REPRESENTATION THEORY OF FINITE GROUPS OF LIE TYPE

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Contents

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1. Week 1: Course overview

1.1. **Introduction.** Suppose that a group G is given and that we want to understand the group G. But then what exactly does it mean to "understand" G? There is a rich framework which enables us to "define" a reasonable answer to this problem; it is *representation theory.* Recall that a *representation* of a group G is a vector space V, say $\mathbb{C}$-coefficient here, equipped with an action of G.

> Let us say that "we understand the group G" when we understand all the representations of G.

The aim of this course is to give an introduction to "Deligne–Lusztig theory" (established in [DL76]), which provides a realization of all representations of finite groups of Lie type.

1.2. **Quick review of representation theory of finite groups.** The basic reference of this subsection is Serre's book [Ser77].

In the following, we let G be a finite group.

Definition 1.1 (representation)**.** We say that (ρ, V) is a *representation* of G if V is a finite-dimensional $\mathbb{C}$-vector space equipped with an action ρ of G, i.e., ρ is a homomorphism $G \to \mathrm{GL}_{\mathbb{C}}(V) := \mathrm{Aut}_{\mathbb{C}}(V)$. We often only write ρ or V for a representation (ρ, V).

Definition 1.2 (homomorphism)**.** Let (ρ_1, V_1) and (ρ_2, V_2) be representations of G. We say that a $\mathbb{C}$-linear map $f\colon V_1 \to V_2$ is a *homomorphism* from (ρ_1, V_1) to (ρ_2, V_2) if it is equivariant with respect to the actions ρ_1 and ρ_2 of G, i.e., we have $f(\rho_1(g)(v)) = \rho_2(g)(f(v))$ for any $g \in G$ and $v \in V_1$.

$$\begin{array}{ccc} V_1 & \xrightarrow{\ f\ } & V_2 \\ {\scriptstyle \rho_1(g)}\big\downarrow & \circlearrowleft & \big\downarrow{\scriptstyle \rho_2(g)} \\ V_1 & \xrightarrow{\ f\ } & V_2 \end{array}$$

We write $\mathrm{Hom}_G(\rho_1, \rho_2)$ for the set of homomorphisms from ρ_1 to ρ_2 (this has a natural $\mathbb{C}$-vector space structure). We say that (ρ_1, V_1) and (ρ_2, V_2) are *isomorphic* if there exists an isomorphism $f\colon V_1 \to V_2$ (i.e., homomorphism which is isomorphic as a $\mathbb{C}$-linear map).

Definition 1.3 (subrepresentation)**.** Let (ρ, V) be a representation of G. We say that a subspace W of V is a *subrepresentation* of V if it is stable under the action ρ of G.

Definition 1.4 (irreducible representation)**.** Let V be a representation of G. We say that V is *irreducible* if $V \neq \{0\}$ and there is no subrepresentation of V except for V itself and $\{0\}$.

Note that basic operations on vector spaces can be considered also for representations. For example, when (ρ_1, V_1) and (ρ_2, V_2) are representations of G, we define their *direct sum* $(\rho_1 \oplus \rho_2, V_1 \oplus V_2)$, which is a representation of G, by

$$(\rho_1 \oplus \rho_2)(g)(v_1 + v_2) := \rho_1(g)(v_1) + \rho_2(g)(v_2)$$

for any $g \in G$ and $v_1 \in V_1$, $v_2 \in V_2$. Similarly, we define the tensor product $\rho_1 \otimes \rho_2$, which is a representation of G, by

$$(\rho_1 \otimes \rho_2)(g)(v_1 \otimes v_2) := \rho_1(g)(v_1) \otimes \rho_2(g)(v_2).$$

We also often use the "box-tensor product" $\rho_1 \boxtimes \rho_2$, which is a representation of $G \times G$ defined by

$$(\rho_1 \boxtimes \rho_2)(g_1, g_2)(v_1 \otimes v_2) := \rho_1(g_1)(v_1) \otimes \rho_2(g_2)(v_2).$$

(Note that this definition works for, more generally, representations ρ_1 of G_1 and ρ_2 of G_2; in this case, $\rho_1 \boxtimes \rho_2$ is a representation of $G_1 \times G_2$.)

The following theorem is very fundamental and important in representation theory of finite groups.

Theorem 1.5 (semisimplicity of representations). *Let V be a representation of G. Then there is a unique (up to permutation) way to write*

$$V \cong \bigoplus_{i=1}^{r} W_i^{\oplus n_i},$$

where W_i's are pairwise inequivalent irreducible representations of G and n_i's are positive integers determined only by V.

By this theorem, the problem of understanding representations of G can be divided into the following two steps:

(1) Classify all irreducible representations of G.
(2) Find a systematic way of determining each n_i from a given V.

Let us list some fundamental facts on the first part (1):

Theorem 1.6. *(1) The number of conjugacy classes of G equals the number of isomorphism classes of irreducible representations of G.*
(2) We have

$$|G| = \sum_{\rho} \dim(\rho)^2,$$

where ρ runs over isomorphism classes of irreducible representations of G.

The key to the part (2) is the following:

Theorem 1.7 (Schur's lemma). *Let (ρ_1, V_1) and (ρ_2, V_2) be irreducible representations of G. Then we have*

$$\mathrm{Hom}_G(\rho_1, \rho_2) \cong \begin{cases} \mathbb{C} & \text{if } \rho_1 \cong \rho_2, \\ 0 & \text{if } \rho_1 \not\cong \rho_2. \end{cases}$$

By Schur's lemma, each multiplicity n_i of an irreducible representation V_i in the irreducible decomposition of a representation V of G is given by $\dim_{\mathbb{C}} \mathrm{Hom}_G(V, V_i)$ (or $\dim_{\mathbb{C}} \mathrm{Hom}_G(V_i, V)$). Then, how can we determine this number for each V_i? Theory of *characters* provides a satisfactory answer to this question.

Definition 1.8 (character). Let (ρ, V) be a representation of G. The *character* of (ρ, V), for which we write Θ_ρ (or Θ_V), is the function $G \to \mathbb{C}$ defined by $\Theta_\rho(g) := \mathrm{Tr}\,\rho(g)$. Namely, $\Theta_\rho(g)$ is the trace of the representation matrix of $\rho(g)$ (with respect to any $\mathbb{C}$-basis of V).

Note that Θ_ρ is constant on each conjugacy class of G. Such a function is called a *class function* on G. Let $C(G)$ denote the set of $\mathbb{C}$-valued class functions on G. Then $C(G)$ has a natural $\mathbb{C}$-vector space structure equipped with an inner product $\langle -, - \rangle$ given by

$$\langle f_1, f_2 \rangle := \frac{1}{|G|} \sum_{g \in G} f_1(g) \cdot \overline{f_2(g)}.$$

Theorem 1.9. *The set of characters of irreducible representations of G forms an orthonormal basis of $C(G)$ with respect the inner product $\langle -, - \rangle$. In particular, for irreducible representations (ρ_1, V_1) and (ρ_2, V_2) of G, we have*

$$\langle \Theta_{\rho_1}, \Theta_{\rho_2} \rangle = \begin{cases} 1 & \text{if } \rho_1 \cong \rho_2, \\ 0 & \text{if } \rho_1 \ncong \rho_2. \end{cases}$$

Note that, by this theorem, it is enough to compute $\langle \Theta_V, \Theta_{W_i} \rangle$ to get the multiplicity n_i of W_i in V.

From these discussion, we could say that our ultimate goal in representation theory of G is to get a list of the character values of all irreducible representations on all conjugacy classes of G. Such a list is called the *character table* of G.

1.3. **Warmup example: $\mathfrak{S}_3$.** When G is a finite abelian group, all irreducible representations of G are 1-dimensional, i.e., characters. Thus there exists $|G|$ irreducible representations of G; all of them can be described explicitly by, e.g., the structure theorem of finite abelian groups.

So let us look at the non-abelian group of the smallest order, i.e., the permutation group of three letters:

$$\mathfrak{S}_3 = \{1, (12), (23), (31), (123), (132)\}.$$

Since this group has three conjugacy classes

$$\{1\}, \quad \{(12), (23), (31)\}, \quad \{(123), (132)\},$$

there should be three irreducible representations. Firstly, we have the trivial representation of $\mathfrak{S}_3$, which is 1-dimensional. Secondly, the signature character $\mathrm{sgn}\colon \mathfrak{S}_3 \to \{\pm 1\}$ gives another 1-dimensional representation. [1]

So, what is the remaining representation? We let r be its dimension. Then we should have

$$1^2 + 1^2 + r^2 = |\mathfrak{S}_3| = 6,$$

i.e., r must be 2. Let us find the remaining 2-dimensional irreducible representation. Almost by definition, $\mathfrak{S}_3$ acts on the set of three letters $X := \{1, 2, 3\}$. Thus, if we let $V := \mathbb{C}[X]$ be the space of $\mathbb{C}$-valued functions on X, then $\mathfrak{S}_3$ also acts on V (via pull-back of functions). This representation is 3-dimensional and contains the trivial representation as its subrepresentation. Indeed, the subspace of constant functions on X is stable under the action $\mathfrak{S}_3$; let us write W for it. We claim that V/W, which is 2-dimensional, is an irreducible representation of $\mathfrak{S}_3$. To check this, it is enough to show that $\langle \Theta_{V/W}, \Theta_{V/W} \rangle = 1$.

Let us first compute the character Θ_V of V. Since Θ_V is a class function, it is enough to compute the traces of the actions of 1, (12), and (123). Let $\mathbb{1}_i$ denote the characteristic function of $\{i\} \subset X$ for $i = 1, 2, 3$. Then $\{\mathbb{1}_i \mid i = 1, 2, 3\}$ is a $\mathbb{C}$-basis of V and the representation matrices of the actions of 1, (12), and (123) with respect to this basis is given by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Hence we have

$$\Theta_V(1) = 3, \quad \Theta_V((12)) = 1, \quad \Theta_V((123)) = 0.$$

[1] Recall that, in general, the signature character of $\mathfrak{S}_3$ associates $+1$ (resp. -1) to a permutation expressed by the product of even (resp. odd) number of transpositions.

As we have $\Theta_W(1)=1, \Theta_W((12))=1, \Theta_W((123))=1$, we get

$$\Theta_{V/W}(1)=2, \quad \Theta_{V/W}((12))=0, \quad \Theta_{V/W}((123))=-1.$$

Therefore, we have

$$\begin{aligned}\langle\Theta_{V/W},\Theta_{V/W}\rangle &= \frac{1}{6}\sum_{g\in\mathfrak{S}_3}\Theta_{V/W}(g)\cdot\overline{\Theta_{V/W}(g)}\\ &= \frac{1}{6}\bigl(2^2+0^2+0^2+0^2+(-1)^2+(-1)^2\bigr)=1.\end{aligned}$$

1.4. **What is Deligne–Lusztig theory?** When a group G is finite, we win if we can find all irreducible representations "by hand" in any way. However, we immediately notice that it's not easy in general. (We will look at the example of $\mathrm{GL}_2(\mathbb{F}_q)$ in the next week. We construct its all irreducible representations by hand, by we can see that it's already not obvious at all.)

In fact, we can find an idea in the above example of $\mathfrak{S}_3$. This example suggests that, more generally, we can try to construct representations of a given group G according to the following steps:

(1) First, introduce a "space" X equipped with an action of G.
(2) Second, find a "functorial linearization" $X\mapsto V_X$, i.e., an operation which associates a vector space to each space X which is functorial in X. Then, the action of G on X induces an action of G on V_X.

Deligne–Lusztig theory exactly realizes this idea for so-called finite groups of Lie type. What is a finite group of Lie type? To explain this, let us first recall the definition of a *general linear group*:

$$\mathrm{GL}_n(\mathbb{C}) := \{g\in M_n(\mathbb{C}) \mid g \text{ is invertible}\}.$$

So $\mathrm{GL}_n(\mathbb{C})$ is the set of all invertible n-by-n matrices whose entries are complex numbers; this has a group structure with respect to the usual multiplication of matrices. The point here is that the definition of a general linear group completely makes sense even if we replace the field $\mathbb{C}$ with any field (or even any ring!). Thus, in some sense, we may think of GL_n as a "machine" which associates a group to any ring;

$$R\mapsto \mathrm{GL}_n(R) := \{g\in M_n(R) \mid g \text{ is invertible}\}.$$

In particular, by taking R to be a finite field $\mathbb{F}_q$, we obtain a finite group $\mathrm{GL}_n(\mathbb{F}_q)$.

In general, this kind of machine is called an *algebraic group*. Among algebraic groups, there is a particular class called *reductive groups*. The general linear group is one of the most typical examples of a reductive group. A finite group of Lie type is a finite group obtained by letting R be a finite field $\mathbb{F}_q$ for a reductive group G which can be "defined over $\mathbb{F}_q$". (In the case of GL_n, its definition makes sense over $\mathbb{Z}$, hence also over $\mathbb{F}_q$.)

Let us introduce more examples. Recall that the symplectic (resp. orthogonal) group is the group consisting of symplectic (resp. orthogonal) matrices:

$$\mathrm{Sp}_{2n}(\mathbb{C}) := \{g\in \mathrm{GL}_{2n}(\mathbb{C}) \mid {}^t g J_{2n} g = J_{2n}\},$$

$$\mathrm{O}_n(\mathbb{C}) := \{g\in \mathrm{GL}_n(\mathbb{C}) \mid {}^t g g = I_n\}.$$

Here, J_{2n} (resp. I_n) denotes the anti-diagonal matrix whose $(i, 2n+1-i)$-entry is given by $(-1)^{i-1}$ (resp. the identity matrix). The defining equations of these groups only uses 1 and -1, hence it makes sense to replace $\mathbb{C}$ with $\mathbb{F}_q$; then we get $\mathrm{Sp}_{2n}(\mathbb{F}_q)$ and $\mathrm{O}_n(\mathbb{F}_q)$.

Let us also introduce a bit more tricky example. The unitary group is the group consisting of unitary matrices:

$$\mathrm{U}_n := \{g \in \mathrm{GL}_n(\mathbb{C}) \mid {}^t\overline{g}g = I_n\}.$$

Here, $\overline{g}$ denotes the entry-wise complex conjugate of g. Note that the complex conjugation is nothing but the nontrivial element of the Galois group of the quadratic extension $\mathbb{C}/\mathbb{R}$. This viewpoint suggests that we can define a unitary group in the same way as long as a quadratic extension of fields is given. In particular, by taking a finite field $\mathbb{F}_q$ and its quadratic extension $\mathbb{F}_{q^2}$, we can define

$$\mathrm{U}_n(\mathbb{F}_q) := \{g \in \mathrm{GL}_n(\mathbb{F}_{q^2}) \mid {}^tF(g)g = I_n\}.$$

Here, F denotes the nontrivial element of $\mathrm{Gal}(\mathbb{F}_{q^2}/\mathbb{F}_q)$; this is so-called the Frobenius, which is given by taking (entry-wise) q-th power.

Now let us also mention the "space X" and the "functorial linearization $X \mapsto V_X$". The space X in the context of Deligne–Lusztig theory is called the *Deligne–Lusztig variety*. The definition of the Deligne–Lusztig variety depends on a finite group of Lie type $G(\mathbb{F}_q)$ (with its additional structure). It originates from a very concrete curve with $\overline{\mathbb{F}}_q$-coefficient called the Drinfeld curve, whose defining equation is given by $xy^q - x^qy = 1$. However, the general Deligne–Lusztig variety is defined based on a very sophisticated language of the theory of reductive groups. We have to make full use of the structure theory of reductive groups to analyze its geometric structure.

On the other hand, the role of "functorial linearization $X \mapsto V_X$" is played by the theory of étale cohomology. More precisely, by choosing a prime number ℓ different to the characteristic p of $\mathbb{F}_q$, we obtain the (compactly supported) ℓ-adic cohomology $H^i_c(X, \overline{\mathbb{Q}}_\ell)$ of X. This cohomology $H^i_c(X, \overline{\mathbb{Q}}_\ell)$ is a finite-dimensional $\overline{\mathbb{Q}}_\ell$-vector space, where $\overline{\mathbb{Q}}_\ell$ is an algebraic closure of the ℓ-adic number field $\mathbb{Q}_\ell$. The point here is that $\overline{\mathbb{Q}}_\ell$ is abstractly isomorphic to $\mathbb{C}$, hence we can regard $H^i_c(X, \overline{\mathbb{Q}}_\ell)$ as a finite-dimensional $\mathbb{C}$-vector space. In particular, we obtain a representation of $G(\mathbb{F}_q)$. In order to analyze the structure of $H^i_c(X, \overline{\mathbb{Q}}_\ell)$ as a representation of $G(\mathbb{F}_q)$, we also need to appeal to various fundamental properties of the étale cohomology.

1.5. **Why Deligne–Lusztig theory?** Then, why is Deligne–Lusztig theory so important? The first reason is that Deligne–Lusztig theory is only a framework (at present) which enables us to construct all irreducible representations of finite groups of Lie type in a uniform way. A lot of important examples of finite groups are contained in the class "finite groups of Lie type". However, irreducible representations had been classified only in the case of $\mathrm{GL}_n(\mathbb{F}_q)$ (due to Green in 1955) before the work of Deligne–Lusztig. Moreover, even in that case, Green's method is based on heavy combinatorial arguments, hence it is quite nontrivial whether it can be generalized to other finite groups of Lie type. Let us cite a comment of Shoji from his book [庄 04] (written in Japanese):

> *A preprint by Deligne–Lusztig was released when I was a student. I was shocked about it; it was like that an iron-made steamship suddenly appeared in a peaceful small village which was only based on the handicraft industry before. For people peacefully living with* $\mathrm{GL}_n(\mathbb{F}_q)$ *at that time, Deligne–Lusztig theory was so surprising, almost like the devil's work.*

The second reason is that Deligne–Lusztig theory is expected to have an application to the local Langlands correspondence. The local Langlands correspondence is also called the non-abelian class field theory; roughly speaking, it predicts a natural connection between representations of p-adic reductive groups (such as $\mathrm{GL}_n(\mathbb{Q}_p)$, $\mathrm{Sp}_{2n}(\mathbb{Q}_p)$, etc...) and Galois representations. The expectation is that a certain case of the local Langlands correspondence can be made from Deligne–Lusztig theory (e.g., [DR09]). [2]

[2]But nothing about this will be explained in this course! Maybe next semester???

2. Week 2: Representations of $\mathrm{GL}_2(\mathbb{F}_q)$

Aim of this week. The aim of this week is to construct/classify all irreducible representations of $\mathrm{GL}_2(\mathbb{F}_q)$, especially, write the character table. Through this example, we should be able to encounter various basic notions on reductive groups and representation theory of finite groups of Lie type. The explanation given here follows [BH06, Section 6].

2.1. **Group structure of $\mathrm{GL}_2(\mathbb{F}_q)$.** Let $\mathbb{F}_q$ be a finite field of order q and characteristic $p > 0$ (hence q is a power of p). Let $\mathrm{GL}_2(\mathbb{F}_q)$ denote the general linear group of size 2 with $\mathbb{F}_q$-coefficients, i.e.,

$$\mathrm{GL}_2(\mathbb{F}_q) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{F}_q) \,\middle|\, ad - bc \in \mathbb{F}_q^\times \right\}.$$

In the following, we simply write G for $\mathrm{GL}_2(\mathbb{F}_q)$. It is a basic fact that the order of $\mathrm{GL}_2(\mathbb{F}_q)$ is given by $(q^2-1)(q^2-q)$.

Exercise 2.1. More generally, it is known that the order of $\mathrm{GL}_n(\mathbb{F}_q)$ is given by $\prod_{i=0}^{n-1}(q^n - q^i)$. Prove this.

We can classify the conjugacy classes of $\mathrm{GL}_2(\mathbb{F}_q)$ by looking at the characteristic polynomials as follows. For an element $g \in \mathrm{GL}_2(\mathbb{F}_q)$, let $\phi_g(x) \in \mathbb{F}_q[x]$ denote its characteristic polynomial. Then we have the following three possibilities:

1. $\phi_g(x)$ is of the form $(x-a)^2$ for some $a \in \mathbb{F}_q^\times$.
2. $\phi_g(x)$ is of the form $(x-a)(x-b)$ for some distinct $a, b \in \mathbb{F}_q^\times$.
3. $\phi_g(x)$ is an irreducible monic of degree 2.

We first consider the case (1). If $\phi_g(x) = (x-a)^2$, then the minimal polynomial of g is either $x - a$ or $(x-a)^2$. In the former case, g is equal to

$$z_a := \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}.$$

This element is central in G. Thus the conjugacy class of g is simply given by $\{z_a\}$.

In the latter case, by theory of Jordan normal form, g is conjugate to

$$u_a := \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}.$$

By a simple computation, we can check that the centralizer of u_g in G is given by ZU (see Section 2.3 for the notation), which is of order $q(q-1)$. Hence the conjugacy class of u_g is of order $|G|/q(q-1) = q^2 - 1$.

We next consider the case (2). In this case, g is necessarily conjugate to

$$t_{a,b} := \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.$$

The centralizer of $t_{a,b}$ is given by T (see Section 2.3 for the notation), which is of order $(q-1)^2$. Hence the conjugacy class of $t_{a,b}$ is of order $|G|/(q-1)^2 = q^2 + q$. We caution that $t_{a,b}$ and $t_{a',b'}$ are conjugate if and only if $(a', b') = (a, b), (b, a)$. In particular, there are $\binom{q-1}{2} = \frac{(q-1)(q-2)}{2}$ conjugacy classes of this type.

We finally consider the case (3). Suppose that ϕ_g is an irreducible monic of degree 2. The subring $\mathbb{F}_q[g]$ of $M_2(\mathbb{F}_q)$ is a degree 2 extension of $\mathbb{F}_q$ (given by the minimal polynomial ϕ_g), hence isomorphic to $\mathbb{F}_{q^2}$. The centralizer of g in G is given by $\mathbb{F}_q[g]^\times$, which is of order $q^2 - 1$. (Indeed, the centralizer of g in $M_2(\mathbb{F}_q)$ (let us write E) must be a commutative subring containing $\mathbb{F}_q[g]$. Since it can be regarded as a $\mathbb{F}_q[g]$-vector space, by counting the dimensions, we see that E must be $\mathbb{F}_q[g]$ or $M_2(\mathbb{F}_q)$. However, the latter case is impossible as $M_2(\mathbb{F}_q)$ is not commutative. Thus $E = \mathbb{F}_q[g]$, hence the centralizer of g in G is given by $E^\times = \mathbb{F}_q[g]^\times$.) Hence the conjugacy class of g is of order $|G|/(q^2-1) = q^2 - q$. Note that, by choosing an $\mathbb{F}_q$-basis of $\mathbb{F}_q[g]$ to be $\{1, g\}$, then the g-multiplication action on $\mathbb{F}_q[g]$ is represented by

$$s_{a,b} := \begin{pmatrix} 0 & -b \\ 1 & -a \end{pmatrix},$$

where we write $\phi_g(x) = x^2 + ax + b$. This matrix represents the conjugacy class of g. An easy computation shows that there are $\frac{q^2-q}{2}$ irreducible degree 2 monics in total. Hence the number of conjugacy classes of this type is also given by $\frac{q^2-q}{2}$.

Now we see that there are

$$(q-1) + (q-1) + \frac{(q-1)(q-2)}{2} + \frac{q^2-q}{2} = q^2 - 1$$

conjugacy classes of G in total. Hence, the number of irreducible representations of G must be $q^2 - 1$.

Table 1. Conjugacy classes of $\mathrm{GL}_2(\mathbb{F}_q)$

representative	order of the conjugacy class	parameter	# of parameters
z_a	1	$a \in \mathbb{F}_q^\times$	$q-1$
u_a	q^2-1	$a \in \mathbb{F}_q^\times$	$q-1$
$t_{a,b}$	q^2+q	$a, b \in \mathbb{F}_q^\times$, $a \neq b$	$\frac{(q-1)(q-2)}{2}$
$s_{a,b}$	q^2-q	irr. deg. 2 monic	$\frac{q^2-q}{2}$

2.2. **Philosophy of induction.** We next give some explanation on a general strategy to construct irreducible representations. For this, here let G temporarily denote any finite group.

Definition 2.2. For a representation (σ, W) of a subgroup H of G, its *induction* to G is defined by

$$\mathrm{Ind}_H^G \sigma := \{f\colon G \to W \mid f(hg) = \sigma(h)(f(g)) \text{ for any } h \in H \text{ and } g \in G\},$$

where G acts via right translation, i.e.,

$$(x \cdot f)(g) := f(gx)$$

for any $x \in G$ and $g \in G$.

Recall that the character of the induced representation $\mathrm{Ind}_H^G \sigma$ can be expressed in terms of the character of σ and the group-theoretic relation between G and H as follows:

Theorem 2.3 (Frobenius formula)**.** *For any $g \in G$, we have*

$$\Theta_{\mathrm{Ind}_H^G \sigma}(g) = \sum_{\substack{x \in H\backslash G \\ xgx^{-1} \in H}} \Theta_\sigma(xgx^{-1}).$$

So, in principle, we should be able to know all about the induced representation $\mathrm{Ind}_H^G \sigma$ as long as the subgroup H and its representation σ are "well-understood". Based on this idea, one can try to construct irreducible representations of G using "well-understood" irreducible representations of subgroups of G. Note that the dimension of $\mathrm{Ind}_H^G \sigma$ is given by $[G : H] \cdot \dim \sigma$. Especially, if H is smaller, then the dimension of $\mathrm{Ind}_H^G \sigma$ is larger. Thus it is possible to expect that we can find more irreducible representations in $\mathrm{Ind}_H^G \sigma$ for small H. Indeed, we have the following fundamental theorem:

Theorem 2.4. *Let G be a finite group. Then the induction of the trivial representation of the trivial subgroup to G decomposes as follows:*

$$\mathrm{Ind}_{\{1\}}^G \mathbb{1} \cong \bigoplus_\rho \rho^{\oplus \dim \rho},$$

where the direct sum is over the isomorphism classes of all irreducible representations of G.

It is beautiful that every irreducible representation is realized in the induction of the trivial representation. However, we can also think that here too many irreducible representations are mixed together, hence it's difficult to distinguish them. So, for example, it would be great if we could find a subgroup H of G which is simultaneously

- small enough that the induction to G can produce various irreducible representations and
- large enough that the inductions are irreducible (or "almost" irreducible).

What we will see in the next section is an example of such a nice subgroup for $\mathrm{GL}_2(\mathbb{F}_q)$, which is called a "Borel subgroup". (In fact, we can also find a family of such nice subgroups for any finite group of Lie type, called "parabolic subgroups".)

