}$.

Let us explain how to introduce the notion of a "smooth point" and also a "singular point" for general schemes in the following.

Let X be a scheme. For any point $x \in X$, we define a ring $\mathcal{O}_{X,x}$ by

$$\mathcal{O}_{X,x} := \varinjlim_{x \in U} \mathcal{O}_X(U),$$

where the inductive limit is over open sets U of X containing $x \in X$ (the structure morphisms are given by the restriction maps $\mathcal{O}_X(V) \to \mathcal{O}_X(U)$ for any $x \in V \subset U$). This ring is a local ring and called the *stalk* of X at $x \in X$. If $x \in X$ is contained in an affine open subscheme $U \subset X$ isomorphic to $\operatorname{Spec} A$, where x is identified with a prime ideal $\mathfrak{p}$ of A, then the stalk $\mathcal{O}_{X,x}$ is nothing but the localization $A_{\mathfrak{p}}$ of A with respect to $\mathfrak{p}$.

For any $x \in X$, we write $\mathfrak{m}_x$ for the unique maximal ideal of the stalk $\mathcal{O}_{X,x}$. We put $\kappa(x) := \mathcal{O}_{X,x}/\mathfrak{m}_x$ and call $\kappa(x)$ the *residue field* of X at $x \in X$.

Definition 2.3. Let X be an algebraic variety over k.

(1) We say that a point $x \in X$ is *smooth* if the local ring $\mathcal{O}_{X,x}$ of X at x is a regular local ring, i.e., we have

$$\dim(\mathcal{O}_{X,x}) = \dim_{\kappa(x)}(\mathfrak{m}_x/\mathfrak{m}_x^2).$$

Here, the left-hand side denotes the Krull dimension of the ring $\mathcal{O}_{X,x}$ and the right-hand side denotes the dimension of $\mathfrak{m}_x/\mathfrak{m}_x^2$ as a $\kappa(x)$-vector space.

(2) We say that X is *smooth* if every point of X is smooth.

Fact 2.4. *Let X be an algebraic variety over k. Then the subset of smooth points of X is open dense in X.*

The subset of smooth point of X is often referred to as the *smooth locus* of X.

Proposition 2.5. *Let G be an algebraic group over k. Then G is smooth.*

Proof. Let U be the smooth locus of G, which is open dense in G by the above fact. Let us show that any closed point g of G is contained in U. If we can show this, then the assertion follows. Indeed, the complement $G \setminus U$ is a closed subset of G; if this is not empty, then it contains at least one closed point of G, hence a contradiction.

Firstly, U contains at least one closed point g_0 of G because, otherwise, $G \setminus U$ is a closed subset of G containing all closed points, hence equal to G by the density of closed points. Next, for any closed point g of G, we consider the (gg_0^{-1})-multiplication morphism

$$G \to G \colon x \mapsto gg_0^{-1}x.$$

(Precisely speaking, for any $h \in G(k)$, the h-multiplication morphism is defined to be the composition $G \cong \operatorname{Spec} k \times_k G \to G \times_k G \to G$, where the second arrow is the fibered product of $h\colon \operatorname{Spec} k \to G$ and id_G and the last arrow is the multiplication morphism of G. At the level of k-rational points, this realizes the intuitive map $x \mapsto hx$.) Then, because this is an isomoprhism of algebraic varieties, any smooth point is mapped to a smooth point. In particular, g, which is the image of the smooth point g_0, is also smooth. Thus U contains g. □

Remark 2.6. The word "smooth" usually means a property of a morphism of schemes $f\colon X \to Y$; the definition introduced above is usually referred to as the regularity (non-singularity) of X (at x), which is an "absolute" notion depending only on X. When $Y = \operatorname{Spec} k$ (where k is an algebraically closed field), the smoothness for the morphism f is equivalent to the regularity (non-singularity) of X. In general, we must be careful about the difference between the regularity and the smoothness; see, e.g., [Mil17, §1.b].

2.3. **Homomorphism between algebraic groups.** Let us investigate a homomorphism between algebraic groups over k.

Proposition 2.7. *Let $\alpha\colon G \to G'$ be a homomorphism between algebraic groups over k. Then the image $\alpha(G)$ is a closed subgroup of G'.*

To show this proposition, let us first review some general notions for topological spaces.

Definition 2.8. Let X be a topological spaces.

(1) We say that a subset Z of X is *locally closed* if Z is an intersection of an open subset of X and a closed subset of X.
(2) We say that a subset Z of X is *constructible* if Z is a finite union of locally closed subsets of X.
(3) We say that X is *noetherian* if any open subset of X is quasi-compact.

Remark 2.9. In the above definition, the word "quasi-compact" just means "compact", i.e., any open covering has a finite subcovering. This is because, sometimes (depending on areas), the word "compact" is used to mean "Hausdorff and compact". In the context of algebraic geometry, we often use the word "quasi-compact" to emphasize that the Hausdorff property is not assumed.