2.3. Principal representations of $\mathrm{GL}_2(\mathbb{F}_q)$.

We introduce the subgroups B, T, U of $\mathrm{GL}_2(\mathbb{F}_q)$ as follows:

$$B := \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in M_2(\mathbb{F}_q) \,\middle|\, a, d \in \mathbb{F}_q^\times,\, b \in \mathbb{F}_q \right\},$$

$$T := \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in M_2(\mathbb{F}_q) \,\middle|\, a, d \in \mathbb{F}_q^\times \right\},$$

$$U := \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in M_2(\mathbb{F}_q) \,\middle|\, b \in \mathbb{F}_q \right\}.$$

Note that U is a normal subgroup in B and that we have the semi-direct decomposition $B = T \ltimes U$. In particular, we have a natural surjection $B \twoheadrightarrow T$ by quotienting by $U \triangleleft B$. We let Z denote the center of G, which consists of scalar matrices:

$$Z := \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in M_2(\mathbb{F}_q) \,\middle|\, a \in \mathbb{F}_q^\times \right\}.$$

Remark 2.5. In the context of theory of reductive groups, the subgroups B, T, and U are called a *Borel subgroup*, a *maximal torus*, and the *unipotent radical (of B)*, respectively.

Definition 2.6 (Principal series representation)**.** Suppose that $\boldsymbol{\chi}\colon T \to \mathbb{C}^\times$ is a character. Then, by pulling back $\boldsymbol{\chi}$ via $B \twoheadrightarrow T$, we may regard it as a character of B (this procedure is called the *inflation*). We call the induction $\mathrm{Ind}_B^G \boldsymbol{\chi}$ of $\boldsymbol{\chi}$ from B to G a *principal series representation* (for $\boldsymbol{\chi}$).

Note that we have $T \cong \mathbb{F}_q^\times \times \mathbb{F}_q^\times$, hence any character $\boldsymbol{\chi}$ of T can be expressed as $\boldsymbol{\chi} = \chi_1 \boxtimes \chi_2$ with some characters χ_1 and χ_2 of $\mathbb{F}_q^\times$, i.e., for any $(t_1, t_2) \in T$, we have $\boldsymbol{\chi}(t_1, t_2) = \chi_1(t_1) \cdot \chi_2(t_2)$. We shortly write $\chi_1 \times \chi_2$ for $\mathrm{Ind}_B^G \boldsymbol{\chi} = \mathrm{Ind}_B^G(\chi_1 \boxtimes \chi_2)$. Since the dimension of $\chi_1 \times \chi_2$ is equal to the index of B in G, we have

$$\dim(\chi_1 \times \chi_2) = \frac{|G|}{|B|} = \frac{(q^2-1)(q^2-q)}{(q-1)^2 q} = q+1.$$

Let us first investigate the principal series representations for $\boldsymbol{\chi} = \chi_1 \boxtimes \chi_2$ such that $\chi_1 \neq \chi_2$.

Proposition 2.7. *If $\chi_1 \neq \chi_2$, then $\chi_1 \times \chi_2$ is an irreducible representation of G of dimension $q+1$. Moreover, for two characters $\chi_1 \boxtimes \chi_2$ and $\chi_1' \boxtimes \chi_2'$ of T,*

$$\chi_1 \times \chi_2 \cong \chi_1' \times \chi_2' \iff \chi_1' \boxtimes \chi_2' = \chi_1 \boxtimes \chi_2 \text{ or } \chi_2 \boxtimes \chi_1.$$

We next consider the case where $\chi_1 = \chi_2$.

Proposition 2.8. *(1) The principal series representation $\mathbb{1} \times \mathbb{1} = \mathrm{Ind}_B^G \mathbb{1}$ associated to the trivial character of T is the sum of two irreducible representations of G:*
- *one is the trivial representation of G;*
- *the other is a q-dimensional irreducible representation of G, for which we write St_G (we call the "Steinberg representation" of G).*

(2) For any character χ of $\mathbb{F}_q^\times$, we have $\chi \times \chi \cong (\mathbb{1} \times \mathbb{1}) \otimes (\chi \circ \det)$. In particular, we have

$$\chi \times \chi \cong (\chi \circ \det) \oplus \mathrm{St}_G \otimes (\chi \circ \det).$$

We prove Propositions 2.7 and 2.8 simultaneously.

Proof. Fisrt, by Frobenius reciprocity (the adjunction formula between the induction and restriction), we have

$$\mathrm{Hom}_G(\chi_1 \times \chi_2, \chi_1' \times \chi_2') \cong \mathrm{Hom}_B\big(\mathrm{Res}_B^G(\chi_1 \times \chi_2), \chi_1' \boxtimes \chi_2'\big).$$

By applying the Mackey decomposition formula, we have

$$\mathrm{Res}_B^G(\chi_1 \times \chi_2) \cong \bigoplus_{s \in B\backslash G/B} \mathrm{Ind}_{B\cap s^{-1}Bs}^B \mathrm{Res}_{B\cap s^{-1}Bs}^{s^{-1}Bs} (\chi_1 \boxtimes \chi_2)^s,$$

where $(\chi_1 \boxtimes \chi_2)^s$ denotes the character of $s^{-1}Bs$ defined by $(\chi_1 \boxtimes \chi_2)^s(s^{-1}bs) = (\chi_1 \boxtimes \chi_2)(b)$. Now we use the *Bruhat decomposition*:

$$G = B \sqcup BwB, \quad w := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Since w maps B to its transpose $\overline{B}$ and swaps the first and second factors of $T \cong \mathbb{F}_q^\times \times \mathbb{F}_q^\times$, we get

$$\bigoplus_{s \in B\backslash G/B} \mathrm{Ind}_{B\cap s^{-1}Bs}^B \mathrm{Res}_{B\cap s^{-1}Bs}^{s^{-1}Bs} (\chi_1 \boxtimes \chi_2)^s \cong \underbrace{(\chi_1 \boxtimes \chi_2)}_{s=1} \oplus \underbrace{\mathrm{Ind}_T^B \mathrm{Res}_T^{\overline{B}}(\chi_2 \boxtimes \chi_1)}_{s=w}.$$

(Note that, on the right-hand side, $\chi_1 \boxtimes \chi_2$ and $\chi_2 \boxtimes \chi_1$ are regarded as characters of B and $\overline{B}$ by inflation, respectively.) This implies that

$$\begin{aligned} &\mathrm{Hom}_B\big(\mathrm{Res}_B^G(\chi_1 \times \chi_2), \chi_1' \boxtimes \chi_2'\big) \\ &\qquad \cong \mathrm{Hom}_B(\chi_1 \boxtimes \chi_2, \chi_1' \boxtimes \chi_2') \oplus \mathrm{Hom}_B\big(\mathrm{Ind}_T^B \mathrm{Res}_T^{\overline{B}}(\chi_2 \boxtimes \chi_1), \chi_1' \boxtimes \chi_2'\big). \end{aligned}$$

The first summand on the right-hand side is equal to $\mathrm{Hom}_T(\chi_1 \boxtimes \chi_2, \chi_1' \boxtimes \chi_2')$. By Frobenius reciprocity, the second summand is equal to $\mathrm{Hom}_T\big(\chi_2 \boxtimes \chi_1, \chi_1' \boxtimes \chi_2'\big)$. So we conclude that

$$(\star) \qquad \mathrm{Hom}_G(\chi_1 \times \chi_2, \chi_1' \times \chi_2') \cong \mathrm{Hom}_T\big(\chi_1 \boxtimes \chi_2, \chi_1' \boxtimes \chi_2'\big) \oplus \mathrm{Hom}_T\big(\chi_2 \boxtimes \chi_1, \chi_1' \boxtimes \chi_2'\big).$$

In particular, this implies that

$$\mathrm{End}_G(\chi_1 \times \chi_2) \cong \begin{cases} \mathbb{C} & \chi_1 \neq \chi_2, \\ \mathbb{C} \oplus \mathbb{C} & \chi_1 = \chi_2. \end{cases}$$

By Schur's lemma, this says that $\chi_1 \times \chi_2$ is irreducible when $\chi_1 \neq \chi_2$ and decomposes into a sum of two irreducible representations when $\chi_1 = \chi_2$. So we obtained Proposition 2.7. (The latter assertion of Proposition 2.7 can be checked by the formula $(\star)$). When $\chi_1 = \chi_2 = \chi$, we can easily check that $\chi \times \chi$ contains $\chi \circ \det$. It's also not difficult to check that $\chi \times \chi$ is isomorphic to $(\mathbb{1} \times \mathbb{1}) \otimes (\chi \circ \det)$. (For example, again use Frobenius reciprocity.) Then Proposition 2.8 follows. $\square$

So, how many irreducible representations have we obtained so far? Since there are $(q-1)$ characters of $\mathbb{F}_q^\times$, the principal series construction produces

$$\underbrace{\binom{q-1}{2}}_{\chi_1 \neq \chi_2} + \underbrace{2 \cdot (q-1)}_{\chi_1 = \chi_2} = \frac{q^2+q-2}{2}$$

irreducible representations of $\mathrm{GL}_2(\mathbb{F}_q)$ in total. Thus there should be exactly

$$(q^2-1) - \frac{q^2+q-2}{2} = \frac{q^2-q}{2}$$

more irreducible representations! These are called "cuspidal" representations.

2.4. Cuspidal representations of $\mathrm{GL}_2(\mathbb{F}_q)$.

Definition 2.9 (Cuspidal representations)**.** Let ρ be an irreducible representation of G. We say that ρ is *cuspidal* if ρ is not contained in any principal series representation.

Remark 2.10. We caution that this definition is somehow misleading for understanding the definition of a cuspidal representation in general. In general, there is a notion of a "parabolic subgroup" of a finite group of Lie type. When G is a finite group of Lie type, we say that its irreducible representation is cuspidal if it is not contained in the induction of any representation of the "reductive part" of any nontrivial parabolic subgroup of G (so-called "parabolic induction"). A Borel subgroup is a minimal parabolic subgroup. Because any nontrivial parabolic subgroup is Borel when $G = \mathrm{GL}_2(\mathbb{F}_q)$, we only have to care about principal series representations in the above definition.

Lemma 2.11. *Suppose that ρ is an irreducible representation of G. The following are equivalent:*

(1) ρ is cuspidal.
(2) The U-coinvariant ρ_U of ρ is zero.
(3) The U-invariant ρ^U of ρ is zero.
(4) $\langle \mathrm{Res}_U^G \rho, \mathbb{1}_U \rangle = 0$.

Proof. We first note that

$$\mathrm{Ind}_U^G \mathbb{1}_U = \mathrm{Ind}_B^G(\mathrm{Ind}_U^B \mathbb{1}_U) = \mathrm{Ind}_B^G\Big(\bigoplus_{\boldsymbol{\chi}\colon T \to \mathbb{C}^\times} \boldsymbol{\chi}\Big). = \bigoplus_{\boldsymbol{\chi}\colon T \to \mathbb{C}^\times} (\mathrm{Ind}_B^G \boldsymbol{\chi}).$$

Thus, by definition, ρ is cuspidal if and only if $\mathrm{Hom}_G(\rho, \mathrm{Ind}_U^G \mathbb{1}_U) = 0$. By Frobenius reciprocity, this is equivalent to that $\mathrm{Hom}_U(\mathrm{Res}_U^G \rho, \mathbb{1}_U) = 0$. As ρ is semisimple as representation of U, this is also equivalent to $\mathrm{Hom}_U(\mathbb{1}_U, \mathrm{Res}_U^G \rho) = 0$. The equivalences between (1)–(4) all follows from these observations. ☐

Now we construct all cuspidal irreducible representations of G "by hand". By regarding $\mathbb{F}_q^2$ as a 2-dimensional $\mathbb{F}_q$-vector space, we embed $\mathbb{F}_{q^2}$ into $M_2(\mathbb{F}_q)$. To be more precise, by choosing an $\mathbb{F}_q$-basis of $\mathbb{F}_{q^2}$ (hence get $\mathbb{F}_{q^2} \cong \mathbb{F}_q^{\oplus 2}$, which is regarded as the space of rank 2 column vectors), the multiplication of $\alpha \in \mathbb{F}_{q^2}$ on $\mathbb{F}_{q^2} \cong \mathbb{F}_q^{\oplus 2}$ itself can be written by

$$\alpha \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}.$$

Then the image of $\alpha \in \mathbb{F}_{q^2}$ in $M_2(\mathbb{F}_q)$ is given by $\left(\begin{smallmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{smallmatrix}\right)$. Note that this embedding depends on the choice of an $\mathbb{F}_q$-basis of $\mathbb{F}_{q^2}$, hence not canonical. We fix such an embedding and define a subgroup $S \subset G$ to be the image of $\mathbb{F}_{q^2}^\times$ (note that S contains Z; it is nothing but $\mathbb{F}_q^\times$ contained in $\mathbb{F}_{q^2}^\times$). We also fix a nontrivial character $\psi\colon U \to \mathbb{C}^\times$.

Definition 2.12. For any character $\theta\colon S \to \mathbb{C}^\times$ satisfying $\theta^{q-1} \neq \mathbb{1}$, we define a virtual representation π_θ of G by

$$\pi_\theta := \mathrm{Ind}_{ZU}^G(\theta|_Z \boxtimes \psi) - \mathrm{Ind}_S^G \theta.$$

Here, the right-hand side is considered in the Grothendieck group of representations of G (or, π_θ can be simply regarded as a class function on G).

Proposition 2.13. *The virtual representation π_θ is a $(q-1)$-dimensional irreducible cuspidal representation.*

To prove this proposition, let us first investigate the characters of π_θ.

Lemma 2.14. *The character values of π_θ are given as follows:*

1. $\Theta_{\pi_\theta}(z_a) = (q-1)\theta(a)$ *for* $a \in \mathbb{F}_q^\times$,
2. $\Theta_{\pi_\theta}(u_a) = -\theta(a)$ *for* $a \in \mathbb{F}_q^\times$,
3. $\Theta_{\pi_\theta}(t_{a,b}) = 0$ *for distinct* $a, b \in \mathbb{F}_q^\times$,
4. $\Theta_{\pi_\theta}(s) = -\theta(s) - \theta(s)^q$ *for* $s \in S \smallsetminus Z$.

Proof. The idea is to apply the Frobenius formula. Here let us only check (4).

First recall that $S \subset G$ is defined by the multiplication action of $\mathbb{F}_{q^2}$ on $\mathbb{F}_{q^2}$ itself. This implies that if $s \in S$ does not lie in $Z \subset S$, then the characteristic polynomial of s is an irreducible monic of degree 2. Conversely, for any irreducible monic of degree 2, there exists an $s \in S$ having the monic as its characteristic polynomial. (The point of this argument is that any irreducible monic of degree 2 generates the degree 2 extension $\mathbb{F}_{q^2}$ of $\mathbb{F}_q$ in $\overline{\mathbb{F}}_q$.)

Now let $s \in S$ be an element with irreducible characteristic polynomial $x^2 + ax + b$, hence conjugate to $s_{a,b}$. We first compute the character of $\mathrm{Ind}_{ZU}^G(\theta|_Z \boxtimes \psi)$ at s. By Frobenius formula, we have

$$\Theta_{\mathrm{Ind}_{ZU}^G(\theta|_Z \boxtimes \psi)}(s) = \sum_{\substack{x \in ZU \backslash G \\ xsx^{-1} \in ZU}} (\theta|_Z \boxtimes \psi)(xsx^{-1}).$$

However, since any element of ZU cannot have $x^2 + ax + b$ as its characteristic polynomial, s cannot be conjugate to an element of ZU. In other words, the index set of the above sum must be empty, hence $\Theta_{\mathrm{Ind}_{ZU}^G(\theta|_Z \boxtimes \psi)}(s) = 0$.

We next compute the character of $\mathrm{Ind}_S^G \theta$ at s. Again by Frobenius formula, we have

$$\Theta_{\mathrm{Ind}_S^G \theta}(s) = \sum_{\substack{x \in S\backslash G \\ xsx^{-1} \in S}} \theta(xsx^{-1}).$$

Let us determine the index set. Note that $S = \mathbb{F}_q[s]^\times$. In particular, if $xsx^{-1} \in S$, then we have $xSx^{-1} \subset S$, which furthermore implies that $xSx^{-1} = S$, i.e., $x \in N_G(S)$. Suppose that we have an element $x \in N_G(S) \smallsetminus S$. Then the conjugation via x should induce a nontrivial $\mathbb{F}_q$-automorphism of $\mathbb{F}_q[s]$ ($\cong \mathbb{F}_{q^2}$) (otherwise, x must be in $Z_G(S)$, which equals S). From this, we see that the index set can be regarded as a subset of $\mathrm{Gal}(\mathbb{F}_{q^2}/\mathbb{F}_q)$. In fact, there indeed exists an element $x \in N_G(S) \smallsetminus S$. To see this, let us note that s and s^q have the same characteristic polynomials. Especially, there exists an element $x \in G$ satisfying $xsx^{-1} = s^q$. Since both s and s^q generate $\mathbb{F}_q[s]$, this implies that $xSx^{-1} = S$, hence $x \in N_G(S)$. Of course, this element x cannot be in S. In summary, we get

$$\sum_{\substack{x \in S\backslash G \\ xsx^{-1} \in S}} \Theta_\theta(xsx^{-1}) = \theta(s) + \theta(s^q).$$

Finally, recalling that π_θ is defined to be $\mathrm{Ind}_{ZU}^G(\theta|_Z \boxtimes \psi) - \mathrm{Ind}_S^G \theta$, we get the result. $\square$

Exercise 2.15. Check (1), (2), and (3).

Now let us prove Proposition 2.13.

Proof of Proposition 2.13. To show that π_θ, it suffices to check that $\langle \pi_\theta, \pi_\theta \rangle = 1$. Note that, even if we can show this, there is a possibility that π_θ is the "minus" of an irreducible representation. However, this possibility is excluded since the character value of π_θ at the unit element z_1 is given by $(q-1)$. (Also, we see that the dimension is $(q-1)$ from this.)

Recall that

$$\langle \pi_\theta, \pi_\theta \rangle = \frac{1}{|G|} \sum_{g \in G} \Theta_{\pi_\theta}(g) \cdot \overline{\Theta_{\pi_\theta}(g)}.$$

(1) The sum (not divided by $|G|$) over the conjugacy classes of z_a is

$$\sum_{a \in \mathbb{F}_q^\times} 1 \cdot (q-1)\theta(a) \cdot \overline{(q-1)\theta(a)} = \sum_{a \in \mathbb{F}_q^\times} 1 \cdot (q-1)^2 = (q-1)^3.$$

(2) The sum (not divided by $|G|$) over the conjugacy classes of u_a is

$$\sum_{a \in \mathbb{F}_q^\times} (q^2-1) \cdot (-\theta(a)) \cdot \overline{(-\theta(a))} = \sum_{a \in \mathbb{F}_q^\times} (q^2-1) = (q^2-1)(q-1).$$

(3) The sum (not divided by $|G|$) over the conjugacy classes of $t_{a,b}$ is zero since each character value is zero.

(4) By noting that the orbits of $(S \smallsetminus Z)$ by the action of $\mathrm{Gal}(\mathbb{F}_{q^2}/\mathbb{F}_q)$ bijectively correspond to the conjugacy classes of elements of the form $s_{a,b}$. Hence, the sum (note divided by $|G|$) over the conjugacy classes of $s_{a,b}$ is

$$\begin{aligned} &\frac{1}{2} \sum_{s \in S \smallsetminus Z} (q^2-q) \cdot (-\theta(s) - \theta(s)^q) \cdot \overline{(-\theta(s) - \theta(s)^q)} \\ &= \frac{q^2-q}{2} \sum_{s \in \mathbb{F}_{q^2}^\times \smallsetminus \mathbb{F}_q^\times} (-\theta(s) - \theta(s)^q) \cdot \overline{(-\theta(s) - \theta(s)^q)}. \end{aligned}$$

By an elementary computation, we can check that this equals $(q^2-q)(q-1)^2$. Therefore, we get

$$\langle \pi_\theta, \pi_\theta \rangle = \frac{1}{|G|} \cdot \Big((q-1)^3 + (q^2-1)(q-1) + 0 + (q^2-q)(q-1)^2 \Big) = 1.$$

Finally, let us check the cuspidality of π_θ. It suffices to show that

$$\langle \mathrm{Res}_U^G \pi_\theta, \mathbb{1}_U \rangle = \frac{1}{|U|} \sum_{u \in U} \Theta_{\pi_\theta} = 0.$$

Any element $\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)$ with $a \in \mathbb{F}_q^\times$ is conjugate to $u_1 = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Hence,

$$\sum_{u \in U} \Theta_{\pi_\theta} = \Theta_{\pi_\theta}(z_1) + (q-1)\Theta_{\pi_\theta}(u_1) = (q-1)\theta(1) - (q-1)\theta(1) = 0.$$

□

Exercise 2.16. Complete the computation skipped in the above proof.

Proposition 2.17. *For any $\theta_i \colon S \to \mathbb{C}^\times$ satisfying $\theta_i^{q-1} \neq \mathbb{1}$ $(i = 1, 2)$, we have $\pi_{\theta_1} \cong \pi_{\theta_2}$ if and only if $\theta_1 = \theta_2$ or $\theta_1 = \theta_2^q$.*

Proof. By the character formulas of Θ_{π_θ}, we see that $\pi_{\theta_1} \cong \pi_{\theta_2}$ only if

$$\theta_1(s) + \theta_1(s)^q = \theta_2(s) + \theta_2(s)^q$$

for any $s \in S$. Recall that *Artin's lemma* says that distinct characters of any finite group are linear independent. Hence, by noting that $\theta^q \neq \theta$, the above condition is equivalent to that $\theta_1 = \theta_2$ or $\theta_1 = \theta_2^q$. Conversely, if this is satisfied, then we have $\pi_{\theta_1} \cong \pi_{\theta_2}$ by the character formula of Θ_{π_θ}. □

Here, note that S is of order $q^2 - 1$, hence any character θ of S satisfies $\theta^{q^2} = \theta$. Thus the condition $\theta_1 = \theta_2^q$ is also equivalent to $\theta_2 = \theta_1^q$. Proposition 2.17 enables us to count the number of irreducible cuspidal representations obtained in this way. The group S is cyclic of order $q^2 - 1$, thus there exactly $(q-1)$ characters of S satisfying $\theta^{q-1} = \mathbb{1}$. In other words, there exactly $q^2 - q$ characters of S satisfying $\theta^{q-1} \neq \mathbb{1}$. Therefore the above construction provides $\frac{q^2-q}{2}$ irreducible cuspidal representations, hence all!

2.5. **What is Deligne–Lusztig theory? (a bit more precisely).** The construction of π_θ presented above is somehow mysterious and seems difficult to generalize. So, we want a more conceptual construction of π_θ, which could work in a more general setting. We can find a hint in Drinfeld's observation.

Before we talk about "the curve" of Drinfeld, let us introduce the groups $G' := \mathrm{SL}_2(\mathbb{F}_q)$ and $S' := S \cap G'$. Note that S' is identified with the norm 1 subgroup of $\mathbb{F}_{q^2}^\times$, i.e.,

$$S' \cong \mathrm{Ker}(\mathrm{Nr} \colon \mathbb{F}_{q^2}^\times \to \mathbb{F}_q^\times).$$

In particular, S' is cyclic of order $(q+1)$. We can also introduce the notions of principal series or cuspidal representations for G' in a similar way. Basically, the representation theory of G' can be "derived" from that of G. Especially, the cuspidal representations of G' can be constructed by restricting those of G to G'. Thus let's talk about how to understand cuspidal representations of G' in the following.

Drinfeld investigated the following curve (see [**?**, Chapter 2]).

Definition 2.18 (Drinfeld curve)**.** Let X be the curve defined by

$$X := \{(x, y) \in \mathbb{A}^2_{\mathbb{F}_q} \mid xy^q - x^q y = 1\}.$$

The curve X has the following properties:

- G' acts on X by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot (x, y) = (ax + by, cx + dy)$;
- S' acts on X by $s \cdot (x, y) = (sx, sy)$;
- the actions of G' and S' commute.

Because the étale cohomology has the functoriality in spaces, we can obtain a representation of $G' \times S'$ on the étale cohomology of X. By cutting it along any character θ of S', we get a representation of G'. In fact, this resulting representation is nothing but "π_θ". In other words, Drinfeld's curve gives a geometric realization of the cuspidal representation π_θ which was constructed in a mysterious way previously!

Deligne–Lusztig theory exactly generalizes this idea. Let G be a finite group of Lie type. The input/output of Deligne–Lusztig theory are as follows:

Input: a pair (S, θ) of

- a "maximal torus" S of G and
- a character $\theta \colon S(\mathbb{F}_q) \to \mathbb{C}^\times$.

Output: a virtual representation $R^G_S(\theta)$ of $G(\mathbb{F}_q)$ ("Deligne–Lusztig virtual representaiton").

For a given input (S, θ), Deligne–Lusztig first defined an algebraic variety X^G_S over $\overline{\mathbb{F}}_q$ equipped with an action of $G(\mathbb{F}_q) \times S(\mathbb{F}_q)$. This is called the Deligne–Lusztig variety (associated to (G, S)); this is a far generalization of the Drinfeld curve. Deligne–Lusztig considered its ℓ-adic étale cohomology $H^i_c(X^G_S, \overline{\mathbb{Q}}_\ell)$. Then, as explained above, we obtain a representation of $G(\mathbb{F}_q) \times S(\mathbb{F}_q)$ on $H^i_c(X^G_S, \overline{\mathbb{Q}}_\ell)$. By taking the alternating sum of the θ-isotypic part of each degree, we get the "output":

$$R^G_S(\theta) := \sum_{i \geq 0} (-1)^i H^i_c(X^G_S, \overline{\mathbb{Q}}_\ell)[\theta]$$

Remark 2.19.

1. At this point, you do not have to be able to understand the meaning of the terminologies such as "finite group of Lie type" or "maximal torus". It is also one of the purposes of this course to get familiar with these notions (through various examples).[3]
2. As its symbol suggests, $H^i_c(X^G_S, \overline{\mathbb{Q}}_\ell)$ is a $\overline{\mathbb{Q}}_\ell$-vector space; not a $\mathbb{C}$-vector space. However, by choosing an isomorphism $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$, we may convert $H^i_c(X^G_S, \overline{\mathbb{Q}}_\ell)$ to a $\mathbb{C}$-vector space. In fact, the resulting representation $R^G_S(\theta)$ with $\mathbb{C}$-coefficients is independent of the choice of such an isomorphism ("ℓ-independence", which is an important part of Deligne–Lusztig theory).
3. The subgroup S of $\mathrm{GL}_2(\mathbb{F}_q)$ introduced in the previous section (or S' of $\mathrm{SL}_2(\mathbb{F}_q)$) is an example of a "maximal torus". With the above notation, we have $\pi_\theta \cong R^G_S(\theta)$ for $G = \mathrm{GL}_2(\mathbb{F}_q)$[4]. Recall that $T \cong \mathbb{F}_q^\times \times \mathbb{F}_q^\times$ is another example of a maximal torus of $\mathrm{GL}_2(\mathbb{F}_q)$. One surprising point is that Deligne–Lusztig theory also naturally generalizes the principal series construction. Namely, for any character $\boldsymbol{\chi}$ of $T \subset \mathrm{GL}_2$, we have $\mathrm{Ind}^G_B \boldsymbol{\chi} \cong R^G_T(\boldsymbol{\chi})$.

[3] On the other hand, I have to confess that I will only give a few words about the theory of étale cohomology.

[4] Precisely speaking, we need "up to sign" here

One of the highlights of the theory is that there exists an explicit formula of the Deligne–Lusztig virtual representaiton $R_S^G(\theta)$ called "Deligne–Lusztig character formula". We can analyze the representation $R_S^G(\theta)$ through that formula; for example, we can prove that any irreducible representation ρ of $G(\mathbb{F}_q)$ can be realized in $R_S^G(\theta)$ for some pair (S, θ).

3. Week 3: Algebraic groups

Aim of this week. The aim of this week is to introduce the notion of an algebraic group and its fundamental properties. The main references of this week are [Spr09] and [Bor91].

3.1. **Comments on scheme theory.** Let $\overline{k}$ be an algebraically closed field. Let $\mathbb{A}^n_{\overline{k}}$ be the n-dimensional affine space over k (here, let us simply understand that $\mathbb{A}^n_{\overline{k}}$ is the set of n-tuples of elements of $\overline{k}$). Roughly speaking, an *affine algebraic variety* is a subset of $\mathbb{A}^n_{\overline{k}}$ consisting of simultaneous solutions to a tuple of polynomials in $k[x_1, \ldots, x_n]$. We can equip an affine variety with a topology called *Zariski topology.* A *algebraic variety* is a space obtained by patching affine algebraic varieties.

From the modern viewpoint, the classical theory of algebraic varieties can be far more generalized by the theory of schemes. For any commutative ring R, the *affine scheme* $\operatorname{Spec} R$ is defined to be the set of prime ideals of R. We can equip $\operatorname{Spec} R$ with the Zariski topology in a similar manner to the classical case. In addition, we can also introduce a further structure on $\operatorname{Spec} R$, that is, a sheaf of rings on $\operatorname{Spec} R$; this makes $\operatorname{Spec} R$ so-called a locally ringed space. A *scheme* is a locally ringed space obtained by patching affine schemes.

When a scheme X equipped with a morphism to $\operatorname{Spec} \overline{k}$ (this amounts to that the rings R defining X are $\overline{k}$-algebras) satisfies certain conditions ("separated, reduced, of finite type"), we can associate an algebraic variety to X. This algebraic variety is given to be the set of all "$\overline{k}$-valued points" of X. We'll give a bit more explanation on the notion of "valued points" later. Conversely, any algebraic variety can be realized in this way from a scheme. Roughly speaking, an *algebraic group* is an algebraic variety equipped with a group structure. Thus we have two choices of languages to study algebraic groups; the classical theory of algebraic varieties and the modern theory of schemes.[5]

When an algebraic variety X has defining polynomials whose coefficients are in a subfield k of $\overline{k}$, we say that X *is defined over* k. In the language of scheme theory, this amounts to that there exists a scheme X_0 equipped with a morphism to $\operatorname{Spec} k$ such that its *base change to* $\overline{k}$ (i.e., the fibered product of $X_0 \to \operatorname{Spec} k$ and $\operatorname{Spec} \overline{k} \to \operatorname{Spec} k$) is isomorphic to X. One advantage of using scheme theory is that it makes it theoretically easier to treat algebraic varieties over a field k which is not necessarily algebraically closed. This is particularly important for us because eventually we want to discuss algebraic groups defined over a finite field. On the other hand, we can understand algebraic groups in a more intuitive way by appealing to the classical theory of algebraic varieties.

In any case, it is unavoidable to rely on these languages of algebraic geometry, but we do not go into the details of algebraic geometry in this course.[6] Rather, our aim is to get familiar with algebraic groups through several concrete examples.

3.2. **Definition and examples of algebraic groups.** Let k be a field. In the following, let us furthermore assume that k is perfect. (In this course, eventually, k will be taken to be a finite field $\mathbb{F}_q$.) We write Γ_k for the absolute Galois group $\operatorname{Gal}(\overline{k}/k)$ of k.

By "an algebraic variety over k", we mean a scheme X equipped with a morphism to $\operatorname{Spec} k$ such that its base change $X_{\overline{k}}$ to $\operatorname{Spec} \overline{k}$ is an algebraic variety.

Definition 3.1 (algebraic group)**.** Let G be an algebraic variety over k. We say that G is an *algebraic group over* k if G is equipped with a group structure, i.e., morphisms of schemes over k

[5]Indeed, [Spr09] is written via the theory of algebraic varieties while [Bor91] is written via scheme theory.
[6]For example, see [Spr09, Chapter 1] or [Bor91, Chapter AG] for a summary on algebraic geometry.

- $m\colon G \times_k G \to G$ ("multiplication morphism"),
- $i\colon G \to G$ ("inversion morphism"), and
- $e\colon \operatorname{Spec} k \to G$ ("unit element")

satisfying the axioms of groups. More precisely, the following diagrams are commutative:

$$\begin{array}{ccc} G\times_k G\times_k G & \xrightarrow{m\times \mathrm{id}} & G\times_k G \\ {\scriptstyle \mathrm{id}\times m}\big\downarrow & \circlearrowleft & \big\downarrow{\scriptstyle m} \\ G\times_k G & \xrightarrow{\quad m \quad} & G\end{array} \qquad \begin{array}{ccc} G & \xrightarrow{\mathrm{id}\times e} & G\times_k G \\ {\scriptstyle e\times \mathrm{id}}\big\downarrow & \overset{\circlearrowleft}{\underset{\circlearrowleft}{\searrow^{\mathrm{id}}}} & \big\downarrow{\scriptstyle m} \\ G\times_k G & \xrightarrow{\quad m \quad} & G\end{array}$$

$$\begin{array}{ccccc} G\times_k G & \xleftarrow{\Delta} & G & \xrightarrow{\Delta} & G\times_k G \\ {\scriptstyle \mathrm{id}\times i}\big\downarrow & \circlearrowleft & {\scriptstyle \epsilon}\big\downarrow & \circlearrowleft & \big\downarrow{\scriptstyle i\times \mathrm{id}} \\ G\times_k G & \xrightarrow{m} & G & \xleftarrow{m} & G\times_k G\end{array}$$

Here, ϵ denotes the composition of the structure morphism $G \to \operatorname{Spec} k$ and $e\colon \operatorname{Spec} k \to G$.

Remark 3.2. Suppose that G is an affine algebraic variety with coordinate ring $k[G]$ (i.e., $G = \operatorname{Spec} k[G]$). Recall that the category of affine schemes is (anti-) equivalent to the category of commutative rings. Thus giving G an algebraic group structure is equivalent to defining ring homomorphisms corresponding to m, i, e and satisfying analogous axioms. For example, the ring homomorphism corresponding to m must be a k-algebra homomorphism $R \to R \otimes_k R$. In general, a commutative ring equipped with such an additional structure is called a *Hopf algebra*.

Various notions in the usual group theory can be formulated also for algebraic groups. For example, for an algebraic group G over k, we can define its center $Z(G)$, its derived subgroup (commutator subgroup) $G_{\mathrm{der}} = [G, G]$, and so on, as algebraic groups over k. The notion of a homomorphism between algebraic groups is also defined in a natural way. For an algebraic group G over k, its Zariski-connected component containing (the image of) the unit element e is closed under the multiplication, i.e., G° is an algebraic subgroup of G over k. We refer *the identity component of* G to it.

Example 3.3. (1) We put $\mathbb{G}_{\mathrm{a}} := \operatorname{Spec} k[x]$ and define m, i, and e at the level of rings as follows:
- $m\colon k[x] \to k[x] \otimes_k k[x]; \quad x \mapsto x \otimes 1 + 1 \otimes x$,
- $i\colon k[x] \to k[x]; \quad x \mapsto -x$,
- $e\colon k[x] \to k; \quad x \mapsto 0$.

Then $\mathbb{G}_{\mathrm{a}}$ is an algebraic group over k with respect to these operations. We call $\mathbb{G}_{\mathrm{a}}$ the *additive group* over k.

(2) We put $\mathbb{G}_{\mathrm{m}} := \operatorname{Spec} k[x, x^{-1}]$ and define m, i, and e at the level of rings as follows:
- $m\colon k[x] \to k[x, x^{-1}] \otimes_k k[x, x^{-1}]; \quad x \mapsto x \otimes x$,
- $i\colon k[x] \to k[x]; \quad x \mapsto x^{-1}$,
- $e\colon k[x] \to k; \quad x \mapsto 1$.

Then $\mathbb{G}_{\mathrm{m}}$ is an algebraic group over k with respect to these operations. We call $\mathbb{G}_{\mathrm{m}}$ the *multiplicative group* over k.

(3) We put $\mathrm{GL}_n := \operatorname{Spec} k[x_{ij}, D^{-1} \mid 1 \leq i, j \leq n]$, where $D := \det(x_{ij})_{1\leq i,j\leq n}$. We define m, i, and e at the level of rings as follows:
- $m(x_{ij}) := \sum_{k=1}^{n} x_{ik} \otimes x_{kj}$,
- $i(x_{ij}) :=$ the (i, j)-entry of the inverse of the matrix $(x_{ij})_{1\leq i,j\leq n}$,

- $e(x_{ij}) := \delta_{ij}$ (Kronecker's delta).

Then GL_n is an algebraic group over k with respect to these operations. We call GL_n the *general linear group (of rank n)* over k. (Note that $\mathrm{GL}_1 \cong \mathbb{G}_{\mathrm{m}}$.)

In fact, it is not always practical to know the structure ring of an algebraic group and the ring homomorphisms defining the algebraic group structure. Instead, by relying on the philosophy of "the functor of points", we may understand algebraic groups over k intuitively as follows. Recall that any affine scheme $X = \operatorname{Spec} k[X]$ over k defines the following functor (*functor of points*) from the category of k-algebras to the category of sets:

$$(k\text{-algebras}) \to (\text{sets})\colon R \mapsto X(R) := \mathrm{Hom}_k(\operatorname{Spec} R, X)\,(\cong \mathrm{Hom}_k(k[X], R)).$$

(The set $X(R)$ is called the *set of R-valued points of X*.) By Yoneda's lemma, regarding X as a functor in this way does not lose any information of X essentially. Moreover, if X is an affine algebraic group over k, then the morphisms m, i, and e induce a group structure on the set $X(R)$ of R-valued points of X. Hence the above functor takes values in the category of groups. In other words, we may regard an affine algebraic group over k as a "machine" which associates a group to each k-algebra. One practical way of treating (affine) algebraic groups over k is to care only about the groups associated to (all) k-algebras. Recall that, in our convention, an algebraic variety X over k is a scheme whose base change to $\overline{k}$ can be regarded as an algebraic variety in the classical sense; as a set, this algebraic variety is nothing but $X(\overline{k})$.

Let us present several basic examples:

Example 3.4.
(1) For a k-algebra R, we have $\mathbb{G}_{\mathrm{a}}(R) \cong R$, where the group structure on R is given by the additive structure of R. Indeed, we have

$$\mathbb{G}_{\mathrm{a}}(R) = \mathrm{Hom}_k(\operatorname{Spec} R, \mathbb{G}_{\mathrm{a}}) \cong \mathrm{Hom}_k(k[x], R) \cong R,$$

where the last map is given by $f \mapsto f(x)$. This is why $\mathbb{G}_{\mathrm{a}}$ is called the "additive group".

(2) For a k-algebra R, we have $\mathbb{G}_{\mathrm{m}}(R) \cong R^\times$, where $R^\times$ denotes the unit group of R with respect to the multiplicative structure of R. Indeed, we have

$$\mathbb{G}_{\mathrm{m}}(R) = \mathrm{Hom}_k(\operatorname{Spec} R, \mathbb{G}_{\mathrm{m}}) \cong \mathrm{Hom}_k(k[x, x^{-1}], R) \cong R^\times,$$

where the last map is given by $f \mapsto f(x)$. This is why $\mathbb{G}_{\mathrm{m}}$ is called the "multiplicative group".

(3) For a k-algebra R, we have

$$\mathrm{GL}_n(R) \cong \{g = (g_{ij})_{i,j} \in M_n(R) \mid \det(g) \in R^\times\}.$$

Indeed, by definition, we have

$$\mathrm{GL}_n(R) = \mathrm{Hom}_k(\operatorname{Spec} R, \mathrm{GL}_n) \cong \mathrm{Hom}_k(k[x_{ij}, D^{-1} \mid 1 \leq i, j \leq n], R).$$

The right-hand side is isomorphic to (at least as sets) $\{g = (g_{ij})_{i,j} \in M_n(R) \mid \det(g) \in R^\times\}$ by the map $f \mapsto (f(x_{ij}))_{i,j}$. It is a routine work to check that this bijection is indeed a group isomorphism.

(4) The *symplectic group* Sp_{2n} is an affine algebraic group such that the group of its R-valued points is given as follows:

$$\mathrm{Sp}_{2n}(R) \cong \{g = (g_{ij})_{i,j} \in \mathrm{GL}_{2n}(R) \mid {}^t g J_{2n} g = J_{2n}\},$$

where J_{2n} denotes the antidiagonal matrix whose antidiagonal entries are given by 1 and -1 alternatively:

$$J_{2n} := \begin{pmatrix} & & & 1 \\ & & -1 & \\ & 1 & & \\ \iddots & & & \end{pmatrix}.$$

(5) Here let's assume that the characteristic of k is not 2. Let J be an element of $\mathrm{GL}_n(k)$ which is symmetric, i.e., its transpose ${}^tJ$ equals J. Then the *orthogonal group (associated to J)* $\mathrm{O}(J)$ is an affine algebraic group such that the group of its R-valued points is given as follows:

$$\mathrm{O}(J)(R) \cong \{g = (g_{ij})_{i,j} \in \mathrm{GL}_n(R) \mid {}^tgJg = J\}.$$

This group is disconnected and has 2 connected components. The identity component of $\mathrm{O}(J)$ is denoted by $\mathrm{SO}(J)$ and called the *special orthogonal group (associated to J)*.[7] When J is taken to be the anti-diagonal matrix whose anti-diagonal entries are all given by 1, we simply write O_n and SO_n.

Here, we don't explain how to define the structure rings of $\mathrm{SO}(J)$ or Sp_{2n} and also how to introduce the group structure at the level of their structure rings. Only the important viewpoint here is what kind of groups are associated as the groups of R-valued points! (When we are only interested in the algebro-geometric nature of a given algebraic group, we even look at only its $\overline{k}$-valued points.) So, in this course, let us just believe that the "functors" $\mathrm{SO}(J)$ or Sp_{2n} are indeed *representable*, i.e., realized as the functors of points of some affine algebraic groups. This remark is always applied to any affine algebraic group which we will encounter in the future.

3.3. **Jordan decomposition.** We first begin with the following proposition, which is a consequence of the theory of Jordan normal form in linear algebra.

Proposition 3.5. *Let g be an element of $\mathrm{GL}_n(k)$. Then there exists a unique decomposition $g = g_s + g_n$ such that*

- $g_sg_n = g_ng_s$,
- $g_s \in \mathrm{GL}_n(k)$ *is semisimple, i.e., diagonalizable in* $\mathrm{GL}_n(\overline{k})$, *and*
- $g_n \in \mathrm{GL}_n(k)$ *is nilpotent, i.e., all the eigenvalues are* 0 *(equivalently, some power of g_n is zero).*

Proof. Let us briefly the sketch of the proof. We first work over the algebraic closure $\overline{k}$ (this is the same as the separable closure of k since we assume that k is perfect).

We regard $g \in \mathrm{GL}_n(\overline{k})$ as an endomorphism of $V := \overline{k}^{\otimes n}$. We let $\{\alpha_1, \ldots, \alpha_r\}$ be the set of eigenvalues of g. Recall that the generalized eigenspace of g with respect to its eigenvalue α_i is defined by

$$V_i := \mathrm{Ker}(g - \alpha_i \cdot I_n)^{n_i},$$

where n_i is any sufficiently large integer (then V_i is equal to the subspace $\{v \in V \mid (g - \alpha_i \cdot I_n)^m(v) = 0 \text{ for some } m > 0\}$). Then the theorem of Cayley–Hamilton implies that we have $V = \bigoplus_{i=1}^r V_i$.

[7]Note that J_{2n} is symmetric if the characteristic of k is 2 since -1 equals 1! When the characteristic is 2, we have to define orthogonal groups in terms of quadratic forms; so the point is that the notion of a quadratic form is not equivalent to the notion of a symmetric bilinear form when the characteristic is 2.

We put $g_i := g|_{V_i} \in \mathrm{End}_k(V_i)$. If we put $g_{i,s} := \alpha_i \cdot I_{\dim V_i}$ and $g_{i,n} := g_i - g_{i,s}$, then we have

- $g_{i,s}$ is semisimple,
- $g_{i,n}$ is nilpotent, and
- $g_{i,s}g_{i,n} = g_{i,n}g_{i,s}$.

Thus, by putting $g_s := \bigoplus_{i=1}^r g_{i,s}$ and $g_n := \bigoplus_{i=1}^r g_{i,n}$, we get a decomposition $g = g_s + g_n$ satisfying the desired conditions. To check the uniqueness of such a decomposition, suppose that we also have another such decomposition $g = g'_s + g'_n$. Then, since g'_s commutes with g, g'_s preserves each V_i. By noting that $g_i - (g'_s)|_{V_i} = (g'_n)|_{V_i}$, which is nilpotent, we see that g and g'_s have the same eigenvalues on V_i. As g'_s is semisimple, this implies that g'_s must be equal to $\alpha_i \cdot I_{\dim V_i}$. Hence we also get $g_n = g'_n$.

Next suppose that $g \in \mathrm{GL}_n(k)$. Then, by what we proved so far, we can find a decomposition $g = g_s + g_n$ satisfying the desired conditions in $\mathrm{GL}_n(\overline{k})$. For any $\sigma \in \mathrm{Gal}(\overline{k}/k)$, we have $\sigma(g) = \sigma(g_s) + \sigma(g_n)$. However, as we have $\sigma(g) = g$ and this decomposition also satisfies the desired conditions, the uniqueness property implies that $\sigma(g_s) = g_s$ and $\sigma(g_n) = g_n$. In other words, g_s and g_n belong to $\mathrm{GL}_n(k)$. □

The decomposition $g = g_s + g_n$ here is called the *additive Jordan decomposition* of g.

Corollary 3.6. *Let g be an element of $\mathrm{GL}_n(k)$. Then there exists a unique decomposition $g = g_s g_u$ such that*

- $g_s g_u = g_u g_s$,
- *g_s is semisimple, and*
- *g_u is unipotent, i.e., all the eigenvalues are 1 (equivalently, $g_u - 1$ is nilpotent).*

Proof. Let $g = g_s + g_n$ be the additive Jordan decomposition of g. Then we have $g = g_s(1 + g_s^{-1}g_n)$. Since $g_s^{-1}g_n$ is nilpotent (use that g_s and g_n commute), $1 + g_s^{-1}g_n$ is unipotent. Let us put $g_u := 1 + g_s^{-1}g_n$. As g_s commutes with g_u, $g = g_s g_u$ is a desired decomposition.

To check the uniqueness, let us assume that $g = g'_s g'_u$ is another such decomposition. Then, by putting $g'_n := g'_s(g'_u - 1)$, we get the additive Jordan decomposition $g = g'_s + g'_n$. By the uniqueness of the additive Jordan decomposition, we have $g'_s = g_s$ and $g'_u = g_u$. □

The decomposition $g = g_s g_u$ is called the *Jordan decomposition* of g.

In fact, the notion of the Jordan decomposition can be extended to much more general class of algebraic groups. The idea is to reduce the problem to the case of GL_n.

Definition 3.7. When an algebraic group G is isomorphic to a closed subgroup of GL_n for some n, we say that G is a *linear algebraic group*.

Definition 3.8 (Jordan decomposition)**.** Let G be a linear algebraic group over k. Let $\rho\colon G \hookrightarrow \mathrm{GL}_n$ be a closed embedding of algebraic group.

1. We say that an element s of $G(k)$ is *semisimple* if $\rho(s) \in \mathrm{GL}_n(k)$ is semisimple.
2. We say that an element u of $G(k)$ is *unipotent* if $\rho(u) \in \mathrm{GL}_n(k)$ is unipotent.
3. For $g \in G(k)$, we say that g has a *Jordan decomposition* if there exist a semisimple $g_s \in G(k)$ and a unipotent $g_u \in G(k)$ satisfying $g = g_s g_u = g_u g_s$.

Proposition 3.9. *Being semisimple/unipotent is independent of the choice of ρ. Moreover, every element of $G(k)$ has a Jordan decomposition uniquely.*

Then, when can an algebraic group be linear? In fact, we have the following:

Proposition 3.10. *Let G be an algebraic group. Then G is affine if and only if G is linear.*

We don't give proofs of Propositions 3.9 and 3.10. See, for example, [Spr09, Section 2.4]. (In both propositions, the point of the proofs is to consider the action of G on its coordinate ring $k[G]$, which gives rise to a faithful representation of G.)

Remark 3.11. The Jordan decomposition can be explained in a quite simple way when the base field k is a finite field. Let us suppose that $k = \mathbb{F}_q$, whose characteristic is $p > 0$. Note that then, for any linear algebraic group G, the group $G(k)$ of its k-valued points is a finite group. In particular, any element $g \in G(k)$ is of finite order. In fact, we can show that $g \in G(k)$ is semisimple (resp. unipotent) if and only if the order of g is prime to p (resp. p-power). Furthermore, appealing to these characterizations, we can show the unique existence of the Jordan decomposition by an elementary arithmetic argument.

Exercise 3.12. Give a proof to the statement given in the above remark. To be more precise, prove that, for any element $g \in G(k)$,

(1) $g \in G(k)$ is semisimple if and only if the order of g is prime to p,
(2) $g \in G(k)$ is unipotent if and only if the order of g is p-power,
(3) there exists a unique decomposition $g = g_s g_u$ such that $g_s g_u = g_u g_s$, g_s is of prime-to-p order, and g_u is of p-power order.

3.4. **Tori.** We investigate linear algebraic groups consisting only of semisimple elements:

Definition 3.13 (tori/diagonalizable groups)**.**
(1) We say that an algebraic group T over k is a *(k-rational) torus* if it is isomorphic to $\mathbb{G}_{\mathrm{m}}^r$ for some r (called the *rank* of T) over $\overline{k}$.
(2) We say that an algebraic group D over k is *diagonalizable* if it is isomorphic to a closed subgroup of a k-rational torus.

Proposition 3.14. *A connected linear algebraic group G over k is a torus if and only if $G(\overline{k})$ consists only of semisimple elements.*

Proof. See, for example, [Spr09, Corollary 6.3.6]. □

For an algebraic group G over k, we put

$$X^*(G) := \mathrm{Hom}_{\overline{k}}(G_{\overline{k}}, \mathbb{G}_{\mathrm{m}}),$$

i.e., the set of homomorphisms (as algebraic groups) from $G_{\overline{k}}$ to $\mathbb{G}_{\mathrm{m}}$ over $\overline{k}$. Such a homomorphism is called a *(absolute) character* of G. As $X^*(G)$ has a natural group structure, $X^*(G)$ is called the *(absolute) character group* of G. We also define the (absolute) cocharacter group of G by

$$X_*(G) := \mathrm{Hom}_{\overline{k}}(\mathbb{G}_{\mathrm{m}}, G_{\overline{k}})$$

(any homomorphism from $\mathbb{G}_{\mathrm{m}}$ to $G_{\overline{k}}$ is called a (absolute) cocharacter).

Suppose that T is a k-rational torus of rank r. Then $X^*(T)$ is a free abelian group of rank r equipped with an action of Γ_k defined by

$$\sigma(\chi) := \sigma_T \circ \chi \circ \sigma_T^{-1}$$

for any $\sigma \in \Gamma_k$ and $\chi \in X^*(T)$. Here, the symbol "σ_T" on the right-hand side denotes the isomorphism of $T_{\overline{k}}$ obtained by the pull-back of $\sigma\colon \operatorname{Spec} \overline{k} \to \operatorname{Spec} \overline{k}$ along the structure morphism (say $f\colon T_{\overline{k}} \to \operatorname{Spec}\overline{k}$):

$$\begin{array}{ccc} T_{\overline{k}} & \xrightarrow{\sigma_T} & T_{\overline{k}} \\ \downarrow & & \downarrow f \\ \operatorname{Spec}\overline{k} & \xrightarrow{\sigma} & \operatorname{Spec}\overline{k} \end{array}$$

In fact, we have the following:

Proposition 3.15. *The association $T \mapsto X^*(T)$ defines an equivalence of categories between*

- *the category of tori over k and*
- *the category of free abelian groups of finite rank equipped with a Γ_k-action.*

Although any k-rational torus T is isomorphic to $\mathbb{G}_{\mathrm{m}}^r$ over $\overline{k}$ by definition, it might happen (quite often!) that T is not isomorphic to $\mathbb{G}_{\mathrm{m}}^r$ over k. In the above equivalence, $\mathbb{G}_{\mathrm{m}}^r$ corresponds to the free abelian group $\mathbb{Z}^{\oplus r}$ with trivial Galois action. We call the k-rational torus $\mathbb{G}_{\mathrm{m}}^r$ the *split torus (of rank r)*. In some sense, the nontriviality of the action of Γ_k on $X^*(T)$ exactly measures how T is far from being split.

Note that, for any k-rational torus T of rank r, its cocharacter group is also a free abelian group of rank r equipped with a Galois action. If we define a pairing $\langle -,-\rangle$ between $X^*(T)$ and $X_*(T)$ by

$$\operatorname{Hom}_{\overline{k}}(G_{\overline{k}}, \mathbb{G}_{\mathrm{m}}) \times \operatorname{Hom}_{\overline{k}}(\mathbb{G}_{\mathrm{m}}, G_{\overline{k}}) \to \operatorname{Hom}_{\overline{k}}(\mathbb{G}_{\mathrm{m}}, \mathbb{G}_{\mathrm{m}}) \cong \mathbb{Z}\colon (\chi, \chi^\vee) \mapsto \chi \circ \chi^\vee,$$

then $\langle -,-\rangle$ is perfect and equivariant with respect to the Galois actions. Here, the identification $\operatorname{Hom}_{\overline{k}}(\mathbb{G}_{\mathrm{m}}, \mathbb{G}_{\mathrm{m}}) \cong \mathbb{Z}$ is given by $[x \mapsto x^n] \leftrightarrow n$.