The following fact is a general nonsense on topological spaces:

Lemma 2.10. *Let X be a noetherian topological space. Let Y be a constructible subset of X. Then Y contains an open dense subset of its closure $\overline{Y}$ in X.*

Exercise 2.11. Prove the above lemma.

Note that, any algebraic variety over k is a noetherian topological space, hence the above lemma can be applied.

On the other hand, the following fact is much deeper:

Fact 2.12. *Let $f\colon X \to Y$ be a morphism between algebraic varieties over k. Then the image of any constructible subset under f is a constructible subset of Y.*

Let us utilize these facts to deduce some useful facts on algebraic groups.

Lemma 2.13. *Let G be an algebraic group over k. Then, for any open dense sets U and V of G, we have $U \cdot V = G$, where we put $U \cdot V := \{u \cdot v \in G \mid u \in U, v \in V\}$.*

Proof. It is enough to show that the open subset $U \cdot V$ contains every closed point g of G. Let $g \in G$ be a closed point. Then both U and $g \cdot V^{-1}$ are dense open subsets of G, hence have a nonempty open intersection. By the density of closed points, there exists a closed point in $U \cap (g \cdot V^{-1})$. In other words, there exists closed points $u \in U$ and $v \in V$ satisfying $u = hv^{-1}$, hence $h = uv \in U \cdot V$. □

Proposition 2.14. *Let G be an algebraic group over k. Then any constructible subgroup H of G is closed.*

Proof. By Lemma 2.10, H contains an open dense subset U of its closure $\overline{H}$ in G. Since H is a subgroup of G, we obtain

$$U \cdot U \subset H \cdot H \subset H.$$

By the above lemma, we have $U \cdot U = \overline{H}$, hence $H = \overline{H}$. □

Corollary 2.15. *Let $\alpha\colon G \to G'$ be a homomorphism between algebraic groups over k. Then the image $\alpha(G)$ is a closed subgroup of G'.*

Proof. By Fact 2.12, $\alpha(G)$ is a constructible subset of G'. Since $\alpha(G)$ is a subgroup of G', the above proposition implies that $\alpha(G)$ is closed. $\square$

Remark 2.16. The notion of a "kernel" in the context of algebraic groups is quite subtle. Scheme-theoretically, the kernel of α is defined to be the fibered product of $\alpha\colon G \to G'$ and $e'\colon \operatorname{Spec} k \to G'$, where e' denotes the unit element of G'. However, the problem is that this fibered product is not necessarily reduced, hence not necessarily an algebraic variety in our sense. For example, consider the morphism $\mathbb{G}_{\mathrm{m}} \to \mathbb{G}_{\mathrm{m}}\colon x \mapsto x^p$ for the multiplicative group defined over an algebraically closed field k of characteristic $p > 0$. Then, as "points", its kernel is equal to $\mu_p(k) := \{x \in k \mid x^p = 1\} = \{1\}$. However, the fibered product is isomorphic to $\operatorname{Spec} k[x]/(x-1)^p$, which is not reduced. This observation suggests that, for a better treatment of algebraic groups, we should work with more general notion of group schemes.

2.4. **Dimension of algebraic groups.**

Definition 2.17. Let X be an algebraic variety. We say that a closed subset Y of X is *irreducible* if Y is non-empty and cannot be written as $Y = Z_1 \cup Z_2$ for non-empty proper closed subsets $Z_1, Z_2 \subsetneq Y$. We call a maximal irreducible subset of X an *irreducible component* of X.

Definition 2.18. For an algebraic variety X, we define the *dimension* $\dim X$ of X to be the maximum of the length d of a chain

$$Y_0 \subsetneq Y_1 \subsetneq \cdots \subsetneq Y_d$$

of irreducible subsets of Y_d.

In fact, the dimension of an algebraic variety is related to the Krull dimension of its stalks in the following sense: let Y be an irreducible component of X. Then, for any $x \in X$, we have $\dim \mathcal{O}_{X,x} = \dim Y$.

Fact 2.19. *Let $\alpha\colon G \to G'$ be a homomorphism between algebraic groups over k. Then we have*

$$\dim G = \dim \operatorname{Ker}(\alpha) + \dim \alpha(G).$$

Here, as noted above, $\alpha(G)$ is a closed subgroup of G while $\operatorname{Ker}(\alpha)$ is not in general because it might not be reduced. So the (ad hoc) meaning of "$\operatorname{Ker}(\alpha)$" is that it is the set-theoretic preimage of the unit element $e' \in G'$ under α. Since α is continuous and e' is closed, the preimage is closed in G, hence it makes sense to talk about its dimension.

For the proof of this fact, see [Mil17, Proposition 1.63].