Example 3.16. Let k'/k be a finite extension. In general, for any linear algebraic group G' over k', there exists a linear algebraic group over k denoted by $\operatorname{Res}_{k'/k} G'$ and called *the Weil restriction (along k'/k) of G'*. As a functor of points, this linear algebraic group associates $G'(R \otimes_k k')$ to any k-algebra R. By applying this construction to the multiplicative group $\mathbb{G}_{\mathrm{m}}$ over k', we obtain a linear algebraic group $\operatorname{Res}_{k'/k} \mathbb{G}_{\mathrm{m}}$ such that $(\operatorname{Res}_{k'/k} \mathbb{G}_{\mathrm{m}})(R) = \mathbb{G}_{\mathrm{m}}(R \otimes_k k') = (R \otimes_k k')^\times$. (Note that, in particular, we have $(\operatorname{Res}_{k'/k} \mathbb{G}_{\mathrm{m}})(k) = k'^\times$.) In fact, $\operatorname{Res}_{k'/k} \mathbb{G}_{\mathrm{m}}$ is a k-rational torus whose character group is given by $\operatorname{Ind}_{\Gamma_{k'}}^{\Gamma_k} \mathbb{Z}$ as a free abelian group equipped with a Γ_k-action. We call a torus which is isomorphic to a product of tori of this form an *induced torus*.

Definition 3.17. Let G be a linear algebraic group over k. We say that a k-rational subtori T of G is a *(k-rational) maximal torus of G* if it is maximal among all k-rational subtori of G.

Example 3.18. Let $G := \mathrm{GL}_n$. Let T be the subgroup of G consisting of diagonal matrices. Then it is obvious that T is defined over k and isomorphic to $\mathbb{G}_{\mathrm{m}}^r$; especially, T is a k-rational subtorus of G. Let us check that T is a maximal torus. To do this, we suppose that T is contained in another k-rational subtorus T' of G. By taking the centralizers of T and T' in G, we get an inclusion $Z_G(T) \supset Z_G(T')$. (Recall that $Z_G(T) = \{g \in G \mid gtg^{-1} = t \text{ for any } t \in T\}$.) By an elementary computation, we can directly check that $Z_G(T)$ is equal to T itself. On the other hand, since T' is commutative, $Z_G(T')$ must include T'. Thus we get $T \supset T'$, which implies that $T = T'$.

Exercise 3.19. Prove the fact $Z_{\mathrm{GL}_n}(T) = T$, which is used in the above example.

Proposition 3.20. *Let G be a linear algebraic group over k. Then there exists a k-rational maximal torus of G. Moreover, all k-rational maximal tori of G are conjugate over $\overline{k}$. More precisely, if T_1 and T_2 are k-rational maximal tori of G, then there exists an element $g \in G(\overline{k})$ satisfying $T_2 = gT_1g^{-1}$.*

Proof. See, for example, [Spr09, 13.3.6. and 6.4.1.]. □

Note that this proposition does **not** say that all k-rational maximal tori are conjugate over k.

Example 3.21. Suppose that k'/k is a finite extension of degree n. If we choose a k-basis of k', then we can embed k' into $M_n(k)$ by sending $x \in k'$ to the matrix representation of the x-multiplication endomorphism of $k' \cong k^{\oplus n}$. This embedding induces an injective group homomorphism $(k' \otimes_k R)^\times \hookrightarrow \mathrm{GL}_n(R)$ for any k-algebra R functorially. In other words, we get an embedding of a torus $\mathrm{Res}_{k'/k}\,\mathbb{G}_{\mathrm{m}}$ into GL_n. The image of this embedding gives a k-rational maximal torus of GL_n which is not conjugate to the diagonal maximal torus over k. Indeed, it has the same rank as the split diagonal maximal torus, it must be maximal. But the Galois action on its character group is not trivial as explained in Example 3.16. Thus it cannot be conjugate to the split diagonal maximal torus over k.

In general, classifying all $G(k)$-conjugacy classes of k-rational maximal tori of a linear algebraic group over k could be a very deep problem. However, when $k = \mathbb{F}_q$ and G is "reductive", we can classify them in a simple and beautiful way. Because this classification is an important step for understanding Deligne–Lusztig theory, we will investigate it in detail later (2 or 3 weeks later?).

4. Week 4: Reductive groups

4.1. Definition of a reductive group.

Proposition/Definition 4.1 ([Spr09, 6.4.14])**.** Let G be a connected linear algebraic group over k.

(1) There uniquely exists a maximal closed connected normal solvable[8] subgroup of G defined over k, which is called the *radical* of G. We write $R(G)$ for the radical of G.
(2) There uniquely exists a maximal closed connected normal unipotent[9] subgroup of G defined over k, which is called the *unipotent radical* of G. We write $R_u(G)$ for the unipotent radical of G.

Definition 4.2 (semisimple/reductive groups)**.** Let G be a connected linear algebraic group over k.

(1) We say that G is *semisimple* if $R(G)$ is trivial.
(2) We say that G is *reductive* if $R_u(G)$ is trivial.

Remark 4.3. In general, any unipotent group is solvable (see [Spr09, 2.4.13]). In particular, $R_u(G)$ is contained in $R(G)$. This means that if G is semisimple, then G is reductive.

Remark 4.4. In general, $R_u(G_{\overline{k}})$ could be different from the base change of $R_u(G)$ from k to $\overline{k}$. This means that the condition that a connected linear algebraic G group over k is reductive in the above sense is not equivalent to the condition that $G_{\overline{k}}$ is reductive. However, such a phenomenon does not happen as long as k is perfect, i.e., we have $R_u(G)_{\overline{k}} = R_u(G_{\overline{k}})$ for any perfect k. In the situation where k is not perfect, a connected linear algebraic group over k with trivial $R_u(G)$ is called a *pseudo-reductive* group. See [CGP15, Section 1.1] for details.

The following proposition basically follows from the definition of being solvable/unipotent.

Proposition 4.5. *The unipotent radical $R_u(G)$ is the set of unipotent elements of $R(G)$.*

Proposition 4.6. *Let G be a connected reductive group over k.*

(1) The center $Z(G)$ of G is finite if and only if G is semisimple.
(2) The derived subgroup $G_{\mathrm{der}} := [G, G]$ is a connected semisimple group over k. Moreover, we have $G = Z(G) \cdot G_{\mathrm{der}}$.

Proof. See [Spr09, 7.3.1 and 8.1.6]. □

Now, let us introduce several practical propositions to determine the unipotent radical of a given connected reductive group. As mentioned above, the unipotent radical behaves consistently with the base change of the field k as long as it is perfect. Thus, in the rest of this section, let us assume that k is algebraically closed and omit the word "over k". (But sometimes we will temporarily assume that k is not algebraically closed, e.g., when we discuss the rationality.)

Definition 4.7 (Borel subgroup)**.** Let G be a linear algebraic group. A subgroup B of G is called a *Borel subgroup of G* if it is a maximal connected solvable closed subgroup of G.

[8]Solvability is defined in the same way as in the usual group theory, i.e., an algebraic group G is said to be solvable when $G_n = \{1\}$ for sufficiently large n, where $G_n := [G_{n-1}, G_{n-1}]$ and $G_1 := G$.

[9]i.e., all elements are unipotent

Theorem 4.8 (Lie–Kolchin's theorem, [Spr09, 6.3.1])**.** *Let B be a connected solvable closed subgroup of GL_n. Let B_n be the group of upper triangular matrices of GL_n. Then B is conjugate to a subgroup of B_n.*

Note that, in particular, B_n is a Borel subgroup of GL_n by Lie–Kolchin's theorem.

Proposition 4.9. *Let G be a connected linear algebraic group. All Borel subgroups of G are conjugate.*

Proof. See [Spr09, 6.2.7]. □

Corollary 4.10. *Let G be a connected linear algebraic group. Then its radical $R(G)$ equals the identity component of the intersection of all Borel subgroups of G.*

Proof. By definition, $R(G)$ is contained in a Borel subgroup. Since $R(G)$ is normal in G and all Borel subgroups of G are conjugate, $R(G)$ is contained in the intersection of all Borel subgroups of G. As $R(G)$ is connected, it must be contained in the identity component of the intersection. Since the identity component of the intersection of all Borel subgroups of G is closed, connected, normal, and solvable, it must be equal to $R(G)$ by the maximality of $R(G)$. □

4.2. **Examples of reductive groups.**

Example 4.11 (tori)**.** Any torus T is reductive. Indeed, since T is commutative, hence solvable, $R(T)$ is T itself. Since all elements of T are semi-simple, $R_u(T)$ is trivial.

Non-Example 4.12 (additive group)**.** The additive group $\mathbb{G}_\mathrm{a}$ is not reductive. Indeed, since $\mathbb{G}_\mathrm{a}$ is commutative, hence solvable, $R(\mathbb{G}_\mathrm{a})$ is $\mathbb{G}_\mathrm{a}$ itself. However, since $\mathbb{G}_\mathrm{a}$ is a unipotent group[10], $R_u(\mathbb{G}_\mathrm{a})$ also equals $\mathbb{G}_\mathrm{a}$.

Example 4.13 (general linear group)**.** The general linear group GL_n is reductive. To check this, note that B_n is a Borel subgroup of GL_n, hence its any conjugate is also a Borel subgroup of GL_n. In particular, the opposite $\overline{B}_n$ (i.e., the subgroup of lower triangular matrices) is also Borel. Hence their intersection, which is the diagonal subgroup T of GL_n, must contain $R(\mathrm{GL}_n)$. This implies that all elements of $R(\mathrm{GL}_n)$ is semisimple, hence $R_u(\mathrm{GL}_n)$ is trivial.

Exercise 4.14. Prove that $R(\mathrm{GL}_n) = Z(\mathrm{GL}_n)$.

Example 4.15 (symplectic group)**.** The symplectic group Sp_{2n} is reductive. Indeed, if we put B to be $B_{2n} \cap \mathrm{Sp}_{2n}$ (i.e., the subgroup of Sp_{2n} consisting of matrices of the upper-triangular form), then we can show that B is a Borel subgroup of Sp_{2n}. (See the following exercise.) Similarly, its opposite $\overline{B} := \overline{B}_{2n} \cap \mathrm{Sp}_{2n}$ is also a Borel subgroup of Sp_{2n}, Thus the same argument as in the case of GL_n implies that $R_u(\mathrm{Sp}_{2n})$ is trivial.

Example 4.16 (orthogonal group)**.** Let us assume that the characteristic of k is not 2. Let $J'_n \in \mathrm{GL}_n(k)$ be the anti-diagonal matrix whose anti-diagonal entries are given by 1. Then, by the same argument as in the previous case, we can show that the special orthogonal group $\mathrm{SO}_{2n} = \mathrm{SO}(J'_n)$ is reductive. (Note that, for any symmetric matrix J, the special orthogonal group $\mathrm{SO}(J)$ is reductive. But an explicit description of its Borel subgroups depends on the choice of J and more complicated.)

[10]For example, this can be seen by choosing an embedding of $\mathbb{G}_\mathrm{a}$ into a general linear group to be $\mathbb{G}_\mathrm{a} \hookrightarrow \mathrm{GL}_2 \colon x \mapsto \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$.

Example 4.17 (unitary group)**.** Here, let us assume that k is not algebraically closed and take a quadratic extension k' of k. Let σ be the nontrivial element of $\mathrm{Gal}(k'/k)$. Let $J \in \mathrm{GL}_n(k')$ be a hermitian matrix, i.e., ${}^t\sigma(J) = J$. We define the *unitary group* $\mathrm{U}(J)$ by

$$\mathrm{U}(J)(R) := \{g \in \mathrm{GL}_n(R \otimes_k k') \mid {}^t\sigma(g)Jg = J\}.$$

(In particular, we have $\mathrm{U}(J)(k) := \{g \in \mathrm{GL}_n(k') \mid {}^t\sigma(g)Jg = J\}$.) Then, by the same argument as in the previous cases, we can show that the special orthogonal group $\mathrm{U}(J)$ is reductive.

Exercise 4.18. We put $B := B_{2n} \cap \mathrm{Sp}_{2n}$. Then prove that B is a Borel subgroup of Sp_{2n}. Hint: let's discuss as follows:

(1) By definition of a Borel subgroup, there exists a Borel subgroup B' of Sp_{2n} containing B. (So our goal is to show that B' is in fact equal to B.) Show that there exists a Borel subgroup B'_{2n} of GL_{2n} containing B' which is given by $B'_{2n} = xB_{2n}x^{-1}$ for some $x \in \mathrm{GL}_{2n}$. (Use: Lie-Kolchin's theorem and the fact that all Borel subgroups are conjugate.)

(2) Check that the following matrix is an element of B.

$$g := \begin{pmatrix} 1 & 1 & 0 & \dots & 0 \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & 0 \\ & 0 & & \ddots & 1 \\ & & & & 1 \end{pmatrix}$$

(Diagonal entries are 1, the entries above the diagonal are 1, and all other entries are 0.)

(3) By (2), in particular, we have $g \in B'_{2n} = xB_{2n}x^{-1}$. From this, deduce that x must belong to B_{2n}, hence B'_{2n} equals B_{2n}.

(4) Show that $B' = B$.

4.3. **Classification of connected reductive groups via root data.** Over an algebraically closed field, isomorphism classes of connected reductive groups can be classified in terms of linear algebraic data called *root data*.

Theorem 4.19 ([Spr09, 9.6.2, 10.1.1])**.** *There exists a bijection between*

- *the set of isomorphism classes of connected reductive groups and*
- *the set of isomorphism classes of reduced root data.*

Let us introduce the definition of a root datum.

Definition 4.20 (root datum)**.** A *root datum* is a quadruple $(X, R, X^\vee, R^\vee)$, where

- X and $X^\vee$ are free abelian groups of finite rank equipped with a perfect pairing $\langle -, - \rangle \colon X \times X^\vee \to \mathbb{Z}$ and
- R and $R^\vee$ are finite subsets of X and $X^\vee$ (called the sets of *roots* and *coroots*) equipped with a bijection $R \leftrightarrow R^\vee \colon \alpha \mapsto \alpha^\vee$

satisfying

(1) for any $\alpha \in R$, we have $\langle \alpha, \alpha^\vee \rangle = 2$,
(2) for any $\alpha \in R$, we have $s_\alpha(R) = R$ and $s_\alpha^\vee(R^\vee) = R^\vee$.

Here, s_α and $s_\alpha^\vee$ denote the automorphisms of X and $X^\vee$ given by

$$s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha \quad \text{and} \quad s_\alpha^\vee(x^\vee) = x^\vee - \langle \alpha, x^\vee \rangle \alpha^\vee.$$

We say that a root datum $(X, R, X^\vee, R^\vee)$ is *reduced* if for any $\alpha \in R$, we have $R \cap \mathbb{Q}\alpha = \{\pm\alpha\}$.

In the following, we explain how to construct the map in Theorem 4.19. Thus our aim is to construct a root datum from a given connected reductive group G. Here, we follow the construction given in [Car85, Section 1.9].

We first take a maximal torus T of G. We put $X := X^*(T)$ and $X^\vee := X_*(T)$. Note that then X and $X^\vee$ have a natural perfect pairing $\langle -, - \rangle \colon X \times X^\vee \to \mathbb{Z}$.

Suppose that U is a minimal nontrivial closed unipotent subgroup of G normalized by T. Then, in fact, U is isomorphic to $\mathbb{G}_{\mathrm{a}}$. By fixing an isomorphism $\iota\colon \mathbb{G}_{\mathrm{a}} \xrightarrow{\cong} U$, we get an element $\alpha \in X$ satisfying

$$t \cdot \iota(x) \cdot t^{-1} = \iota(\alpha(t) \cdot x)$$

for any $x \in \mathbb{G}_{\mathrm{a}}$. This element α is independent of the choice of ι. Furthermore, if U' is another (different to U) minimal nontrivial closed unipotent subgroup of G normalized by T, then the associated element of X is also different. Thus it makes sense to write U_α for U. We call α a *root of T in G* and U_α its *root subgroup*. We put R to be the set of roots of T in G.

It can be proved that $-\alpha$ is also a root when α is a root. Moreover, the subgroup $\langle U_\alpha, U_{-\alpha} \rangle$ generated by U_α and $U_{-\alpha}$ is isomorphic to SL_2 or $\mathrm{PGL}_2 := \mathrm{SL}_2 / \{\pm 1\}$. Furthermore, in any case, there exists a homomorphism $\phi\colon \mathrm{SL}_2 \to \langle U_\alpha, U_{-\alpha} \rangle$ satisfying

$$\phi\left(\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)\right) = U_\alpha \quad \text{and} \quad \phi\left(\left(\begin{smallmatrix} 1 & 0 \\ * & 1 \end{smallmatrix}\right)\right) = U_{-\alpha}.$$

This homomorphism ϕ maps any diagonal element of SL_2 into T. Thus, we can define a cocharacter $\alpha^\vee \in X^\vee$ by

$$\alpha^\vee(y) := \phi\left(\left(\begin{smallmatrix} y & 0 \\ 0 & y^{-1} \end{smallmatrix}\right)\right).$$

We call $\alpha^\vee$ the *coroot associated to* α. We put $R^\vee$ to be the set of all coroots obtained in this way.

Proposition 4.21. *For any connected reductive group G, the quadruple $(X, R, X^\vee, R^\vee)$ forms a reduced root datum.*

Example 4.22. Let $G := \mathrm{GL}_n$. We take T to be the diagonal maximal torus. Then we can choose a basis of $X^*(T)$ to be $\{e_i\}_{i=1}^n$, where $e_i\colon \mathrm{diag}(t_1, \ldots, t_n) \mapsto t_i$. In other words, we have

$$X = X^*(T) \cong \bigoplus_{i=1}^{n} \mathbb{Z}e_i.$$

Similarly, we can choose a basis of $X_*(T)$ to be $\{e_i^\vee\}_{i=1}^n$, where $e_i^\vee\colon t \mapsto \mathrm{diag}(1, \ldots, 1, t, 1, \ldots, 1)$, where t is put on the i-th entry. In other words, we have

$$X^\vee = X_*(T) \cong \bigoplus_{i=1}^{n} \mathbb{Z}e_i^\vee.$$

Any minimal nontrivial closed unipotent subgroup U normalized by T is of the form

$$U_{ij} := \{u_{ij}(x) \mid x \in k\},$$

where $u_{ij}(x)$ denotes the matrix such that the diagonal entries are 1, (i,j)-entry is x, and all other entries are 0. We define an isomorphism between U_{ij} and $\mathbb{G}_{\mathrm{a}}$ by

$$\iota\colon \mathbb{G}_{\mathrm{a}} \to U_{ij}\colon x \mapsto u_{ij}(x).$$

We can easily check that the action of T on U_{ij} is given by

$$t \cdot u_{ij}(x) \cdot t^{-1} = u_{ij}(x \cdot t_i/t_j),$$

where $t = \mathrm{diag}(t_1, \ldots, t_n)$. In other words, the root determined by the subgroup U_{ij} is $e_i - e_j$. We can also check that its corresponding coroot is $e_i^\vee - e_j^\vee$. Therefore we have

$$R = \{e_i - e_j \mid 1 \leq i \neq j \leq n\}, \quad R^\vee = \{e_i^\vee - e_j^\vee \mid 1 \leq i \neq j \leq n\}.$$

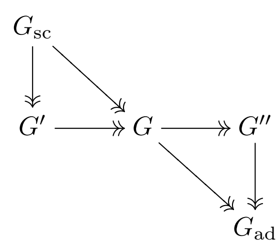
4.4. **Classification of reductive groups: more concrete version.**

Definition 4.23 (isogeny)**.** We say that a homomorphism $f\colon G \to G'$ of algebraic groups is an *isogeny* if it is surjective and has finite kernel. We say that two algebraic groups G and G' are *isogenous* if there exists an isogeny between G and G'.

Recall that, any connected reductive group G can be written as $G = Z(G) \cdot G_{\mathrm{der}}$, where G_{der} is semisimple. Especially, we have a surjective homomorphism $f\colon Z(G) \times G_{\mathrm{der}} \to G\colon (z,g) \mapsto zg$. Since $Z(G) \cap G_{\mathrm{der}}$ is contained in $Z(G_{\mathrm{der}})$, which is finite, f is an isogeny. In other words, any connected reductive group is realized as the quotient of $Z(G) \times G_{\mathrm{der}}$ by its finite subgroup. Thus, let us discuss how to classify semisimple groups in the following. (Being semisimple can be expressed in terms of root data: a connected reductive group G is semisimple if and only if R spans $X_{\mathbb{Q}} := X \otimes_{\mathbb{Z}} \mathbb{Q}$ as a $\mathbb{Q}$-vector space.)

We say that a semisimple group G is *adjoint* if its center $Z(G)$ is trivial. In fact, for any semisimple group G, its quotient $G/Z(G)$ is the unique adjoint group isogenous to G; this is denoted by G_{ad}. The adjoint quotient G_{ad} is a semisimple group whose center is minimal (trivial) among all semisimple groups isogenous to G.

On the other hand, for any semisimple group G, there uniquely exists a semisimple group "G_{sc}" such that any isogeny to G can be lifted to an isogeny from G_{sc} to G; this group is called *the simply-connected cover of* G. The simply-connected cover G_{sc} is a semisimple group whose center is maximal among all semisimple groups isogenous to G.



Proposition 4.24. *Let G be a semisimple group.*

(1) We say that G is simply-connected if $R^\vee$ spans $X^\vee$ over $\mathbb{Z}$.
(2) We say that G is adjoint if R spans X over $\mathbb{Z}$.

Example 4.25. Let $G := \mathrm{GL}_n$ and Z be its center. We put $\mathrm{SL}_n := \{g \in G \mid \det(g) = 1\}$ and $\mathrm{PGL}_n := \mathrm{GL}_n/Z$.[11] Then we obviously have a natural map $\mathrm{SL}_n \to \mathrm{PGL}_n$, which is surjective. Moreover, this map has finite kernel; it is given by $\{z \in Z \mid \det(z) = 1\}$, which

[11] Here, the quotient is taken as an algebraic group. In general, for any linear algebraic group G and its closed subgroup H over k, we can define and prove the existence of the quotient of G by H (see [Spr09, 5.5]). One difficult point to care about is that $(G/H)(R)$ might not be equal to $G(R)/H(R)$. (But at least

is isomorphic to the group of n-th roots of unity. Hence $\mathrm{SL}_n \to \mathrm{PGL}_n$ is an isogeny. On the other hand, the quotient map $\mathrm{GL}_n \twoheadrightarrow \mathrm{PGL}_n$ is not an isogeny since its kernel is given by Z, which is not finite. In fact, SL_n is simply-connected and PGL_n is adjoint.

Definition 4.26 (almost simple group)**.** We say that a semisimple group G is *almost simple* if it does not contain any nontrivial closed normal subgroup of positive dimension.

Proposition 4.27. *Let G be a simply-connected (resp. adjoint) group. Then G is written as a product of almost simple simply-connected (resp. adjoint) subgroups.*

Definition 4.28. We say that a root datum $\Psi = (X, R, X^\vee, R^\vee)$ is *reducible* if there exist nonzero root data $\Psi_1 = (X_1, R_1, X_1^\vee, R_1^\vee)$ and $\Psi_2 = (X_2, R_2, X_2^\vee, R_2^\vee)$ such that $\Psi = \Psi_1 \oplus \Psi_2$ (in the obvious sense) and Ψ_1 and Ψ_2 are orthogonal. We say that Ψ is *irreducible* if it is not reducible.

Proposition 4.29. *Let G be an almost simple simply-connected (or adjoint) group with root data Ψ. Then G is almost simple if and only if Ψ is irreducible.*

By the discussion so far, the classification problem of semisimple groups is now reduced ("modulo isogeny") to classifying all almost simple simply-connected semisimple groups. Moreover, it is equivalent to classifying all irreducible reduced root data such that $R^\vee$ spans $X^\vee$.

The miraculous fact is that there are very limited number of such groups! Such groups can be parametrized by combinatorial objects called *Dynkin diagrams*. Among them, the types A_n, B_n, C_n, and D_n are called *classical types*, and the types E_6, E_7, E_8, F_4, and G_2 are called *exceptional types*.

Example 4.30 (type A_n)**.** Let $G := \mathrm{GL}_{n+1}$. Then we have $G_{\mathrm{der}} = \mathrm{SL}_{n+1}$. It's simply-connected, and its adjoint quotient is PGL_{n+1}. There are of type A_n.

Example 4.31 (type A_n)**.** Here let k be a non-algebraically-closed field. Let k' be a quadratic extension of k and $J \in \mathrm{GL}_{n+1}(k')$ be a hermitian matrix. We put $G := \mathrm{U}(J)$. Then we have $G_{\mathrm{der}} = \mathrm{SU}(J)$ (consisting of determinant 1 matrices). It's simply-connected, and its adjoint quotient is $\mathrm{PU}(J)$.[12] There are of type A_n. Here, note that, the above classification theorem of connected reductive groups is for groups over an algebraically closed field. So the point here is that $\mathrm{U}(J)$ and GL_{n+1} are not isomorphic over k, but isomorphic over $\overline{k}$.

Exercise 4.32. Let k be a non-algebraically-closed field. Let k' be a quadratic extension of k and $J \in \mathrm{GL}_n(k')$ be a hermitian matrix. Prove that $\mathrm{U}(J)$ and GL_n are isomorphic over $\overline{k}$. More concretely, prove that the group

$$\mathrm{U}(J)(\overline{k}) = \{g \in \mathrm{GL}_n(\overline{k} \otimes_k k') \mid {}^t\sigma(g)Jg = J\}$$

is isomorphic to $\mathrm{GL}_n(\overline{k})$. Here, if you want, please choose a hermitian matrix J in any way you prefer.

Example 4.33 (type B_n)**.** Let $G := \mathrm{SO}_{2n+1}$. Then we have $G_{\mathrm{der}} = G$. It's adjoint, and its simply-connected cover is so-called the "spin group" Spin_{2n+1} (two-fold cover of SO_{2n+1}). There are of type B_n.

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we have the equality for $R = \overline{k}$. Thus, in this example, we may think of $\mathrm{PGL}_n(\overline{k})$ as the quotient of $\mathrm{GL}_n(\overline{k})$ by its center.)

[12] I'm not sure if this is a standard notation.

Example 4.34 (type C_n)**.** Let $G := \mathrm{Sp}_{2n}$. Then we have $G_{\mathrm{der}} = G$. It's simply-connected, and its adjoint quotient is PSp_{2n} (Sp_{2n} is its two-fold cover). There are of type C_n.

Example 4.35 (type D_n)**.** Let $G := \mathrm{SO}_{2n}$. Then we have $G_{\mathrm{der}} = G$. Its simply-connected cover is Spin_{2n} (two-fold cover of G), and its adjoint quotient is PSO_{2n} (SO_{2n} is its two-fold cover). There are of type D_n.

4.5. **Rationality.** Let us finally discuss the rationality. From now on, let us again assume that k is a perfect field.

Definition 4.36. Let G and G' be connected reductive groups over k. We say that G is a *k-form* of G' (or G' is a *k-form* of G) is they are isomorphic over $\overline{k}$.

Example 4.37. The previous exercise says that $\mathrm{U}(J)$ is a k-form of GL_n.

Definition 4.38. We say that a connected reductive group G over k is *split* if it has a split maximal torus over k, i.e., a maximal torus which is isomorphic to a product of $\mathbb{G}_{\mathrm{m}}$'s over k.

Proposition 4.39. *For any connected reductive group G over k, there uniquely exists (up to isomorphism) a split connected reductive group G' over k such that G is a k-form of G'.*

Definition 4.40. We call a finite group a *finite group of Lie type* is it is realized as $G(\mathbb{F}_q)$ for some connected reductive group G over $\mathbb{F}_q$. We say that a finite group of Lie type $G(\mathbb{F}_q)$ is

- of Chevalley type if G is split, and
- of Steinberg type if G is not split.

5. Week 5: Deligne–Lusztig varieties

5.1. **Frobenius endomorphism.** In the following, we let $k = \mathbb{F}_q$. Note that then the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ is a pro-cyclic group isomorphic to $\hat{\mathbb{Z}}$. This group has the Frobenius automorphism $F\colon \overline{\mathbb{F}}_q \to \overline{\mathbb{F}}_q; x \mapsto x^q$ as its (topological) generator.

Now let us suppose that X is an affine algebraic variety over k. Recall that, in our sense, this means that X is a scheme equipped with a morphism to $\mathrm{Spec}\, k$ such that its base change $X_{\overline{k}}$ to $\mathrm{Spec}\, \overline{k}$ corresponds to an algebraic variety in the classical sense. Let $k[X]$ be the coordinate ring of X, i.e., $X = \mathrm{Spec}\, k[X]$ (hence $X_{\overline{k}} = \mathrm{Spec}\, \overline{k}[X]$, where $\overline{k}[X] = k[X] \otimes_k \overline{k}$). We define a ring endomorphism F^* of $\overline{k}[X]$ by

$$F^*\colon k[X] \otimes_k \overline{k} \to k[X] \otimes_k \overline{k}; \quad f \otimes a \mapsto f^q \otimes a.$$

(Note that this is a well-defined ring homomorphism since k is of characteristic p!) By abuse of notation, we write $F\colon X_{\overline{k}} \to X_{\overline{k}}$ for the endomorphism of $X_{\overline{k}}$ induced by F^*. Naively, F can be thought of as the entry-wise q-th power map.

In the following (and actually, so far in this course), we often simply write "$g \in X$" to mean that $g \in X(\overline{k}) = X_{\overline{k}}(\overline{k})$. Then it makes sense to talk about the image $F(g)$ of g under the Frobenius morphism. Following the definition, we can easily check that the set of fixed points $X^F = X_{\overline{k}}(\overline{k})^F$ is nothing but $X(k)$.

We finally note that a closed subvariety $Y_{\overline{k}}$ of $X_{\overline{k}}$ is k-rational if and only if $Y_{\overline{k}}$ is stable under F; this fact is a special case of so-called the *Galois descent* (see [Spr09, 11.2]).

5.2. **Definition of a Deligne–Lusztig variety.** Let G be a connected reductive group over $k = \mathbb{F}_q$. Let $F\colon G_{\overline{k}} \to G_{\overline{k}}$ be the Frobenius endomorphism of G. (Note that F is compatible with the Hopf algebra structure of the coordinate ring of $G_{\overline{k}}$, hence F is a group endomorphism of $G_{\overline{k}}$.)

Definition 5.1 (Deligne–Lusztig variety)**.** Let T be a k-rational maximal torus of G. We take a Borel subgroup B of G containing T. Let U be the unipotent radical of B. We define an algebraic variety $\mathcal{X}^G_{T\subset B}$ (over $\overline{k}$) by

$$\mathcal{X}^G_{T\subset B} := \{g \in G \mid g^{-1}F(g) \in F(U)\}.$$

We call $\mathcal{X}^G_{T\subset B}$ the *Deligne–Lusztig variety associated to T (and B).*

Remark 5.2. Recall that a Borel subgroup of G is a maximal connected solvable closed subgroup of G. Since any subtorus of G is connected solvable and closed, we can always find a Borel subgroup B of G containing a given maximal torus T of G. But be careful that B might not be taken to be k-rational even when T is k-rational (hence U also may not be k-rational).

Let us fix a T in the following and shortly write $\mathcal{X}$ for $\mathcal{X}^G_{T\subset B}$.

First suppose that $g \in G^F$ and $x \in \mathcal{X}$. Then we have

$$(gx)^{-1}F(gx) = x^{-1}g^{-1}F(g)F(x) = x^{-1}g^{-1}gF(x) = x^{-1}F(x) \in F(U).$$

In other words, the element $gx \in G$ again belongs to $\mathcal{X}$. Thus we get an action of G^F on $\mathcal{X}$ by left multiplication.

Next suppose that $t \in T^F$ and $x \in \mathcal{X}$. Then we have

$$(xt)^{-1}F(xt) = t^{-1}x^{-1}F(x)F(t) = t^{-1}x^{-1}F(x)t \in t^{-1}F(U)t = F(U),$$

where we used that T normalizes $F(U)$ in the last equality. In other words, the element $xt \in G$ again belongs to $\mathcal{X}$. Thus we get an action of T^F on $\mathcal{X}$ by right multiplication.

Note that the actions of G^F and T^F on $\mathcal{X}$ obviously commute. Hence we get an action of the direct product group $G^F \times T^F$ on $\mathcal{X}$.

This observation is very important; by the functoriality, the étale cohomology of $\mathcal{X}$ also has an action of $G^F \times T^F$. In other words, we can construct a representation of $G^F \times T^F$. The aim of this course (Deligne–Lusztig theory) is to investigate the representations of G^F realized in this way through the geometry of $\mathcal{X}$.

5.3. **Classification of maximal tori.** Deligne–Lusztig varieties are determined by the choice of a k-rational maximal torus of G. Then, how many k-rational maximal tori does G have (up to k-conjugacy)? Let us investigate it (following [Car85, 3.3]).

We first note the following fact:

Proposition 5.3. *Any connected reductive group G over k possesses a k-rational Borel subgroup.* [13]

Let us fix a k-rational Borel subgroup B_0 of G. Let T_0 be a k-rational maximal torus of G contained in B_0. We call this maximal torus T_0 the "base torus" (this is our temporary terminology). We write $N_G(T_0)/T_0$ for the normalizer group of T_0 in G and $W_G(T_0) := N_G(T_0)/T_0$ for the *Weyl group* of T_0 in G. We often write W_0 for $W_G(T_0) := N_G(T_0)/T_0$ in short. Note that, since T_0 is k-rational, so is $N_G(T_0)$. Hence we have a natural action of F on W_0. We say that two elements $w_1, w_2 \in W_0$ are *F-conjugate* if there exists an element $v \in W_0$ satisfying $w_2 = vw_1F(v)^{-1}$. Note that this is an equivalence relation on W_0.

Now let T be a k-rational maximal torus of G. Recall that all maximal tori of G are conjugate (over $\overline{k}$). Thus let us choose an element $g \in G$ satisfying $T = {}^{g}T_0$, where ${}^{g}(-) := g(-)g^{-1}$. Since both T and T_0 are k-rational subgroups of G, T and T_0 are stable under F. Hence we get

$$ {}^{F(g)}T_0 = F({}^{g}T_0) = F(T) = T = {}^{g}T_0. $$

In particular, we have ${}^{g^{-1}F(g)}T_0 = T_0$. In other words, the element $g^{-1}F(g)$ belongs to the normalizer $N_G(T_0)$ of T_0 in G. We let w be the image of $g^{-1}F(g) \in N_G(T_0)$ in the Weyl group $W_G(T_0)$.

Lemma 5.4. *The F-conjugacy class of $w \in W_0$ is well-defined, i.e., independent of the choice of $g \in G$ satisfying ${}^{g}T_0 = T$. Moreover, two G^F-conjugate k-rational maximal tori of G give rise to the same F-conjugacy class of W_0.*

Proof. Suppose that $g_1, g_2 \in G$ are elements satisfying ${}^{g_1}T_0 = T$ and ${}^{g_2}T_0 = T$. Let w_1 and w_2 be the images of $g_1^{-1}F(g_1)$ and $g_2^{-1}F(g_2)$ in W_0, respectively.

As we have ${}^{g_1}T_0 = T = {}^{g_2}T_0$, we have $g_1^{-1}g_2 \in N_G(T_0)$. Hence, if we put v to be the image of $g_1^{-1}g_2$ in W_0, we get $w_2 = v^{-1}w_1F(v)$.

It is easy to check the latter assertion. □

By this lemma, we see that the above procedure $T \mapsto w$ induces a well-defined map

$$ \{k\text{-rational maximal tori of } G\}/G^F\text{-conj.} \to W_0/F\text{-conj.} $$

Proposition 5.5. *This map is bijective.*

To show this proposition, we introduce following famous fact, which is known as *Lang's theorem*.

[13] In general, a connected reductive group G over k (any field) is said to be "quasi-split" if it has a k-rational Borel subgroup. The proposition says that any connected reductive group over $\mathbb{F}_q$ is quasi-split.

Theorem 5.6 ([Spr09, 4.4.17])**.** *Let G be a connected algebraic group over $k = \mathbb{F}_q$. Then the map $G_{\overline{k}} \to G_{\overline{k}}\colon g \mapsto g^{-1}F(g)$ is surjective.*

Proof of Proposition 5.5. Let us first show the surjectivity. Let $w \in W_0$ and $n \in N_G(T_0)$ be any its representative. By Lang's theorem for G, we can find an element $g \in G$ satisfying $g^{-1}F(g) = n$. If we put $T := {}^{g}T_0$, then the condition $g^{-1}F(g) = n \in N_G(T_0)$ implies that T is F-stable. Hence T is k-rational.

Let us next show the injectivity. Suppose that T_1 and T_2 are k-rational maximal tori of G which give rise to the same F-conjugacy class of W_0. If we write $T_1 = {}^{g_1}T_0$ and $T_2 = {}^{g_2}T_0$, then we have $g_1^{-1}F(g_1) = n^{-1}g_2^{-1}F(g_2)F(n)t_0$ for some elements $n \in N_G(T_0)$ and $t_0 \in T_0$. By noting that $F(g_2)F(n)t_0 = tF(g_2)F(n)$ for an element t of T_2 and applying Lang's theorem for T_2 to t, we can find an element $s \in T_2$ satisfying $s^{-1}F(s) = t$. Hence we get

$$g_1^{-1}F(g_1) = n^{-1}g_2^{-1}s^{-1}F(s)F(g_2)F(n),$$

which implies that $F(sg_2ng_1^{-1}) = sg_2ng_1^{-1}$, i.e., $sg_2ng_1^{-1} \in G^F$. If we put g to be this element, then we have

$${}^{g}T_1 = {}^{gg_1}T_0 = {}^{sg_2n}T_0 = {}^{s}T_2 = T_2.$$

Hence T_1 and T_2 are G^F-conjugate. □

In the following, for any element $w \in W_0$, let T_w denote a k-rational maximal torus of G corresponding to the F-conjugacy class of w. Let us describe the rational structure of T_w in terms of the base torus T_0. Let $g \in G$ be an element satisfying $T_w = {}^{g}T_0$. By replacing g with an element of $gN_G(T_0)$ if necessary, we may assume that the image of $g^{-1}F(g) \in N_G(T_0)$ in W_0 is exactly w. Then, the action of F on T_w is transferred to the composition of $\mathrm{Int}(w)$ and F on T_0 through the isomorphism $\mathrm{Int}(g)^{-1}\colon T_w \to T_0$:

$$\begin{array}{ccc} T_w & \xleftarrow{\mathrm{Int}(g)} & T_0 \\ {\scriptstyle F}\downarrow & & \\ T_w & \xrightarrow[\mathrm{Int}(g)^{-1}]{} & T_0 \end{array} \qquad \begin{array}{ccc} gtg^{-1} & \longleftarrow\!\!\!\shortmid & t \\ \downarrow & & \\ F(g)F(t)F(g)^{-1} & \longmapsto & g^{-1}F(g)F(t)F(g)^{-1}g = \mathrm{Int}(w) \circ F(t) \end{array}$$

Example 5.7. Let $G = \mathrm{GL}_n$. In this case, the base torus T_0 can be taken to be the diagonal maximal torus. Thus we have $T_0 \cong (\overline{\mathbb{F}}_q^{\times})^n$ (if we loosely identify T_0 with $T_0(\overline{\mathbb{F}}_q)$) and the action F on T_0 is given by

$$(t_1, t_2, \ldots, t_n) \mapsto (t_1^q, t_2^q, \ldots, t_n^q).$$

The Weyl group W_0 can be naturally identified with the subgroup of permutation matrices of GL_n, hence isomorphic to $\mathfrak{S}_n$.

(1) We first consider the case where $w \in \mathfrak{S}_n$ is trivial. In this case, T_w is nothing but T_0 itself. Hence $T_w^F = T_0^F \cong (\mathbb{F}_q^{\times})^n$.

(2) We next consider the case where $w \in \mathfrak{S}_n$ is the cyclic permutation $(1\,2\,\ldots\,n)$ of length n (this element is so-called a "Coxeter element"). The action $\mathrm{Int}(w) \circ F$ on T_0 is explicitly written by

$$(t_1, t_2, \ldots, t_n) \mapsto (t_n^q, t_1^q \ldots, t_{n-1}^q).$$

Thus $(t_1, t_2, \ldots, t_n) \in T_0$ is fixed by $\mathrm{Int}(w) \circ F$ if and only if $(t_1, t_2, \ldots, t_n) = (t_n^q, t_1^q \ldots, t_{n-1}^q)$, which is equivalent to

$$(t_1, t_2, \ldots, t_n) = (t_1, t_1^q, \ldots, t_1^{q^{n-1}}) \quad \text{and} \quad t_1^{q^n} = t_1.$$

In other words, T_w^F is identified with $\mathbb{F}_{q^n}^\times$, hence is of order $q^n - 1$.

(3) We finally consider the general case. The Frobenius F acts on W_0 trivially, thus the F-conjugacy of W_0 is nothing but the usual conjugacy. Recall that the conjugacy classes of $\mathfrak{S}_n$ correspond to the partitions of n bijectively. Suppose that the conjugacy class of $w \in \mathfrak{S}_n$ corresponds to a partition $(n_1, n_2, \ldots, n_r)$ of n, where $n_1 \geq \cdots \geq n_r > 0$ and $n_1 + \cdots + n_r = n$. [14] Then, by a similar argument to (2), we can check that T_w^F is identified with $\mathbb{F}_{q^{n_1}}^\times \times \cdots \times \mathbb{F}_{q^{n_r}}^\times$. Hence the order of T_w^F is given by $(q^{n_1} - 1) \cdots (q^{n_r} - 1)$.

As demonstrated in the above example, it is not very difficult to describe k-rational maximal tori of G as long as the descriptions of the base torus T_0 and its Weyl group explicitly.

Let us finally mention a general proposition on the order of T_w. We first note that the actions of F and W_0 on $X^*(T_0)$ are induced as follows:

$$F(\chi)(t) := \chi(F(t)) \qquad \text{for any } \chi \in X^*(T_0),\ t \in T_0,$$
$$w(\chi)(t) := \chi(w^{-1}tw) \qquad \text{for any } \chi \in X^*(T_0),\ t \in T_0.$$

Similarly, the actions of F and W_0 on $X_*(T_0)$ are induced as follows:

$$F(\chi^\vee)(t) := F(\chi^\vee(t)) \qquad \text{for any } \chi^\vee \in X_*(T_0),\ t \in \mathbb{G}_{\mathrm{m}},$$
$$w(\chi^\vee)(t) := w\chi^\vee(t)w^{-1} \qquad \text{for any } \chi^\vee \in X_*(T_0),\ t \in \mathbb{G}_{\mathrm{m}}.$$

Then it is a routine task to check that the maps on $X^*(T)$ and $X_*(T)$ induced by F in a similar way are identified with $F \circ w^{-1}$ and $w^{-1} \circ F$ on $X^*(T_0)$ and $X_*(T_0)$, respectively (see [Car85, Proposition 3.3.4]). This leads to the following (see [Car85, Proposition 3.3.5]):

Proposition 5.8. *The order of T_w^F is given by $|\det(w^{-1} \circ F - \mathrm{id} \mid X_*(T_0)_{\mathbb{R}})|$. More explicitly, if we write $F = qF_0$ (then F_0 is an automorphism of $X_*(T_0)_{\mathbb{R}}$ of finite order) and let $\chi(-)$ be the characteristic polynomial of $F_0^{-1} \circ w$ on $X_*(T_0)_{\mathbb{R}}$, then we have $T_w^F = \chi(q)$.*

Remark 5.9. Note that F_0 is the identity when G is split.

Exercise 5.10. Compute the order of T^F for all k-rational maximal tori T of Sp_{2n}.

5.4. **Some variants.** Now we introduce of several variants of the Deligne–Lustig variety. Later (after next weeks), it will turn out that all of these variants are technically convenient. (The description given here follows [DL76, 1.18–1.20] and [Car85, 7.7].)

Let T be a k-rational maximal torus of G. As before, we take a Borel subgroup B of G containing T. Let U be the unipotent radical of B. Recall that

$$\mathcal{X}_{T\subset B}^G := \{g \in G \mid g^{-1}F(g) \in F(U)\}.$$

Note that $\mathcal{X}_{T\subset B}^G$ is also stable under the right multiplication by $U \cap F(U)$. We define algebraic varieties $\tilde{X}_{T\subset B}^G$ and $X_{T\subset B}^G$ (over $\overline{k}$) by

$$\tilde{X}_{T\subset B}^G := \{g \in G \mid g^{-1}F(g) \in F(U)\}/(U \cap F(U))$$
$$X_{T\subset B}^G := \{g \in G \mid g^{-1}F(g) \in F(U)\}/T^F(U \cap F(U)).$$

Then $\mathcal{X}_{T\subset B}^G$ is a G^F-equivariant $U \cap F(U)$-torsor over $\tilde{X}_{T\subset B}^G$ and $\tilde{X}_{T\subset B}^G$ is a G^F-equivariant T^F-torsor over $X_{T\subset B}^G$.

$$\mathcal{X}_{T\subset B}^G \xrightarrow{(U\cap F(U))\text{-torsor}} \tilde{X}_{T\subset B}^G \xrightarrow{T^F\text{-torsor}} X_{T\subset B}^G.$$

[14] For example, the trivial permutation corresponds to $(1, \ldots, 1)$ and the cyclic permutation $(1\,2\,\ldots\,n)$ of length n corresponds to (n).

Now assume that T corresponds to $w \in W$. What we want to do in the following is to understand the above varieties in a more concrete language based on flag varieties. For this, again let us fix a k-rational Borel subgroup B_0 of G and a base torus $T_0 \subset B_0$. We define the variety $\mathcal{B}$ to be the quotient G/B_0 of G by B_0. (By a fundamental fact in the theory of algebraic groups, this is a projective variety.) Note that the $\overline{k}$-rational points of $\mathcal{B}_0$ can be identified with the set of all Borel subgroups of G via map $g \mapsto {}^{g}B_0$. This can be checked by using the following facts:

(1) all Borel subgroups of G are conjugate, and
(2) for any Borel subgroup B of G, we have $N_G(B) = B$.

We call $\mathcal{B} = G/B_0$ the *flag variety* of G.

Proposition 5.11. *We have bijections*

$$W_0 = N_G(T_0)/T_0 \overset{1:1}{\longleftrightarrow} B_0\backslash G/B_0 \overset{1:1}{\longleftrightarrow} G\backslash(\mathcal{B}\times\mathcal{B}).$$

Here, the first map is $n \mapsto BnB$ and the second map is $g \mapsto G(B_0, {}^{g}B_0)$. (The action of G on $\mathcal{B}\times\mathcal{B}$ is given by a diagonal conjugation, i.e., $g(B_1, B_2) = ({}^{g}B_1, {}^{g}B_2)$).

Proof. The bijectivity of the first map is known as the "Bruhat decomposition". See, for example, [Spr09, 8.3]. The bijectivity of the second map can be checked again by the above-mentioned fundamental properties (1) and (2) of Borel subgroups. □

Let $O(w)$ denote the cell of $\mathcal{B}\times\mathcal{B}$ corresponding to $w \in W_0$ under the above identification; explicitly, this is given by $O(w) = G(B_0, {}^{w}B_0)$. When a pair of two Borel subgroup (B_1, B_2) belongs to $O(w)$, we say that B_1 and B_2 are *in relative position* w.

We define a set $X(w)$ to be the subset of $\mathcal{B}$ consisting of all Borel subgroups B of G such that B and $F(B)$ are in relative position w:

$$\begin{aligned} X(w) &:= \{gB_0 \in G/B_0 \mid ({}^{g}B_0, F({}^{g}B_0)) \in O(w)\} \\ &= \{gB_0 \in G/B_0 \mid g^{-1}F(g) \in B_0wB_0\}. \end{aligned}$$

Since $X(w)$ is locally closed in $\mathcal{B}$, $X(w)$ has a variety structure. We put $\tilde{\mathcal{B}} := G/U_0$; hence $\tilde{\mathcal{B}}$ is a T_0-torsor over $\mathcal{B}$. By choosing a representative $\dot{w} \in N_G(T_0)$ of $w \in W_0$, we define a similar subset $\tilde{X}(\dot{w})$ of $\tilde{\mathcal{B}}$ as follows:

$$\begin{aligned} \tilde{X}(\dot{w}) &:= \{gU_0 \in G/U_0 \mid F(gU_0) = gU_0\dot{w}\} \\ &= \{gU_0 \in G/U_0 \mid g^{-1}F(g) \in U_0\dot{w}U_0\}. \end{aligned}$$

Then the covering $\tilde{\mathcal{B}} \twoheadrightarrow \mathcal{B}$ restricts to a covering $\tilde{X}(\dot{w}) \twoheadrightarrow X(w)$, which is G^F-equivariant. Let us compute the fiber of this map. Suppose that $gU_0 \in \tilde{X}(\dot{w})$, hence $gB_0 \in X(w)$. The fiber of $\tilde{\mathcal{B}} \twoheadrightarrow \mathcal{B}$ at gB_0 is simply given by $\{gtU_0 \mid t \in T_0\}$. It is not difficult to check that gtU_0 belongs to $\tilde{X}(\dot{w})$ if and only if $wF(t)w^{-1} \in U_0t$. By noting that both $wF(t)w^{-1}$ and t belong to T_0, this is furthermore equivalent to that $wF(t)w^{-1} = t$, i.e., $t \in T_0^{\mathrm{Int}(w)\circ F}$. (Indeed, $wF(t)w^{-1}t^{-1}$ must be an element of $T_0 \cap U_0 = \{1\}$.) Therefore, we conclude that

$$\tilde{X}(\dot{w}) \twoheadrightarrow X(w)$$

is a G^F-equivariant $T_0^{\mathrm{Int}(w)\circ F}$-torsor. We note that $T_0^{\mathrm{Int}(w)\circ F}$ is identified with T_w^F by the map $T_0^{\mathrm{Int}(w)\circ F} \to T_w^F : t \mapsto gtg^{-1}$.

All the relations between the varieties we introduced so far are summarized as follows:

Proposition 5.12. *Suppose that $T = T_w$ for a $w \in W$. Let $x \in G$ be an element such that $\dot{w} := x^{-1}F(x)$ belongs to $N_G(T_0)$ and lifts w (hence $T = {}^xT_0$). We take B to be ${}^xB_0$, hence $U = {}^xU_0$. Then the map $g \mapsto gx$ induces a bijection from the G^F-equivariant T^F-torsor $\tilde{X}^G_{T\subset B} \to X^G_{T\subset B}$ to the G^F-equivariant $T_0^{\mathrm{Int}(w)\circ F}$-torsor $\tilde{X}(\dot{w}) \to X(w)$ (T^F and $T_0^{\mathrm{Int}(w)\circ F}$ are identified under the map $t \mapsto g^{-1}tg$).*

$$\begin{array}{ccccc}
\mathcal{X}^G_{T\subset B} & & & & \\
{\scriptstyle /(U\cap F(U))}\big\downarrow & & & & \\
\tilde{X}^G_{T\subset B} & \xrightarrow[g\mapsto gx]{\sim} & \tilde{X}(\dot{w}) & \hookrightarrow & \tilde{\mathcal{B}} \\
{\scriptstyle /T^F}\big\downarrow & & \big\downarrow{\scriptstyle /T_0^{\mathrm{Int}(w)\circ F}} & & \big\downarrow \\
X^G_{T\subset B} & \xrightarrow[g\mapsto gx]{\sim} & X(w) & \hookrightarrow & \mathcal{B}
\end{array}$$

5.5. **Example: GL_n case.** Let us investigate the variety $\tilde{X}(\dot{w})$ in the case where $G = \mathrm{GL}_n$ and $w = (1\,2\,\ldots\,n) \in \mathfrak{S}_n$. Let T_0 be the diagonal maximal torus of G and B_0 the upper-triangular Borel subgroup of G.

Definition 5.13. Let V be a finite-dimensional k-vector space. A *flag* of V is a sequence of subspaces $\mathcal{F} = (0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_r = V)$. We say that a flag $\mathcal{F}$ is *complete* if $\dim V_i = i$.

Let $V := \mathbb{F}_q^{\oplus n}$ and $\{e_i\}_{i=1}^n$ be the standard basis of V (i.e., $e_1 = (1, 0, \ldots, 0)$ and so on). Let $\mathcal{F}_{\mathrm{std}}$ be the complete flag of V given by $V_i = \bigoplus_{j=1}^i \mathbb{F}_q e_j$. We call $\mathcal{F}_{\mathrm{std}}$ the *standard flag* of V. Note that the set of points of $\mathcal{B} = G/B_0$ parametrizes the complete flags of V. Indeed, G acts on the set of complete flags via natural multiplication, i.e., $g \cdot (V_0 \subsetneq \cdots \subsetneq V_n) := (g(V_0) \subsetneq \cdots \subsetneq g(V_n))$. It is easy to see that this action is transitive and that the stabilizer of $\mathcal{F}_{\mathrm{std}}$ is nothing but B_0.

Definition 5.14. Let V be a finite-dimensional k-vector space. A *marked flag* of V is a flag $(0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_r = V)$ equipped with nonzero element $v_i \in V_i/V_{i-1}$ for each $1 \leq i \leq r$.

Note that the standard flag $\mathcal{F}_{\mathrm{std}}$ can be upgraded to a marked complete flag with mark $\{e_i \in V_i/V_{i-1}\}_{i=1}^n$. Then, similarly to above, we see that the set of points of $\tilde{\mathcal{B}} = G/U_0$ parametrizes the marked complete flags of V.