2.5. **Algebraic group action on algebraic varieties.**

Definition 2.20. Let G be an algebraic group over k and X an algebraic variety over k. We say that G *acts on* X if there exists a morphism of algebraic varieties $\alpha\colon G \times X \to X$ satisfying the usual axioms of group actions, i.e., the following diagrams are commutative:

$$\begin{array}{ccc} G \times_k G \times_k X & \xrightarrow{m \times \mathrm{id}} & G \times_k X \\ {\scriptstyle \mathrm{id} \times \alpha} \downarrow & \circlearrowleft & \downarrow {\scriptstyle \alpha} \\ G \times_k X & \xrightarrow{\alpha} & G \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{e \times \mathrm{id}} & G \times_k X \\ & {\scriptstyle \mathrm{id}} \searrow \ \circlearrowleft & \downarrow {\scriptstyle m} \\ & & X \end{array}$$

We can also consider the usual notion on the group action such as normalizer, stabilizer, and so on, in the context of algebraic groups.

Proposition/Definition 2.21. Suppose that an algebraic group G acts on an algebraic variety X.

(1) For any closed subvarieties Y and Z of X, there exists a closed subvariety $N_G(Y,Z)$ satisfying

$$N_G(Y,Z)(R) = N_{G(R)}(Y(R), Z(R)) := \{n \in G(R) \mid nY(R) \subset Z(R)\}$$

for any k-algebra R. We call $N_G(Y,Z)(R)$ the *transporter* from Y to Z in G.

(2) When $Y = Z$, we call the transporter $N_G(Y,Y)$ the *normalizer* of Y in Z and write $N_G(Y) := N_G(Y,Y)$. Note that the normalizer is a subgroup of G.

(3) When Y consists of a single closed point $x \in X$, we call the normalizer group $N_G(\{x\})$ the *stabilizer* group of x in G and write $G_x := N_G(\{x\})$. More generally, for any closed subvariety $Y \subset X$, we put $G_Y := \cap_{x\in Y} G_x$.[4]

The subtle point of the above definition is that, so that the resulting "subfunctor" $N_G(Y,Z)$ is indeed given by a "subvariety" (more naively speaking, the subset $\{n \in G \mid nY \subset Z\}$ has a natural subscheme structure), we need to assume that the subsets Y and Z are *closed* subvarieties of G. See [Mil17, 1.79] for the details.

Proposition 2.22 ("Closed orbit lemma")**.** *Let G be an algberaic group acting on an algebraic variety X. For any closed point $x \in X$, let Gx denote its orbit.*

(1) Each Gx is a smooth variety which is open in its closure $\overline{Gx}$ in X.

(2) The boundary $\overline{Gx} \smallsetminus Gx$ is a union of orbits of strictly smaller dimension.

Proof. Note that $G \cdot x$ is (by definition) the image of the morphism $G \to X : g \mapsto gx$. Using the fact that the image of any constructible set is again constructible (Fact 2.12), we see that Gx contains a dense open subset U of its closure $\overline{Gx}$. Here note that both Gx and $\overline{Gx}$ are stable under the G-action. In particular, we have

$$Gx = \bigcup_{g\in G(k)} gU.$$

(Precisely speaking, we first see that the closed points contained in $\bigcup_{g\in G(k)} gU$ are the same as those of Gx. Then, by the density of closed points, we get the equality as subvarieties.) Each gU is open in $\overline{Gx}$, hence this equality implies that Gx is open in $\overline{Gx}$. The smoothness follows from the same argument as in the proof of the smoothness of algebraic groups, i.e., use the open-density of the smooth locus and that G acts on Gx transitively.

It can be easily checked that any dense open subset of a noetherian space intersects every irreducible component. In particular, the boundary $\overline{Gx} \smallsetminus Gx$ does not contain any irreducible component $\overline{Gx}$. In other words, $\overline{Gx} \smallsetminus Gx$ is a closed subset of $\overline{Gx}$ of strictly smaller dimension. Since $\overline{Gx} \smallsetminus Gx$ is G-stable, it can be written as the union of its G-orbits. □

Corollary 2.23. *Let G be an algberaic group acting on an algebraic variety X. Then any G-orbit of minimal dimension is closed. In particular, X always has a closed G-orbit.*

Proof. If the dimension of a G-orbit Gx is minimal, then the boundary $\overline{Gx} \smallsetminus Gx$ must be empty by the above proposition. Hence Gx is closed. □

[4] When $X = G$ and the action of G on X is the conjugation, we call the stabilizer G_X the *centralizer* of X in G.

Example 2.24. A typical example of the application of the above proposition is the following. Let $G = \mathrm{GL}_n$. We consider $\mathcal{N} := \{N \in M_n \mid (N - I_n)^r = 0 \text{ for some } r \in \mathbb{Z}_{\leq 0}\}$. In other words, $\mathcal{N}$ is an algebraic subvariety of $M_n \cong \mathbb{A}_k^{n^2}$ (the affine space of n-by-n matrices) consisting of nilpotent matrices. Then G acts on $\mathcal{N}$ via conjugation. By the theory of Jordan normal form, each nilpotent G-orbit corresponds to a partition of n. It is known that the "closure relation" on $\mathcal{N}$ (i.e., when a G-orbit Gx is contained in the closure of another G-orbit $\overline{Gy}$) can be described in terms of the combinatorics on the partition of n.

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