Recall that $O(w)$ parametrizes pairs of Borel subgroups of G whose relative position is w. Let (B, B') be a pair of Borel subgroups of G. Let $\mathcal{F}^{(\prime)} = (0 = V_0^{(\prime)} \subsetneq V_1^{(\prime)} \subsetneq \cdots \subsetneq V_n^{(\prime)} = V^{(\prime)})$ be the complete flag of V corresponding to $B^{(\prime)}$.

Exercise 5.15. Check that $(\mathcal{F}, \mathcal{F}')$ is in relative position w if and only if $(\mathcal{F}, \mathcal{F}')$ satisfies the following conditions:

$$\begin{cases} V_i + V_i' = V_{i+1} & (1 \leq i \leq n-1), \\ V_1 + V_{n-1}' = V. \end{cases}$$

Next recall that $X(w)$ parametrizes Borel subgroups B of G such that $(B, F(B))$ belongs to $O(w)$. By the above exercise, this is equivalent to that a complete flag $\mathcal{F} = (0 = V_0 \subsetneq$

$V_1 \subsetneq \cdots \subsetneq V_n = V$) corresponding to B satisfies the following:

$$\begin{cases} V_i + F(V_i) = V_{i+1} & (1 \le i \le n-1), \\ V_1 + F(V_{n-1}) = V. \end{cases}$$

We now consider $\tilde{X}(\dot{w})$. Similarly to above, we can check that $\tilde{X}(\dot{w})$ parametrizes marked complete flags $(\mathcal{F}, \{v_1\}_{i=1}^n)$ satisfying

$$\begin{cases} v_{i+1} \equiv F(v_i) \pmod{V_i} & (1 \le i \le n-1), \\ v_1 \equiv F^n(v_1) \pmod{F(v_1), \ldots, F(v_{n-1})}. \end{cases}$$

Exercise 5.16. Check that this condition is equivalent to that

$$v_1 \wedge F(v_1) \wedge \cdots \wedge F^{n-1}(v_1) = F^n(v_1) \wedge F(v_1) \wedge \cdots \wedge F^{n-1}(v_1)$$

(and both sides are nonzero), which can be also written as

$$F(v_1 \wedge F(v_1) \wedge \cdots \wedge F^{n-1}(v_1)) = (-1)^{n-1} \cdot v_1 \wedge F(v_1) \wedge \cdots \wedge F^{n-1}(v_1).$$

Let us explicate this equality by writing $v_1 \in V$ via the standard basis as $v_1 = \sum_{i=1}^n x_i e_i$. Since F acts on V via q-th power on the coefficients, we have that $F^i(v_1) = \sum_{i=1}^n x_i^{q^i} e_i$. Therefore, the above equality is equivalent to that

$$\left(\det(x_i^{q^{j-1}})_{1 \le i,j \le n}\right)^q = (-1)^{n-1} \cdot \det(x_i^{q^{j-1}})_{1 \le i,j \le n}.$$

Since both sides are necessarily nonzero, this is equivalent to

$$(-1)^{n-1} \cdot \left(\det(x_i^{q^{j-1}})_{1 \le i,j \le n}\right)^{q-1} = 1.$$

This is quite close to (and more complicated than) the Drinfeld curve! In fact, $\tilde{X}(\dot{w})$ exactly generalizes the Drinfeld curve.

Exercise 5.17. Verify that $\tilde{X}(\dot{w})$ exactly coincides with the Drinfeld curve $\{(x, y \in \mathbb{A}^2_{\overline{F}_p}) \mid xy^q - x^q y = 1\}$ when $G = \mathrm{SL}_2$ and w is the Coxeter element, i.e., the unique nontrivial element of the Weyl group. (CAUTION: In the case of special linear groups, we cannot simply take the representatives of the Weyl group elements to be permutation matrices because of the determinant condition. In particular, $\dot{w}$ cannot taken to be $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Instead, for example, we can use $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. But then we get a nontrivial sign contribution to the defining equation of $\tilde{X}(\dot{w})$).

6. Week 6: Deligne–Lusztig representations

6.1. **Quick overview of étale cohomology.** In the following, we quickly introduce the basic properties of the étale cohomology for algebraic varieties. (Here, we do not even give the definition of the étale cohomology. Carter's book [Car85, Appendix] has a beautiful summary of the étale cohomology theory, so please look at it if you want to know more about some details.)

Let us briefly recall the notion of *ℓ-adic numbers*. Let ℓ be a prime number. We consider the inverse system of finite rings

$$\cdots \to \mathbb{Z}/\ell^{n+1}\mathbb{Z} \to \mathbb{Z}/\ell^{n}\mathbb{Z} \to \cdots \to \mathbb{Z}/\ell^{2}\mathbb{Z} \to \mathbb{Z}/\ell\mathbb{Z},$$

where the transition map $\mathbb{Z}/\ell^{n+1}\mathbb{Z} \to \mathbb{Z}/\ell^{n}\mathbb{Z}$ is given by the natural surjection. The inverse limit of this system forms a ring, which is called the *ring of ℓ-adic integers* and denoted by $\mathbb{Z}_\ell$:

$$\mathbb{Z}_\ell := \varprojlim_n \mathbb{Z}/\ell^n\mathbb{Z} := \{(x_n)_n \in \prod_{n\geq 1} \mathbb{Z}/\ell^n\mathbb{Z} \mid \overline{x_{n+1}} = x_n\}.$$

Since $\mathbb{Z}_\ell$ is an integral domain, it makes sense to consider its fractional field; it is called the *field of ℓ-adic numbers* and denoted by $\mathbb{Q}_\ell$. [15]

Lemma 6.1. *Let $\overline{\mathbb{Q}}_\ell$ be an algebraic closure of $\mathbb{Q}_\ell$. Then $\overline{\mathbb{Q}}_\ell$ is isomorphic to the complex number field $\mathbb{C}$ as an abstract field.* [16]

Exercise 6.2. Prove this lemma. Hint: note that both $\overline{\mathbb{Q}}_\ell$ and $\mathbb{C}$ are algebraically closed fields of characteristic 0 and the same cardinality.

Now let k be a finite field $\mathbb{F}_q$ of characteristic $p > 0$. In the following, let ℓ be a prime number distinct to p. For any algebraic variety X over $\overline{k} = \overline{\mathbb{F}}_p$ and for each $i \in \mathbb{Z}_{\geq 0}$, we can associate a $\overline{\mathbb{Q}}_\ell$-vector space $H^i_c(X, \overline{\mathbb{Q}}_\ell)$ called the *compactly supported (i-th) étale cohomology of X with $\overline{\mathbb{Q}}_\ell$-coefficient*. In this course, we simply refer to it by the *ℓ-adic cohomology of X*. [17]

It is known that $H^i_c(X, \overline{\mathbb{Q}}_\ell)$ satisfies various "basic" properties. For a moment, let us introduce only the following:

Theorem 6.3. *(1) For any X, $H^i_c(X, \overline{\mathbb{Q}}_\ell)$ is finite-dimensional.*
(2) For any X, $H^i_c(X, \overline{\mathbb{Q}}_\ell) \neq 0$ only for $0 \leq i \leq 2\dim(X)$.
(3) For any proper[18] morphism of algebraic varieties $f\colon X \to Y$ over $\overline{k}$, a $\overline{\mathbb{Q}}_\ell$-vector space homomorphism $f^*\colon H^i_c(Y, \overline{\mathbb{Q}}_\ell) \to H^i_c(X, \overline{\mathbb{Q}}_\ell)$ is canonically (functorially) associated (for each i).

For references on these facts, see [Car85, Section 7.1].

Now suppose that X is an algebraic variety over k. Then we have the Frobenius endomorphism $F\colon X_{\overline{k}} \to X_{\overline{k}}$. Thus, by the functoriality, we also have an endomorphism F^* of $H^i_c(X, \overline{\mathbb{Q}}_\ell)$.

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[15] Another equivalent way of defining $\mathbb{Q}_\ell$ is to complete the rational number field $\mathbb{Q}$ with respect to the ℓ-adic distance. But the above definition seems better in this context because the ℓ-adic cohomology is defined by taking the limit of torsion coefficient ($\mathbb{Z}/\ell^n\mathbb{Z}$-coefficient) cohomologies.

[16] Note that, however, $\overline{\mathbb{Q}}_\ell$ and $\mathbb{C}$ cannot be topologically isomorphic.

[17] There is also the "(i-th) étale cohomology of X with $\overline{\mathbb{Q}}_\ell$-coefficient", so this terminology is a bit too abbreviated. But we do not mind because we only use the compactly supported one in this course.

[18] Here, we don't give the definition of the properness of a morphism of algebraic varieties. We only note that any isomorphism is proper and also a Frobenius endomorphism is also proper.

Theorem 6.4 (Grothendieck–Lefschetz fixed point theorem)**.** *We have*

$$|X^F| = \sum_{i\geq 0}(-1)^i \operatorname{Tr}(F^* \mid H^i_c(X, \overline{\mathbb{Q}}_\ell)).$$

One of the important application of the fixed point theorem is the following ℓ-independence result: Suppose that X is furthermore equipped with an action of a finite group G. Then, by the functoriality of ℓ-adic cohomology, we obtain a representation of G on a finite-dimensional $\overline{\mathbb{Q}}_\ell$-vector space $g \mapsto (g^{-1})^*$. (Here it is better to take the inverse of g since $(-)^*$ is contravariant.) By abuse of notation, let us simply write "g" for the action $(g^{-1})^*$ on $H^i_c(X, \overline{\mathbb{Q}}_\ell)$.

Theorem 6.5. *Suppose that an element $g \in G$ satisfies $g \circ F = F \circ g$ as an endomorphism of $X_{\overline{k}}$. Then the number*

$$\sum_{i\geq 0}(-1)^i \operatorname{Tr}(g \mid H^i_c(X, \overline{\mathbb{Q}}_\ell))$$

is an integer independent of ℓ (called the "Lefschetz number" of g).

Proof. Here we need the fact that, for any $n \geq 1$, the endomorphism $g \circ F^n$ of $X_{\overline{k}}$ associated to another $\mathbb{F}_{q^n}$-rational structure of $X_{\overline{k}}$. Let us write X_n for the algebraic variety over $\mathbb{F}_{q^n}$ determined by this rational structure. Then $X^{g\circ F^n}$ is the set of $\mathbb{F}_{q^n}$-rational points of X_n, hence finite.

We first investigate the following formal series:

$$R(t) := -\sum_{n=1}^{\infty} |X^{g\circ F^n}| \cdot t^n \in \mathbb{Z}[\![t]\!] \subset \overline{\mathbb{Q}}_\ell(\!(t)\!).$$

Since g and F^* are commuting endomorphism of $V := \bigoplus_{i\geq 0} H^i_c(X, \overline{\mathbb{Q}}_\ell)$ (note that this is finite-dimensional), we can simultaneously triangulate g and F^*. Let $v_1, \ldots, v_k$ be a set of simultaneous eigenvectors ($d := \dim V$) with eigenvalues $\alpha_1, \ldots, \alpha_d \in \overline{\mathbb{Q}}_\ell$ for g^* and $\beta_1, \ldots, \beta_d \in \overline{\mathbb{Q}}_\ell$ for F^*. Here, we may assume that each v_j is contained in $H^i_c(X, \overline{\mathbb{Q}}_\ell)$ for some i. For each $j = 1, \ldots, k$, we define a sign ϵ_j by

$$\epsilon_j := \begin{cases} 1 & \text{if } v_j \text{ is contained in an even degree cohomology,} \\ -1 & \text{if } v_j \text{ is contained in an odd degree cohomology.} \end{cases}$$

Then, by applying the fixed point formula to X_n over $\mathbb{F}_{q^n}$, we get

$$|X^{g\circ F^n}| = \sum_{j=1}^{d} \epsilon_j \alpha_j \beta_j^n.$$

Therefore, we get

$$\begin{aligned} R(t) = -\sum_{n=1}^{\infty} |X^{g\circ F^n}| \cdot t^n &= -\sum_{n=1}^{\infty}\sum_{j=1}^{d} \epsilon_j \alpha_j \beta_j^n \cdot t^n \\ &= -\sum_{j=1}^{d} \epsilon_j \alpha_j \sum_{n=1}^{\infty} \beta_j^n \cdot t^n \\ &= -\sum_{j=1}^{d} \epsilon_j \alpha_j \frac{\beta_j t}{1-\beta_j t} \in \overline{\mathbb{Q}}_\ell(t). \end{aligned}$$

In particular, $R(t)$ is a rational function which does not have a pole at $t=\infty$. Let us write $R(t)=p(t)/q(t)$ with polynomials $p(t),q(t)\in\overline{\mathbb{Q}}_\ell[t]$; then, by noting that $R(t)$ is initially given by a formal series with $\mathbb{Z}$-coefficients, we can easily check that the coefficients of $p(t)$ and $q(t)$ can be taken to be in $\mathbb{Q}$. In other words, we have $R(t)\in\mathbb{Q}(t)$.

On the other hand, we note that $R(\infty)$ is given by $\sum_{j=1}^{d}\epsilon_j\alpha_j$, which is nothing but $\sum_{i\geq 0}(-1)^i\operatorname{Tr}(g\mid H_c^i(X,\overline{\mathbb{Q}}_\ell))$. Since $R(t)$ is independent of ℓ (by its definition) and belongs to $\mathbb{Q}(t)$, we have that $\sum_{i\geq 0}(-1)^i\operatorname{Tr}(g\mid H_c^i(X,\overline{\mathbb{Q}}_\ell))$ is a rational number which is independent of ℓ. Moreover, since g is of finite order, $\alpha_j\in\overline{\mathbb{Q}}_\ell$ also must be of finite order. In particular, $\sum_{i\geq 0}(-1)^i\operatorname{Tr}(g\mid H_c^i(X,\overline{\mathbb{Q}}_\ell))=\sum_{j=1}^{d}\epsilon_j\alpha_j$ is an algebraic integer. As $\mathbb{Q}\cap\overline{\mathbb{Z}}=\mathbb{Z}$, we get $\sum_{i\geq 0}(-1)^i\operatorname{Tr}(g\mid H_c^i(X,\overline{\mathbb{Q}}_\ell))\in\mathbb{Z}$. □

We let $\mathcal{L}(g,X)$ denote the Lefschetz number of g.

6.2. **Deligne–Lusztig representation.** In the following, we let k be a finite field $\mathbb{F}_q$ of characteristic $p>0$. We fix a prime number $\ell\neq p$ and also fix an isomorphism $\iota\colon\overline{\mathbb{Q}}_\ell\xrightarrow{\cong}\mathbb{C}$. Let G be a connected reductive group over k.

Recall that, for any k-rational maximal torus T of G and a Borel subgroup B containing T[19], the Deligne–Lusztig variety $\mathcal{X}_{T\subset B}^G$ is defined; this is an algebraic variety over $\overline{k}$ equipped with an action of $G^F\times T^F$. Therefore, its ℓ-adic cohomology $H_c^i(\mathcal{X}_{T\subset B}^G,\overline{\mathbb{Q}}_\ell)$ is a finite-dimensional representation (on a $\overline{\mathbb{Q}}_\ell$-vector space) of $G^F\times T^F$.

Now suppose that $\theta\colon T^F\to\mathbb{C}^\times$ is a character. Then, through the fixed isomorphism ι, we may regard θ as a $\overline{\mathbb{Q}}_\ell^\times$-valued character of T^F. Let us write $\theta_\iota:=\iota^{-1}\circ\theta\colon T^F\to\overline{\mathbb{Q}}_\ell^\times$. Then it makes sense to consider the θ_ι-isotypic part $H_c^i(\mathcal{X}_{T\subset B}^G,\overline{\mathbb{Q}}_\ell)[\theta_\iota]$ of $H_c^i(\mathcal{X}_{T\subset B}^G,\overline{\mathbb{Q}}_\ell)$, which is a finite-dimensional representation of G^F on a $\overline{\mathbb{Q}}_\ell$-vector space.

Definition 6.6. We call the alternating sum of $H_c^i(\mathcal{X}_{T\subset B}^G,\overline{\mathbb{Q}}_\ell)[\theta_\iota]$ the *Deligne–Lusztig (virtual) representation of G^F associated to* (T,θ_ι) and write $R_T^G(\theta_\iota)$ for it:

$$R_{T\subset B}^G(\theta_\iota):=\sum_{i\geq 0}(-1)^iH_c^i(\mathcal{X}_{T\subset B}^G,\overline{\mathbb{Q}}_\ell)[\theta_\iota].$$

By abuse of notation, we also write $R_{T\subset B}^G(\theta_\iota)$ for the character of the Deligne–Lusztig (virtual) representation (called *Deligne–Lusztig (virtual) character*).

Remark 6.7. Let us say a bit more about the notion of the θ_ι-isotypic part $H_c^i(\mathcal{X}_{T\subset B}^G,\overline{\mathbb{Q}}_\ell)[\theta_\iota]$. By definition, it is the maximal subspace of $H_c^i(\mathcal{X}_{T\subset B}^G,\overline{\mathbb{Q}}_\ell)$ whose action of T^F is given by θ_ι, i.e., $t\cdot v=\theta_\iota(t)v$ for any $t\in T^F$ and $v\in H_c^i(\mathcal{X}_{T\subset B}^G,\overline{\mathbb{Q}}_\ell)$ (such a subspace always uniquely exists since any representation of T^F on a finite-dimensional vector space is semisimple). More explicitly, $H_c^i(\mathcal{X}_{T\subset B}^G,\overline{\mathbb{Q}}_\ell)[\theta_\iota]$ is realized as the image of the following endomorphism of $H_c^i(\mathcal{X}_{T\subset B}^G,\overline{\mathbb{Q}}_\ell)$:

$$\frac{1}{|T^F|}\sum_{t\in T^F}\theta_\iota(t)^{-1}\cdot t.$$

Now let us discuss the ℓ-independence of the Deligne–Lusztig representation. At this point, the coefficients of the Deligne–Lusztig representation is taken to be $\overline{\mathbb{Q}}_\ell$ and its construction depends on $\iota\colon\overline{\mathbb{Q}}_\ell\cong\mathbb{C}$. Hence, the Deligne–Lusztig character is also a class function on G^F valued in $\overline{\mathbb{Q}}_\ell$.

[19]Here, B is a subgroup of $G_{\overline{k}}$ which may not defined over $\overline{k}$. So, precisely speaking, it might be better to write " a Borel subgroup B containing $T_{\overline{k}}$".

We note that the Deligne–Lusztig variety $\mathcal{X}^G_{T\subset B}$ might not be defined over k. However, there exists a finite extension k' of k such that $\mathcal{X}^G_{T\subset B}$ is defined over k. Indeed, suppose that T splits over $k' = \mathbb{F}_{q^n}$. Then we can choose a Borel subgroup B containing T so that it is defined over k'. This is equivalent to that U satisfies $F^n(U) = U$. Hence, if $g \in G$ satisfies $g^{-1}F(g) \in F(U)$, then we have $F^n(g)^{-1}F(F^n(g)) = F^n(g^{-1}F(g)) \in F^n(F(U)) = F(U)$. In other words, $\mathcal{X}^G_{T\subset B}$ is a subset of G which is stable under F^n. Thus, by the Galois descent, $\mathcal{X}^G_{T\subset B}$ is defined over k'. Note that the Frobenius endomorphism of $\mathcal{X}^G_{T\subset B}$ associated to this k'-rational structure is given by F^n.

Now let us apply Theorem 6.5 to the action of $G^F \times T^F$ on $\mathcal{X}^G_{T\subset B}$. Any $(g,t) \in G^F \times T^F$ satisfies $(g,t)\circ F^n = F^n \circ (g,t)$. Indeed, for any $x \in \mathcal{X}^G_{T\subset B}$, we have

$$(g,t)\circ F^n(x) = gF^n(x)t = F^n(gxt) = F^n \circ (g,t)(x)$$

(note that g and t are fixed by F). In other words, the (g,t)-action on $\mathcal{X}^G_{T\subset B}$ satisfies the assumption of Theorem 6.5. Hence the Lefschetz number of (g,t) is an integer independent of ℓ:

$$\mathcal{L}((g,t),\mathcal{X}^G_{T\subset B}) := \sum_{i\geq 0}(-1)^i \operatorname{Tr}((g,t) \mid H^i_c(\mathcal{X}^G_{T\subset B},\overline{\mathbb{Q}}_\ell)) \in \mathbb{Z}.$$

Proposition 6.8. *For any $g \in G^F$, we have*

$$R^G_{T\subset B}(\theta_\iota)(g) = \frac{1}{|T^F|}\sum_{t\in T^F}\theta_\iota(t)^{-1}\cdot\mathcal{L}((g,t),\mathcal{X}^G_{T\subset B}).$$

Proof. By Remark 6.7, we have

$$\begin{aligned}
R^G_{T\subset B}(\theta_\iota)(g) &= \sum_{i\geq 0}(-1)^i \operatorname{Tr}(g \mid H^i_c(\mathcal{X}^G_{T\subset B},\overline{\mathbb{Q}}_\ell)[\theta_\iota]) \\
&= \sum_{i\geq 0}(-1)^i \frac{1}{|T^F|}\sum_{t\in T^F}\theta_\iota(t)^{-1}\operatorname{Tr}((g,t) \mid H^i_c(\mathcal{X}^G_{T\subset B},\overline{\mathbb{Q}}_\ell)) \\
&= \frac{1}{|T^F|}\sum_{t\in T^F}\theta_\iota(t)^{-1}\sum_{i\geq 0}(-1)^i\operatorname{Tr}((g,t) \mid H^i_c(\mathcal{X}^G_{T\subset B},\overline{\mathbb{Q}}_\ell)) \\
&= \frac{1}{|T^F|}\sum_{t\in T^F}\theta_\iota(t)^{-1}\cdot\mathcal{L}((g,t),\mathcal{X}^G_{T\subset B}).
\end{aligned}$$

□

Note that, though the isomorphism $\iota\colon \overline{\mathbb{Q}}_\ell \cong \mathbb{C}$, we can regard $R^G_{T\subset B}(\theta_\iota)$ as a $\mathbb{C}$-valued class function on G^F. By the above proposition, then its values is given by

$$\frac{1}{|T^F|}\sum_{t\in T^F}\theta(t)^{-1}\cdot\mathcal{L}((g,t),\mathcal{X}^G_{T\subset B}),$$

which is independent of ℓ (and also of ι). Let us write $R^G_{T\subset B}(\theta)$ for the virtual representation/character of G^F with $\mathbb{C}$-coefficients obtained in this way.

Example 6.9. Let us present an example in the GL_2-case without any justification. Recall that (Week 2) irreducible representations of $\mathrm{GL}_2(\mathbb{F}_q)$ are constructed by two different kinds of inductions:

(1) To any character $\boldsymbol{\chi}$ of $\mathbb{F}_q^\times \times \mathbb{F}_q^\times$, we can associate a principal series representation $\operatorname{Ind}^{\mathrm{GL}_2(\mathbb{F}_q)}_{B(\mathbb{F}_q)}\boldsymbol{\chi}$.

(2) To any character θ of $\mathbb{F}_{q^2}^\times$ satisfying $\theta^{q-1} \neq \mathbb{1}$, we can associate a cuspidal representation π_θ.

Also recall that (Week 5) G^F-conjugacy classes of k-rational maximal tori of a connected reductive group G over k can be classified by the F-conjugacy classes of Weyl group of G. When $G = \mathrm{GL}_2$, its Weyl group W is equal to $\mathfrak{S}_2 = \{1, s\}$ with trivial F-action. So there exist exactly two G^F-conjugacy classes of k-rational maximal tori of GL_2:

(1) The one T_1 corresponding to the trivial element $1 \in W$ is split; $T_1(\mathbb{F}_q) \cong (\mathbb{F}_q^\times)^2$. For any character $\boldsymbol{\chi}$ of $T_1(\mathbb{F}_q)$, we have $R^G_{T_1\subset B}(\boldsymbol{\chi}) \cong \mathrm{Ind}_{B(\mathbb{F}_q)}^{\mathrm{GL}_2(\mathbb{F}_q)} \boldsymbol{\chi}$.

(2) The other one T_s corresponding to the non-trivial element $s \in W$ is non-split; $T_s(\mathbb{F}_q) \cong \mathbb{F}_{q^2}^\times$. If we take a character θ of $T_s(\mathbb{F}_q)$ satisfying $\theta^{q-1} \neq \mathbb{1}$, then we have $R^G_{T_s\subset B}(\theta) \cong -\pi_\theta$.[20][21]

6.3. **Split case: principal series.** Let us first investigate the Deligne–Lusztig representation in the case where G is split and T is a split maximal torus ("base torus") T_0. Then we can find a Borel subgroup B of G containing T which is defined over k. Let $\theta\colon T^F \to \mathbb{C}^\times$ be any character. Since B is equal to the semi-direct product of its unipotent radical U and T (T normalizes U), we have a natural surjective homomorphism $B \twoheadrightarrow B/U = T$. By inflating through this homomorphism, we can regard θ as a character of B^F. We define the *principal series representation of G^F (associated to θ)* to be $\mathrm{Ind}_{B^F}^{G^F} \theta$.

Proposition 6.10. *We have $R^G_{T\subset B}(\theta) \cong \mathrm{Ind}_{B^F}^{G^F} \theta$.*

Proof. We let $\mathcal{B}^F$ denote the set of k-rational Borel subgroups of G. We note that any two k-rational Borel subgroups of G are G^F-conjugate; in particular, $\mathcal{B}^F$ is a finite set. We define a morphism π from $\mathcal{X}^G_{T\subset B}$ to $\mathcal{B}^F$ by

$$\pi\colon \mathcal{X}^G_{T\subset B} = \{g \in G \mid g^{-1}F(g) \in U\} \to \mathcal{B}^F; \quad g \mapsto gBg^{-1}$$

(note that $F(U)$ in the definition of $\mathcal{X}^G_{T\subset B}$ is equal to U since U is k-rational). This morphism is well-defined; indeed, if $g \in G$ satisfies $g^{-1}F(g) \in U$ (say $g^{-1}F(g) = u$), then we have

$$F(gBg^{-1}) = F(g)BF(g)^{-1} = guBu^{-1}g^{-1} = gBg^{-1}.$$

Hence gBg^{-1} is a k-rational Borel subgroup of G. Moreover, π is surjective. To check this, let us take a k-rational Borel subgroup B' of G. Then there exists an element $g \in G^F$ satisfying $B' = gBg^{-1}$. since $g^{-1}F(g) = 1 \in U$, g belongs to $\mathcal{X}^G_{T\subset B}$ and satisfies $\pi(g) = B'$. Therefore, we obtain a disjoint union decomposition $\mathcal{X}^G_{T\subset B}$ into finite number of closed subvarieties:

$$\mathcal{X}^G_{T\subset B} = \bigsqcup_{B'\in\mathcal{B}^F} \pi^{-1}(B').$$

Recall that, $\mathcal{X}^G_{T\subset B}$ has an action of $G^F \times T^F$ given by $(x,t)\colon g \mapsto xgt$. We introduce an action of $G^F \times T^F$ on $\mathcal{B}^F$ by $(x,t)\colon B' \mapsto xB'x^{-1}$. Then π is $G^F \times T^F$-equivariant, i.e., $\pi((x,t)\cdot g) = (x,t)\cdot\pi(g)$. Note that the action of $G^F \times T^F$ permutes the closed subvarieties $\pi^{-1}(B')$ (for $B' \in \mathcal{B}^F$). The resulting action $G^F \times T^F$ of on the finite set $\{\pi^{-1}(B') \mid B' \in \mathcal{B}^F\}$ is transitive and the stabilizer of $\pi^{-1}(B)$ is given by $B^F \times T^F$. In this setting, we have that the class function

$$G^F \times T^F \to \mathbb{Z}\colon (g,t) \mapsto \mathcal{L}((g,t), \mathcal{X}^G_{T\subset B})$$

[20] Note that the Deligne-Lusztig representation itself can be defined even if θ does not satisfy the condition $\theta^{q-1} \neq \mathbb{1}$.

[21] Here, a Borel subgroup B containing T_s cannot be taken to be the standard upoper-triangular one.

is given by the induction of

$$B^F \times T^F \to \mathbb{Z}\colon (b,t) \mapsto \mathcal{L}((b,t), \pi^{-1}(B))$$

(This is a general fact which holds for the Lefschetz number of a variety equipped with a finite group action; see [Car85, Property 7.1.7]).

Hence, by Proposition 6.8, the Deligne–Lusztig character $R^G_{T\subset B}(\theta)$ is given by the induction of the following class function from B^F to G^F:

$$b \mapsto \frac{1}{|T^F|} \sum_{t\in T^F} \theta(t)^{-1} \cdot \mathcal{L}((b,t), \pi^{-1}(B)).$$

Let us compute $\mathcal{L}((b,t), \pi^{-1}(B))$. By recalling that $N_G(B) = B$, we see that $\pi^{-1}(B)$ is given by

$$\mathcal{X}^G_{T\subset B} \cap N_G(B) = \mathcal{X}^G_{T\subset B} \cap B = T^F U.$$

Note that each fiber of the quotient map $T^F U \twoheadrightarrow T^F U/U$ is isomorphic to U, which is furthermore isomorphic to an affine space $\mathbb{A}^{\dim U}$ (this is a general property of an unipotent group). In fact, it is known that such a map ("affine fibration") does not change the Lefschetz number, i.e., $\mathcal{L}((b,t), \pi^{-1}(B)) = \mathcal{L}((b,t), T^F U/U)$ (see [Car85, Property 7.1.5]). Here, $B^F \times T^F$ acts on $T^F U/U$ in an obvious way, that is, $(b,t)\cdot sU = bstU$.

Now note that $T^F U/U = T^F U^F/U^F$ is a finite set. Thus $\mathcal{L}((b,t), T^F U^F/U^F)$ is equal to the cardinality of the set $(T^F U^F/U^F)^{(b,t)}$ of points of $T^F U^F/U^F$ fixed by (b,t) (see the exercise below). For any $sU^F \in T^F U^F/U^F$, we have $(b,t)\cdot sU^F = sU^F$ if and only if $bstU^F = sU^F$, which is equivalent to $b \in t^{-1}U^F$. This implies that the fixed points set $(T^F U^F/U^F)^{(b,t)}$ is empty if $b \notin t^{-1}U^F$ and equal to $T^F U^F/U^F$ if $b \in t^{-1}U^F$. Since $|T^F U^F/U^F| = |T^F|$, we get

$$\mathcal{L}((b,t), \pi^{-1}(B)) = \begin{cases} |T^F| & \text{if } b \in t^{-1}U^F, \\ 0 & \text{if } b \notin t^{-1}U^F. \end{cases}$$

Therefore, $R^G_{T\subset B}(\theta)$ is given by the induction of

$$b = su \mapsto \frac{1}{|T^F|} \sum_{t\in T^F} \theta(t)^{-1} \cdot \mathcal{L}((su,t), \pi^{-1}(B)) = \theta(s).$$

This means that $R^G_{T\subset B}(\theta)$ is the induction of the inflation of θ, i.e., $\mathrm{Ind}^{G^F}_{B^F}\theta$. □

Exercise 6.11. Prove the following claim:

> Let X be a finite set (this can be regarded as a 0-dimensional algebraic variety $\bigsqcup_{x\in X} \operatorname{Spec} \overline{k}$). Suppose that g is an automorphism of X. Then we have $\mathcal{L}(g, X) = |X^g|$.

Hint:

(1) Show that the Frobenius endomorphism F induced from the obvious k-rational structure $\bigsqcup_{x\in X} \operatorname{Spec} k$ is the identity of X.
(2) Define a formal power series $R(t)$ in the same way as the proof of Theorem 6.5 and do the same argument.

7. Week 7: Deligne–Lusztig character formula

Let G be a connected reductive group over $k = \mathbb{F}_q$ and F its associated Frobenius endomorphism. We fix a k-rational maximal torus T of G and a Borel subgroup B of G containing T. We also fix a character $\theta\colon T^F \to \mathbb{C}^\times$. Then we have the Deligne–Lusztig virtual representation $R^G_{T\subset B}(\theta)$ of G^F. By abuse of notation, we also write $R^G_{T\subset B}(\theta)$ for the Deligne–Lusztig virtual character, which is a class function $G^F \to \mathbb{C}$ defined to be the trace of the Deligne–Lusztig virtual representation. Today's aim is to establish a character formula for $R^G_{T\subset B}(\theta)$.

7.1. Deligne–Lusztig character formula.

We write G^F_{ss} and G^F_{unip} for the set of semisimple (equivalently, prime-to-p order) and unipotent elements of G^F (equivalently, p-power order), respectively. In the following, for any $g \in G$ and $h \in G$, we write ${}^g h = ghg^{-1}$. Similarly, for any $g \in G$ and a subgroup $H \subset G$, we write ${}^g H = gHg^{-1}$.

Definition 7.1. We define a function $Q^G_T\colon G^F_{\mathrm{unip}} \to \mathbb{C}$ by $Q^G_T := R^G_{T\subset B}(\mathbb{1})|_{G^F_{\mathrm{unip}}}$. We call Q^G_T the *Green function (of G associated to T)*.

We note that, for notational convenience, we simply write "Q^G_T" although a priori Q^G_T depends on the choice of a Borel subgroup B containing T. (But, in fact, later it will turn out that Q^G_T does not depend on B!)

To state the Deligne–Lusztig character formula, let us recall that any element $g \in G^F$ has the Jordan decomposition $g = su$, where $s \in G^F$ is a semisimple element and $u \in G^F$ is a unipotent element such that $su = us$.

Theorem 7.2 (Deligne–Lusztig character formula). *Let $g \in G^F$ with Jordan decomposition $g = su$. We shortly write G_s for the centralizer of s in G, i.e., $G_s = Z_G(s) = \{x \in G \mid xs = sx\}$. Then we have*

$$R^G_{T\subset B}(\theta)(g) = \frac{1}{|(G^\circ_s)^F|} \sum_{\substack{x\in G^F\\ x^{-1}sx\in T^F}} \theta(x^{-1}sx)\cdot Q^{G^\circ_s}_{{}^xT}(u).$$

Let us explain why the right-hand side of this formula makes sense. We first note the following result (see [Car85, 1.14]).

Lemma 7.3. *(1) For any $s \in G^F_{\mathrm{ss}}$, the identity component G°_s of its centralizer G_s is a connected reductive group defined over k.*

(2) Any unipotent element of G_s lies in G°_s. In particular, when $g \in G^F$ has the Jordan decomposition $g = su$, its unipotent part u belongs to $(G^\circ_s)^F$.

Let us look at the index set of the sum in the Deligne–Lusztig character formula. When $x^{-1}sx \in T$, we necessarily have the opposite inclusion $Z_G(x^{-1}sx) \supset Z_G(T)$. Here, it is easy to check that $Z_G(x^{-1}sx) = x^{-1}Z_G(s)x$. On the other hand, it is known that the centralizer of a maximal torus in a connected reductive group is the maximal torus itself, i.e., $Z_G(T) = T$ (see [Spr09, 7.6.4]). Hence, we have $x^{-1}Z_G(s)x \supset T$, or equivalently, ${}^xT = xTx^{-1} \subset Z_G(s) = G_s$. Since T is connected, this furthermore implies that ${}^xT \subset G^\circ_s$. Furthermore, it is known that $(B\cap G^\circ_s)^\circ$ is a Borel subgroup of G°_s and $U\cap G^\circ_s$ is its unipotent radical.[22]

In summary, when $x^{-1}sx \in T$, we obtain a k-rational maximal torus ${}^xT$ of a connected reductive group G°_s. Thus it makes sense to consider the Green function $Q^{G^\circ_s}_{{}^xT}$ of G°_s associated

[22] Here, note that $U \cap G^\circ_s$ is already connected!

to ${}^xT$ and $(B \cap G_s^\circ)^\circ$. Since u belongs to $(G_s^\circ)^F_{\mathrm{unip}}$, it also makes sense to look at the value of $Q^{G_s^\circ}_{{}^xT}$ at u.

Thus the Deligne–Lusztig character formula reflects an inductive nature of the theory of reductive groups. The contribution of the semisimple part s is given just by θ, which is very simple. On the other hand, the contribution of the unipotent part u is given by the Green function, which is independent of θ and taken in G_s°. Hence, ultimately, the Deligne–Lusztig characters of G are governed by the Green functions for G and all its "smaller" reductive subgroups.

7.2. **Outline of the proof of DL character formula.** The key of the proof of the Deligne–Lusztig character formula is the following general result, which is called *Deligne–Lusztig's fixed point formula*:

Theorem 7.4 (Deligne–Lusztig fixed point formula)**.** *Let X be an algebraic variety over k and g is an automorphism of X of finite order. Let s and u be automorphisms of X such that s is of prime-to-p order, u is of p-power order, and $g = su = us$. Then we have $\mathcal{L}(g, X) = \mathcal{L}(u, X^s)$, where $X^s := \{x \in X \mid s(x) = x\}$.*

Unfortunately, I cannot explain the proof of this theorem in this course. Please look at [DL76, Theorem 3.2] if you have an interest.

Now suppose that $g \in G^F$ has the Jordan decomposition $g = su = us$. As disxussed in the last week, we have

$$R^G_{T\subset B}(\theta)(g) = \frac{1}{|T^F|}\sum_{t\in T^F}\theta(t)^{-1}\cdot\mathcal{L}((g,t),\mathcal{X}^G_{T\subset B}).$$

Let us compute each $\mathcal{L}((g,t),\mathcal{X}^G_{T\subset B})$ using the Deligne–Lusztig fixed point formula.

Recall that the action of (g,t) on $\mathcal{X}^G_{T\subset B} = \{x \in G \mid x^{-1}F(x) \in F(U)\}$ is given by $x \mapsto gxt$. We note that the order of T^F is prime-to-p. (Indeed, if we suppose that T splits over $\mathbb{F}_{q^n}$, i.e., $T_{\mathbb{F}_{q^n}} = \mathbb{G}_{\mathrm{m}}^r$ for some r, we have $T^F = T(\mathbb{F}_q) \subset T_{\mathbb{F}_{q^n}}(\mathbb{F}_{q^n}) \cong (\mathbb{F}_{q^n}^\times)^r$.) Hence the order of t is also prime-to-p. Thus, the decomposition $(g,t) = (s,t)\circ(u,1)$ satisfies the assumption of the Deligne–Lusztig fixed point formula.

We determine $(\mathcal{X}^G_{T\subset B})^{(s,t)}$. In the following, we simply write $\mathcal{X} := \mathcal{X}^G_{T\subset B}$.

Proposition 7.5. *We have*

$$\mathcal{X}^{(s,t)} = \bigsqcup_{\substack{x\in G^F/(G_t^\circ)^F \\ {}^xt = s^{-1}}} \mathcal{X}^{(s,t)}(x),$$

where we put $\mathcal{X}^{(s,t)}(x) := \mathcal{X}^{(s,t)} \cap xG_t^\circ$.

Proof. Suppose that $y \in \mathcal{X}^{(s,t)}$, i.e., $y \in G$ is an element satisfying $syt = y$ and $y^{-1}F(y) \in F(U)$ (say $y^{-1}F(y) = v$). By applying F to $syt = y$, we get $sF(y)t = F(y)$, thus $syvt = yv$. Combining $syvt = yv$ with $syt = y$, we get $yt^{-1}vt = yv$, hence $t^{-1}vt = v$. This means that u belongs to $G_t = Z_G(t)$. As u is unipotent, u furthermore lies in G_t°. Let us apply Lang's theorem to G_t°, which asserts that the map

$$G_t^\circ \to G_t^\circ \colon z \mapsto z^{-1}F(z)$$

is surjective; we can find an element $z \in G_t^\circ$ satisfying $z^{-1}F(z) = v$.

We put $x := yz^{-1}$. Then $F(x) = F(y)F(z)^{-1} = yvv^{-1}z^{-1} = yz^{-1} = x$, i.e., $x \in G^F$. Note that we have $y \in xG_t^\circ$. Furthermore, we have

$$xz = y = syt = s(xz)t = sxtz$$

(use that $z \in G_t$ in the last equality), hence ${}^x t = s^{-1}$.

From the disxussion so far, we have obtained

$$\mathcal{X}^{(s,t)} = \bigcup_{\substack{x \in G^F/(G_t^\circ)^F \\ {}^x t = s^{-1}}} \mathcal{X}^{(s,t)}(x).$$

It is obvious that the union is disjoint. □

Let us investigate each summand $\mathcal{X}^{(s,t)}(x)$. Note that, since $t \in T^F$, we have $T \subset G_t^\circ$. Moreover, $B_t^\circ := (B \cap G_t^\circ)^\circ$ is a Borel subgroup of G_t° with unipotent radical $U \cap G_t^\circ$ (see the paragraph after Lemma 7.3). Therefore, it makes sense to consider the Deligne–Lusztig variety for G_t° associated to $T \subset B_t^\circ$:

$$\mathcal{X}_{T \subset B_t^\circ}^{G_t^\circ} = \{y' \in G_t^\circ \mid y'^{-1} F(y') \in U \cap G_t^\circ\}.$$

This is a variety equipped with an action of $(G_t^\circ)^F \times T^F$. On the other hand, $\mathcal{X}^{(s,t)}(x)$ is stable under the action of the subgroup $(G_s^\circ)^F \times T^F$ of $G^F \times T^F$ on $\mathcal{X}$.

Proposition 7.6. *Let $x \in G^F$ be an element satisfying ${}^x t = s^{-1}$. Then have an isomorphism of varieties*

$$\varphi_x \colon \mathcal{X}^{(s,t)}(x) \xrightarrow{\cong} \mathcal{X}_{T \subset B_t^\circ}^{G_t^\circ} : y \mapsto x^{-1} y,$$

which is equivariant with respect to the actions of $(G_s^\circ)^F \times T^F$ on $\mathcal{X}^{(s,t)}(x)$ and $(G_t^\circ)^F \times T^F$ on $\mathcal{X}_{T \subset (B \cap G_t^\circ)^\circ}^{G_t^\circ}$. Here, $(G_s^\circ)^F \times T^F$ and $(G_t^\circ)^F \times T^F$ are identified by $(z, t') \mapsto (x^{-1} z x, t')$.

Proof. Suppose that $y \in \mathcal{X}^{(s,t)}(x)$, i.e., $y \in x G_t^\circ$ is an element satisfying $syt = y$ and $y^{-1} F(y) \in F(U)$. Then we have $x^{-1} y \in G_t^\circ$ and thus

$$(x^{-1} y)^{-1} F(x^{-1} y) = y^{-1} F(y) \in F(U) \cap G_t^\circ = F(U \cap G_t^\circ).$$

In other words, $\varphi_x(y) = x^{-1} y$ belongs to $\mathcal{X}_{T \subset B_t^\circ}^{G_t^\circ}$. Conversely, for any element $y' \in \mathcal{X}_{T \subset B_t^\circ}^{G_t^\circ}$, we can check that $\varphi_x^{-1}(y') = x y' \in \mathcal{X}^{(s,t)}(x)$.

Let us check the assertion on the equivariance. What we have to prove is that, for any $(z, t') \in (G_s^\circ)^F \times T^F$ and $y \in \mathcal{X}^{(s,t)}(x)$, we have

$$\varphi_x((z, t') \cdot y) = (x^{-1} z x, t') \cdot \varphi_x(y).$$

The left-hand side is given by $\varphi_x((z, t') \cdot y) = \varphi_x(z y t') = x^{-1} z y t'$. The right-hand side is given by $(x^{-1} z x, t') \cdot \varphi_x(y) = (x^{-1} z x, t') \cdot (x^{-1} y) = x^{-1} z x (x^{-1} y) t' = x^{-1} z y t'$. So these indeed coincide. □

Now let us start the proof of the Deligne–Lusztig character formula:

Proof of Theorem 7.2. We have

$$R_{T \subset B}^G(\theta)(g) = \frac{1}{|T^F|} \sum_{t \in T^F} \theta(t)^{-1} \cdot \mathcal{L}((g, t), \mathcal{X}_{T \subset B}^G).$$

By applying the Deligne–Lusztig fixed point theorem to $(g, t) = (s, t) \circ (u, 1)$, we get

$$\mathcal{L}((g, t), \mathcal{X}_{T \subset B}^G) = \mathcal{L}((u, 1), (\mathcal{X}_{T \subset B}^G)^{(s,t)}).$$

By combining the above propositions, we get

$$\mathcal{L}(u, (\mathcal{X}_{T \subset B}^G)^{(s,t)}) = \sum_{\substack{x \in G^F/(G_t^\circ)^F \\ {}^x t = s^{-1}}} \mathcal{L}(x^{-1} u x, \mathcal{X}_{T \subset B_t^\circ}^{G_t^\circ}).$$

Hence we get

$$R^G_{T\subset B}(\theta)(g) = \frac{1}{|T^F|}\sum_{t\in T^F}\theta(t)^{-1}\sum_{\substack{x\in G^F/(G^\circ_t)^F\\ {}^x t=s^{-1}}}\mathcal{L}(x^{-1}ux,\mathcal{X}^{G^\circ_t}_{T\subset B^\circ_t})$$

$$= \frac{1}{|T^F|}\sum_{t\in T^F}\theta(t)^{-1}\cdot\frac{1}{|(G^\circ_t)^F|}\sum_{\substack{x\in G^F\\ {}^x t=s^{-1}}}\mathcal{L}(x^{-1}ux,\mathcal{X}^{G^\circ_t}_{T\subset B^\circ_t}).$$

Note that the internal sum is nonzero only when there exists an element $x\in G^F$ satisfying $t=x^{-1}s^{-1}x$. In this case, $|(G^\circ_t)^F| = |(G^\circ_s)^F|$, hence the above equals

$$\frac{1}{|T^F|\cdot|(G^\circ_s)^F|}\sum_{t\in T^F}\sum_{\substack{x\in G^F\\ {}^x t=s^{-1}}}\theta(t)^{-1}\cdot\mathcal{L}(x^{-1}ux,\mathcal{X}^{G^\circ_t}_{T\subset B^\circ_t}).$$

We note that the set $\{(t,x)\in T^F\times G^F \mid {}^x t=s^{-1}\}$ is bijective to $\{x\in G^F \mid x^{-1}sx\in T^F\}$ by $(t,x)\mapsto x$. By also noting that $\mathcal{L}(x^{-1}ux,\mathcal{X}^{G^\circ_t}_{T\subset B^\circ_t}) = \mathcal{L}(u,\mathcal{X}^{G^\circ_s}_{{}^x T\subset B^\circ_s})$, we rewrite the above double sum:

$$\frac{1}{|T^F|\cdot|(G^\circ_s)^F|}\sum_{\substack{x\in G^F\\ x^{-1}sx\in T^F}}\theta(x^{-1}s^{-1}x)^{-1}\cdot\mathcal{L}(u,\mathcal{X}^{G^\circ_s}_{{}^x T\subset B^\circ_s})$$

$$= \frac{1}{|T^F|\cdot|(G^\circ_s)^F|}\sum_{\substack{x\in G^F\\ x^{-1}sx\in T^F}}\theta(x^{-1}sx)\cdot\mathcal{L}(u,\mathcal{X}^{G^\circ_s}_{{}^x T\subset B^\circ_s}).$$

Here, in general, we have

$$Q^G_T(u) = \frac{1}{|T^F|}\cdot\mathcal{L}(u,\mathcal{X}^G_{T\subset B}).$$

Indeed, by definition, we have

$$Q^G_T(u) = \frac{1}{|T^F|}\sum_{t\in T^F}\mathcal{L}((u,t),\mathcal{X}^G_{T\subset B}).$$

By using the Deligne–Lusztig fixed point formula to $(u,t)=(u,1)\cdot(1,t)$, we have $\mathcal{L}((u,t),\mathcal{X}^G_{T\subset B}) = \mathcal{L}((u,1),(\mathcal{X}^G_{T\subset B})^{(1,t)})$. However, $(\mathcal{X}^G_{T\subset B})^{(1,t)}$ is nonempty only when $t=1$ (indeed, $x\in\mathcal{X}^G_{T\subset B}$ is fixed by $(1,t)$ if and only if $xt=x$). Thus we get

$$Q^G_T(u) = \frac{1}{|T^F|}\mathcal{L}(u,\mathcal{X}^G_{T\subset B}).$$

Therefore, we finally obtain

$$R^G_{T\subset B}(g) = \frac{1}{|T^F|\cdot|(G^\circ_s)^F|}\sum_{\substack{x\in G^F\\ x^{-1}sx\in T^F}}\theta(x^{-1}sx)\cdot Q^{G^\circ_s}_{{}^x T}(u).$$

□

Corollary 7.7. *We have $R^G_{T\subset B}(\theta)|_{G^F_{\mathrm{unip}}} = Q^G_T$ for any character $\theta\colon T^F\to\mathbb{C}^\times$.*

Proof. Let $g \in G^F_{\mathrm{unip}}$ (hence its semisimple part s is 1 and unipotent part u is g). Then, by applying the Deligne–Lusztig character formula to g, we get

$$\begin{aligned} R^G_{T\subset B}(\theta)(g) &= \frac{1}{|(G^\circ_s)^F|} \sum_{\substack{x\in G^F \\ x^{-1}sx\in T^F}} \theta(x^{-1}sx)\cdot Q^{G^\circ_s}_{{}^xT}(u) \\ &= \frac{1}{|G^F|}\sum_{x\in G^F} Q^G_{{}^xT}(u). \end{aligned}$$

It is not difficult to check that, in general, we have $R^G_{T\subset B}(\theta)(g) = R^G_{{}^xT\subset {}^xB}({}^x\theta)({}^xg)$, where ${}^x\theta$ denotes the character of ${}^xT^F$ defined by ${}^x\theta({}^xt) = \theta(t)$. In particular, when $\theta = \mathbb{1}$, hence get $Q^G_T(u) = Q^G_{{}^xT}({}^xu)$. By also noting that the Green function is invariant under conjugation (since it is the restriction of a Deligne–Lusztig character, which is a class function), we get $Q^G_T(u) = Q^G_{{}^xT}({}^xu) = Q^G_{{}^xT}(u)$. Hence the most right-hand side of the above equalities is $Q^G_T(u)$. □

Definition 7.8. We say that a semisimple element $s \in G$ is *regular* if G°_s is a maximal torus of G.

Example 7.9. Let $G = \mathrm{GL}_2$. Let T be the diagonal maximal torus of G. We consider an element $s = \left(\begin{smallmatrix} a & 0 \\ 0 & b\end{smallmatrix}\right) \in T$. Then, since s is already diagonalized, s is semisimple. Let us compute the centralizer $G_s = Z_G(s)$ of s in G.

- When $a = b$, s commutes with any element of G. Thus $G_s = G$, hence $G^\circ_s = G^\circ = G$. Hence s is not regular in this case.
- Suppose that $a \neq b$. If $g = \left(\begin{smallmatrix} x & y \\ z & w\end{smallmatrix}\right) \in Z_G(s)$, we have $sgs^{-1} = g$. Since

$$sgs^{-1} = \begin{pmatrix} a & 0 \\ 0 & b\end{pmatrix}\begin{pmatrix} x & y \\ z & w\end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b\end{pmatrix}^{-1} = \begin{pmatrix} x & ayb^{-1} \\ a^{-1}zb & w\end{pmatrix},$$

  we necessarily have $y = z = 0$, i.e., $g \in T$. Conversely, we obviously have $T \subset Z_G(s)$. Hence we get $G_s = T$, so $G^\circ_s = T$, which means that s is regular.

Exercise 7.10. Let $G = \mathrm{GL}_n$ and $g \in G$. Prove that g is regular semisimple if and only if the characteristic polynomial of g has n distinct roots. (Hint: compute the centralizer of g in G by looking at the Jordan normal form of g.)

Exercise 7.11. Let $G = \mathrm{GL}_n$. Recall that G^F-conjugacy classes of k-rational maximal tori of G correspond bijectively to the conjugacy classes of $\mathfrak{S}_n$ (see Week 5 notes). Let S be a maximal torus of G corresponding to the cyclic permutation $(1\,2\,\cdots\,n) \in \mathfrak{S}_n$. Count the number of regular semisimple elements in $S^F = \mathbb{F}^\times_{q^n}$.

Corollary 7.12. *Suppose that $s \in G^F$ is a regular semisimple element. If s is not conjugate to any element of T^F, then we have $R^G_{T\subset B}(\theta)(s) = 0$. If s is conjugate to any element of T^F (suppose that s itself belongs to T^F), then we have*

$$R^G_{T\subset B}(\theta)(s) = \sum_{x\in W_{G^F}(T)} \theta(x^{-1}sx),$$

where $W_{G^F}(T) := N_{G^F}(T)/T^F$.

Proof. By the Deligne–Lusztig character formula, we have

$$R^G_{T\subset B}(\theta)(s) = \frac{1}{|(G^\circ_s)^F|} \sum_{\substack{x\in G^F \\ x^{-1}sx\in T^F}} \theta(x^{-1}sx)\cdot Q^{G^\circ_s}_{{}^xT}(1).$$

Since the index set is empty if s is not conjugate to any element of T^F, we get the first assertion.

To show the second assertion, let us suppose that $s \in T^F$. Then, we must have $G_s^\circ = Z_G(s)^\circ \supset Z_G(T) = T$. Since G_s° is a maximal torus of G, this implies that $G_s^\circ = T$. By the same argument, the condition $x^{-1}sx \in T^F$ of the index set implies that $T = x^{-1}Tx$. In other words, $x \in N_{G^F}(T)$. Conversely, any element $x \in N_{G^F}(T)$ satisfies $x^{-1}sx \in T^F$. Thus, by noting that $Q_T^T(1) = 1$ (this can be checked by going back to the definition), we get

$$R_{T\subset B}^G(\theta)(s) = \frac{1}{|T^F|} \sum_{x \in N_{G^F}(T)} \theta(x^{-1}sx) = \sum_{x \in W_{G^F}(T)} \theta(x^{-1}sx).$$

□

8. Week 8: Inner product formula for Deligne–Lusztig representations

8.1. **Inner product formula for Deligne–Lusztig representations.** Let G be a connected reductive group over $k=\mathbb{F}_q$. Recall that the $\mathbb{C}$-vector space $C(G^F)$ of class functions on G^F has an inner product $\langle -,-\rangle$ given by

$$\langle f_1, f_2\rangle := \frac{1}{|G^F|}\sum_{g\in G^F} f_1(g)\cdot\overline{f_2(g)}.$$

Our next aim is to compute the inner product of two Deligne–Lusztig representations. To state the theorem, we introduce some notations. For k-rational maximal tori T and T' of G, we put

$$N_{G^F}(T,T') := \{n\in G^F \mid {}^nT = T'\},$$
$$W_{G^F}(T,T') := N_{G^F}(T,T')/T^F \cong T'^F\backslash N_{G^F}(T,T').$$

(Recall that, in our notation, ${}^nT$ denotes nTn^{-1}.) Note that, for any $w\in W_{G^F}(T,T')$ and a character $\theta\colon T^F\to\mathbb{C}^\times$, we can define a character ${}^w\theta$ of T'^F by

$${}^w\theta(t') := \theta(w^{-1}t'w).$$

(This definition is independent of the choice of a representative of w.)

Theorem 8.1 (Inner product formula)**.** *Let T and T' be k-rational maximal tori of G. Let $B=TU$ and $B'=T'U'$ be Borel subgroups of G containing T and T', respectively. For any characters $\theta\colon T^F\to\mathbb{C}^\times$ and $\theta'\colon T'^F\to\mathbb{C}^\times$, we have*

$$\langle R^G_{T\subset B}(\theta), R^G_{T'\subset B'}(\theta')\rangle = |\{w\in W_{G^F}(T,T') \mid {}^w\theta=\theta'\}|.$$

Before we prove this theorem, we explain several important consequences.

Corollary 8.2. *The Deligne–Lusztig representation $R^G_{T\subset B}(\theta)$ is independent of the choice of a Bore subgroup $B\subset T$. The Green function Q^G_T is also independent of $B\subset T$.*

Proof. Recall that $Q^G_T := R^G_{T\subset B}(\mathbb{1})|_{G^F_{\mathrm{unip}}}$. Thus it is enough to show the first assertion.

Let us take any Borel subgroup B and B' containing T. Our task is to show that $R^G_{T\subset B}(\theta) = R^G_{T\subset B'}(\theta)$ (here, both are regarded as class functions on G^F). Equivalently, it suffices to show that

$$\langle R^G_{T\subset B}(\theta) - R^G_{T\subset B'}(\theta), R^G_{T\subset B}(\theta) - R^G_{T\subset B'}(\theta)\rangle = 0.$$

The left-hand side equals

$$\langle R^G_{T\subset B}(\theta), R^G_{T\subset B}(\theta)\rangle - 2\langle R^G_{T\subset B}(\theta), R^G_{T\subset B'}(\theta)\rangle + \langle R^G_{T\subset B'}(\theta), R^G_{T\subset B'}(\theta)\rangle.$$

This equals 0 since we have

$$\langle R^G_{T\subset B}(\theta), R^G_{T\subset B}(\theta)\rangle = \langle R^G_{T\subset B}(\theta), R^G_{T\subset B'}(\theta)\rangle = \langle R^G_{T\subset B'}(\theta), R^G_{T\subset B'}(\theta)\rangle$$

by the inner product formula. □

From now on, let us simply write $R^G_T(\theta)$ instead of $R^G_{T\subset B}(\theta)$. (But, in the proof of the inner product formula, we will again write $R^G_{T\subset B}(\theta)$.)

Corollary 8.3. *Suppose that T and T' are k-rational maximal tori of G which are not G^F-conjugate. Then, for any characters θ of T and θ' of T', we have*

$$\langle R^G_T(\theta), R^G_{T'}(\theta')\rangle = 0.$$

Proof. This is clear from the inner product formula; if T and T' are not G^F-conjugate, then $N_{G^F}(T,T')$ is empty. □

Remark 8.4. Note that even if $\langle R_T^G(\theta), R_{T'}^G(\theta')\rangle = 0$, it might happen that $R_T^G(\theta)$ and $R_{T'}^G(\theta')$ have a common irreducible constituent. For example, the inner product of virtual representations $\pi_1 + \pi_2$ and $\pi_1 - \pi_2$ is zero, when π_1 and π_2 are irreducible.

Corollary 8.5. *If we write*

$$R_T^G(\theta) = \sum_\rho n_\rho \rho,$$

where ρ runs all isomorphism classes of irreducible representations of G^F, we have

$$\sum_\rho n_\rho^2 = |\{w \in W_{G^F}(T) \mid {}^w\theta = \theta\}|.$$

In particular, $R_T^G(\theta)$ is irreducible up to sign if and only if we have $\{w \in W_{G^F}(T) \mid {}^w\theta = \theta\} = \{1\}$.

Proof. This follows from the inner product formula (choose (T', θ') to be (T, θ)) and the general fact that, for irreducible representations ρ_1 and ρ_2 of G^F, we have

$$\langle \Theta_{\rho_1}, \Theta_{\rho_2}\rangle = \begin{cases} 1 & \text{if } \rho_1 \cong \rho_2, \\ 0 & \text{if } \rho_1 \ncong \rho_2. \end{cases}$$

□

Definition 8.6. We say that a character $\theta\colon T^F \to \mathbb{C}^\times$ is *regular (in general position)* if $\{w \in W_{G^F}(T) \mid {}^w\theta = \theta\} = \{1\}$. (Note that, by the above corollary, this is equivalent to that $R_T^G(\theta)$ is irreducible up to sign.)

8.2. **Weyl groups of k-rational maximal tori.** The inner product formula suggests that it is practically very important to determine the set $W_{G^F}(T, T')$ and its "action" on T[23]. Suppose that $N_{G^F}(T, T')$ is not empty. If we fix any element $n_0 \in N_{G^F}(T, T')$, then we get a bijection

$$N_{G^F}(T) \xrightarrow{\cong} N_{G^F}(T, T')\colon n \mapsto n_0 n.$$

Similarly, if we fix any element $w_0 \in W_{G^F}(T, T')$ (as long as this set is not empty), then we get a bijection

$$W_{G^F}(T) \xrightarrow{\cong} W_{G^F}(T, T')\colon w \mapsto w_0 w.$$

Therefore, it is essentially enough to investigate the action of $W_{G^F}(T)$ on T.

Recall that $W_{G^F}(T) := N_{G^F}(T)/T^F$. We also introduce $W_G(T)^F := (N_G(T)/T)^F$. Note the following lemma:

Lemma 8.7. *We have $W_{G^F}(T) \cong W_G(T)^F$.*

Proof. Let $N_{G^F}(T) \hookrightarrow N_G(T)$ be the natural inclusion, which induces an inclusion $N_{G^F}(T)/T^F \hookrightarrow N_G(T)/T$. The image of this inclusion is obviously fixed by F, thus we get a natural inclusion

$$W_{G^F}(T) = N_{G^F}(T)/T^F \hookrightarrow (N_G(T)/T)^F = W_G(T)^F.$$

To show the surjectivity, let us take an element $w \in W_G(T)^F$ and its representative $n \in N_G(T)$. Since w is fixed by F, there exists an element $t \in T$ satisfying $F(n) = nt$. We apply Lang's theorem to $t \in T$; then we can find an element $s \in T$ satisfying $s^{-1}F(s) = t$. We let $n' := ns^{-1}$. As we have $F(n') = F(n)F(s)^{-1} = F(n)t^{-1}s^{-1} = ns^{-1} = n'$, we have $n' \in N_{G^F}(T)$. Moreover, obviously n' and n maps to w. This completes the proof. □

[23]Since $W_{G^F}(T, T')$ is not a group, it is better to say "how $W_{G^F}(T, T')$ transports T to T'"

Based on this lemma, let us consider $W_G(T)^F$ instead of $W_{G^F}(T)$. We review how the (G^F-conjugacy classes of) k-rational maximal tori of G are classified. Let B_0 be a k-rational Borel subgroup G and T_0 be a k-rational maximal torus of G contained in B_0. We write W_0 for the Weyl group $W_0 := W_G(T_0) := N_G(T_0)/T_0$.[24] Note that this is a finite group on which F (the Frobenius endomorphism of G) acts. In Week 5, we (Cheng-Chiang) discussed that there exists a bijection

$$\{k\text{-rational maximal tori of } G\}/G^F\text{-conj.} \to W_0/F\text{-conj.}$$

Let $w \in W_0$. Let us recall how to produce a k-rational maximal torus T_w corresponding to w. We take a representative $n \in N_G(T_0)$ of w and apply the Lang's theorem to n; we can find $g \in G$ satisfying $g^{-1}F(g) = n$. If we put $T_w := {}^gT_0 = gT_0g^{-1}$, then T gives a k-rational maximal torus of G corresponding to (the F-conjugacy class of) w under the above bijection. The action of F on T_w is described as follows:

$$\begin{array}{ccc} T_w & \xleftarrow{\mathrm{Int}(g)} & T_0 \\ {\scriptstyle F}\downarrow & & \\ T_w & \xrightarrow[\mathrm{Int}(g)^{-1}]{} & T_0 \end{array} \qquad \begin{array}{ccc} gtg^{-1} & \longleftarrow\!\!\!\!\dashv & t \\ \downarrow & & \\ F(g)F(t)F(g)^{-1} & \longmapsto & g^{-1}F(g)F(t)F(g)^{-1}g = \mathrm{Int}(w) \circ F(t) \end{array}$$

Hence, in particular, we have an isomorphism

$$\mathrm{Int}(g)\colon T_0^{\mathrm{Int}(w)\circ F} \xrightarrow{\cong} T_w^F; \quad t \mapsto gtg^{-1}.$$

Note that $\mathrm{Int}(g)$ also gives an identification $W_0 = W_G(T_0) \xrightarrow{\cong} W_G(T_w)\colon w \mapsto gwg^{-1}$, which induces

$$\mathrm{Int}(g)\colon W_0^{\mathrm{Int}(w)\circ F} \xrightarrow{\cong} W_G(T_w)^F; \quad w \mapsto gwg^{-1}.$$

Example 8.8. Let $G = \mathrm{GL}_n$ and T_0 be the diagonal maximal torus of G. Then W_0 is naturally identified with $\mathfrak{S}_n$, which is realized as the subgroup of permutation matrices in $\mathrm{GL}_n(\mathbb{F}_q)$. In this case, the Frobenius action F on W_0 is trivial.

(1) When $w = 1$, we have

$$T_0^{\mathrm{Int}(w)\circ F} = T_0^F = \{\mathrm{diag}(t_1, \ldots, t_n) \mid t_i \in \mathbb{F}_q^\times\}.$$

The action of $W_0^{\mathrm{Int}(w)\circ F} = W_0 = \mathfrak{S}_n$ on this group is given by the natural permutation action.

(2) When w is the cyclic permutation $(1\,2\,\ldots\,n)$, we have

$$T_0^{\mathrm{Int}(w)\circ F} = \{\mathrm{diag}(t_1, t_1^q \ldots, t_1^{q^{n-1}}) \mid t_1 \in \mathbb{F}_{q^n}^\times\}$$

(see Week 5 notes for details). Note that $W_0^{\mathrm{Int}(w)\circ F} = W_0^{\mathrm{Int}(w)}$ is nothing but the centralizer of $w = (1\,2\,\ldots\,n)$ in $\mathfrak{S}_n$. We can check that it is the subgroup $\langle w\rangle$ generated by w. Since $w(t_1, t_1^q \ldots, t_1^{q^{n-1}}) = (t_1^q \ldots, t_1^{q^{n-1}}, t_1) = (t_1^q \ldots, t_1^{q^{n-1}}, t_1^{q^n})$, the action of $\langle w\rangle$ on $T_0^{\mathrm{Int}(w)\circ F}$ is identified with the action of $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$ on $\mathbb{F}_{q^n}^\times$.

[24] Caution: this is the "absolute" Weyl group taken in G, while we consider the "relative" Weyl group taken in G^F in the inner product formula.

8.3. **Example: the case of** GL_2**.** Let $G = \mathrm{GL}_2$. Recall that we exactly have two non-isomorphic k-rational maximal tori of G (up to G^F-conjugacy): the split one T and the non-split one S.

(1) For the split one T, we have $T^F = T(\mathbb{F}_q) \cong (\mathbb{F}_q^\times)^2$ and $W_{G^F}(T) \cong \mathfrak{S}_2$; $\mathfrak{S}_2$ acts on $(\mathbb{F}_q^\times)^2$ by swapping two entries. Therefore, for any character $\boldsymbol{\chi} = \chi_1 \boxtimes \chi_2$ of $(\mathbb{F}_q^\times)^2$, we have that
   - $R_T^G(\boldsymbol{\chi})$ is irreducible (up to sign) if $\chi_1 \neq \chi_2$ ($\boldsymbol{\chi}$ is regular), and
   - $R_T^G(\boldsymbol{\chi})$ consists of two irreducible representations (up to sign) if $\chi_1 = \chi_2$.

(2) For the non-split one S, we have $S^F = S(\mathbb{F}_q) \cong \mathbb{F}_{q^2}^\times$ and $W_{G^F}(S) = \mathbb{Z}/2\mathbb{Z}$; $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathbb{F}_{q^2}^\times$ via $\mathrm{Gal}(\mathbb{F}_{q^2}/\mathbb{F}_q)$. Therefore, for any character θ of $\mathbb{F}_{q^2}^\times$, we have that
   - $R_S^G(\theta)$ is irreducible (up to sign) if $\theta^q \neq \theta$ (θ is regular), and
   - $R_S^G(\theta)$ consists of two irreducible representations (up to sign) if $\theta^q = \theta$.

Recall that, in Week 6, we proved that $R_{T\subset B}^G(\boldsymbol{\chi}) \cong \mathrm{Ind}_B^G(\boldsymbol{\chi})$. Also recall that, in Week 2, we proved that $\mathrm{Ind}_{B(\mathbb{F}_q)}^{\mathrm{GL}_2(\mathbb{F}_q)} \boldsymbol{\chi}$ is irreducible when $\chi_1 \neq \chi_2$ and consists of two irreducible representations when $\chi_1 = \chi_2$. Therefore, the computation in the bove example is perfectly consistent with those!

Exercise 8.9. For any θ of S^F satisfying $\theta^{q-1} \neq \mathbb{1}$, we have $R_S^G(\theta) \cong -\pi_\theta$.

Hint: Recall that the irreducible representations of $\mathrm{GL}_2(\mathbb{F}_q)$ are classified as follows (see Week 2 notes):

(1) Characters of $\mathrm{GL}_2(\mathbb{F}_q)$; $\chi \circ \det$ for a character $\chi \colon \mathbb{F}_q^\times$.
(2) Character twists of the Steinberg representation; $\mathrm{St}_G \otimes (\chi \circ \det)$ for a character $\chi \colon \mathbb{F}_q^\times$.
(3) Irreducible principal series representations; $\mathrm{Ind}_B^G \boldsymbol{\chi}$ for $\boldsymbol{\chi} = \chi_1 \boxtimes \chi_2$ where $\chi_1 \neq \chi_2$.
(4) Irreducible cuspidal representations; $\pi_{\theta'}$ for a character θ' of $\mathbb{F}_{q^2}^\times$ satisfying $\theta'^q \neq \theta'$.

Exclude the first three possibilities by using the inner product formula for $R_T^G(\boldsymbol{\chi})$ and $R_T^G(\theta)$, which implies that necessarily have $R_S^G(\theta) \cong \pm\pi_{\theta'}$ for some θ'. Then compute the characters of $R_S^G(\theta)$ at regular semisimple elements using the Deligne–Lusztig character formula. Compare it with the character computation on $\pi_{\theta'}$ demonstrated in Week 2.

8.4. **Proof of inner product formula for DL representations.** We first prove the inner product formula for Deligne–Lusztig representations by admitting the following:

Theorem 8.10 (Orthogonality relation for Green functions)**.** *Let T and T' be k-rational maximal tori of G. Let B and B' be Borel subgroup of G containing T and Q_T^G and $Q_{T'}^G$ associated Green functions. Then we have*

$$\frac{1}{|G^F|} \sum_{u \in G^F_{\mathrm{unip}}} Q_T^G(u) \cdot Q_{T'}^G(u) = \frac{|N_{G^F}(T,T')|}{|T^F| \cdot |T'^F|}.$$

Proof of Theorem 8.1. Recall that the Jordan decomposition implies that we have the following bijection:

$$\bigsqcup_{s \in G^F_{\mathrm{ss}}} (G_s^\circ)^F_{\mathrm{unip}} \xrightarrow{1:1} G^F \colon u \mapsto su.$$

By using the Deligne–Lusztig character formula, we have

$$\begin{aligned}
&\langle R^G_{T\subset B}(\theta), R^G_{T'\subset B'}(\theta')\rangle\\
&= \frac{1}{|G^F|}\sum_{g\in G^F} R^G_{T\subset B}(\theta)(g)\cdot\overline{R^G_{T'\subset B'}(\theta')(g)}\\
&= \frac{1}{|G^F|}\sum_{s\in G^F_{\mathrm{ss}}}\sum_{u\in (G^\circ_s)^F_{\mathrm{unip}}}\frac{1}{|(G^\circ_s)^F|^2}\sum_{\substack{x\in G^F\\ x^{-1}sx\in T^F}}\theta(x^{-1}sx)Q^{G^\circ_s}_{{}^xT}(u)\sum_{\substack{y\in G^F\\ y^{-1}sy\in T'^F}}\overline{\theta'(y^{-1}sy)Q^{G^\circ_s}_{{}^yT'}(u)}\\
&= \frac{1}{|G^F|}\sum_{s\in G^F_{\mathrm{ss}}}\frac{1}{|(G^\circ_s)^F|^2}\sum_{\substack{x,y\in G^F\\ x^{-1}sx\in T^F\\ y^{-1}sy\in T'^F}}\theta(x^{-1}sx)\overline{\theta'(y^{-1}sy)}\sum_{u\in (G^\circ_s)^F_{\mathrm{unip}}}Q^{G^\circ_s}_{{}^xT}(u)\overline{Q^{G^\circ_s}_{{}^yT'}(u)}.
\end{aligned}$$

Here, note that the values of Green functions are integer (exercise). By applying the orthogonality relation for Green functions (for G°_s), this equals

$$\frac{1}{|G^F|}\sum_{s\in G^F_{\mathrm{ss}}}\frac{1}{|(G^\circ_s)^F|}\sum_{\substack{x,y\in G^F\\ x^{-1}sx\in T^F\\ y^{-1}sy\in T'^F}}\theta(x^{-1}sx)\overline{\theta'(y^{-1}sy)}\frac{|N_{(G^\circ_s)^F}({}^xT,{}^yT')|}{|{}^xT^F|\cdot|{}^yT'^F|}$$

$$\frac{1}{|G^F|}\sum_{s\in G^F_{\mathrm{ss}}}\frac{1}{|(G^\circ_s)^F|\cdot|T^F|^2}\sum_{\substack{x,y\in G^F\\ x^{-1}sx\in T^F\\ y^{-1}sy\in T'^F}}\theta(x^{-1}sx)\overline{\theta'(y^{-1}sy)}\cdot|N_{(G^\circ_s)^F}({}^xT,{}^yT')|.$$

Here, we note that the following two sets are bijective by the map $(x,y,n)\mapsto(x,y^{-1}nx,n)$ and its inverse $(x,nxn'^{-1},n)\leftarrow\!\shortmid(x,n',n)$:

$$\{(x,y,n)\in G^F\times G^F\times G^F \mid x^{-1}sx\in T^F, y^{-1}sy\in T'^F, n\in N_{(G^\circ_s)^F}({}^xT,{}^yT')\},$$

$$\{(x,n',n)\in G^F\times N_{G^F}(T,T')\times (G^\circ_s)^F \mid x^{-1}sx\in T^F\}.$$

Hence, the above sum equals

$$\frac{1}{|G^F|}\sum_{s\in G^F_{\mathrm{ss}}}\frac{1}{|(G^\circ_s)^F|\cdot|T^F|^2}\sum_{\substack{x\in G^F\\ n'\in N_{G^F}(T,T')\\ n\in (G^\circ_s)^F\\ x^{-1}sx\in T^F}}\theta(x^{-1}sx)\overline{\theta'((nxn'^{-1})^{-1}s(nxn'^{-1}))}.$$

As n commutes with s, we have

$$\theta'((nxn'^{-1})^{-1}s(nxn'^{-1}))=\theta'(n'x^{-1}sxn'^{-1})={}^{n'^{-1}}\theta'(x^{-1}sx).$$

In particular, this is independent of $n\in(G^\circ_s)^F$. Thus we get

$$\frac{1}{|G^F|\cdot|T^F|^2}\sum_{s\in G^F_{\mathrm{ss}}}\sum_{\substack{x\in G^F\\ n'\in N_{G^F}(T,T')\\ x^{-1}sx\in T^F}}\theta(x^{-1}sx)\overline{{}^{n'^{-1}}\theta'(x^{-1}sx)}.$$

We finally note that the following map is surjective

$$\{(s,x)\in G^F_{\mathrm{ss}}\times G^F \mid x^{-1}sx\in T^F\}\twoheadrightarrow T^F\colon (s,x)\mapsto x^{-1}sx.$$

Moreover, each fiber is of order $|G^F|$. Therefore, we get

$$\begin{aligned}
&\frac{1}{|G^F|\cdot|T^F|^2}\sum_{s\in G^F_{\mathrm{ss}}}\sum_{\substack{x\in G^F\\ n'\in N_{G^F}(T,T')\\ x^{-1}sx\in T^F}}\theta(x^{-1}sx)\overline{{}^{n'^{-1}}\theta'(x^{-1}sx)}\\
&=\frac{1}{|T^F|^2}\sum_{t\in T^F}\sum_{n'\in N_{G^F}(T,T')}\theta(t)\overline{{}^{n'^{-1}}\theta'(t)}\\
&=\sum_{w\in W_{G^F}(T,T')}\frac{1}{|T^F|}\sum_{t\in T^F}\theta(t)\overline{{}^{w^{-1}}\theta'(t)}\\
&=\sum_{w\in W_{G^F}(T,T')}\begin{cases}1 & \text{if } \theta={}^{w^{-1}}\theta',\\ 0 & \text{if } \theta\neq{}^{w^{-1}}\theta',\end{cases}\\
&=|\{w\in W_{G^F}(T,T')\mid {}^{w}\theta=\theta'\}|.
\end{aligned}$$

□

Exercise 8.11. For any connected reductive group G over k and its k-rational maximal torus T, prove that the Green function $Q^G_T(-)$ is $\mathbb{Z}$-valued.

Hint: Describe the Green function using a Lefschetz number by going back to the definition. Then utilize the fact that the Lefschetz number is an integer.

9. Week 9: Semisimple character formula and exhaustion theorem

Recall that we proved the inner product formula for Deligne–Lusztig representations by assuming the orthogonality relation for Green functions. The aim of this week is to partially prove the orthogonality relation. More precisely, we introduce another result which we call the "disjointness theorem" and then deduce the orthogonality relation from the disjointness theorem.

Chart: Disjointness Theorem (Theorem 9.4, not proved this week)

$\overset{\text{this week}}{\Longrightarrow}$ Orthogonality relation for Green functions

$\overset{\text{last week}}{\Longrightarrow}$ Inner product formula for Deligne–Lusztig representaitons

(the second $\Longrightarrow$ is in fact $\Longleftrightarrow$).

After that, we also prove that any irreducible representation can be realized in some Deligne–Lusztig representation.

9.1. **Geometric conjugacy and disjointness theorem.** Let G be a connected reductive group over $k = \mathbb{F}_q$ with associated Frobenius endomorphism F. Suppose that T is a k-rational maximal torus of G. Note that then we have $T^{F^r} = T(\mathbb{F}_{q^r})$ for any $r \in \mathbb{Z}_{>0}$. We define the *norm map* N_r from T^{F^r} to T^F by

$$N_r\colon T^{F^r} \to T^F; \quad t \mapsto t \cdot F(t) \cdots F^{r-1}(t).$$

Note that, if $T = \mathbb{G}_{\mathrm{m}}$, then N_r is nothing but the usual norm map from $T^{F^r} = \mathbb{F}_{q^r}^\times$ to $T^F = \mathbb{F}_q^\times$. Recall that the norm map from $\mathbb{F}_{q^r}^\times$ to $T^F = \mathbb{F}_q^\times$ is surjective. In fact, the same property holds for the norm map for any T:

Lemma 9.1. *The norm map $N_r\colon T^{F^r} \to T^F$ is surjective.*

Exercise 9.2. Prove this lemma.

Hint: Suppose $t \in T^F$. Apply Lang's theorem for $F^r\colon T \to T$ to t; then we get an $s \in T$ satisfying $F^r(s)s^{-1} = t$. Show that $F(s)s^{-1}$ belongs to T^{F^r} and maps to t under N_r.

Definition 9.3. Let T and T' be k-rational maximal tori of G. We say that characters θ of T^F and θ' of T'^F are *geometrically conjugate* if $(T, \theta\circ N_r)$ and $(T', \theta'\circ N_r)$ are G^{F^r}-conjugate for some $r \in \mathbb{Z}_{>0}$, i.e., there exists $x \in G^{F^r}$ satisfying $T' = {}^xT$ and $\theta' \circ N_r = {}^x(\theta \circ N_r)$.

Note that if θ and θ' are conjugate, then they are geometrically conjugate (r can be taken to be 1).

The following theorem is a key to the proof of the orthogonality relation (for convenience, let us call the following the "disjointness theorem"):

Theorem 9.4 (Disjointness theorem)**.** *Let T and T' be k-rational maximal tori of G. Suppose that characters θ of T^F and θ' of T'^F are not geometrically conjugate. Then $R^G_{T\subset B}(\theta)$ and $R^G_{T'\subset B'}(\theta')$ do not contain a common irreducible representation.*

Remark 9.5. (1) The precise meaning of "a virtual representation R contains an irreducible representation σ" is that "if we write R as the sum $\sum_\rho n_\rho \rho$ over all (isomorphism classes of) irreducible representations ($n_\rho \in \mathbb{Z}$), then $n_\sigma \neq 0$". Each coefficient n_ρ is often called the "multiplicity" of ρ in R.

(2) Recall that, as a corollary of the inner product formula, we obtained that "if θ and θ' are not G^F-conjugate, then $\langle R^G_{T\subset B}(\theta), R^G_{T'\subset B'}(\theta')\rangle = 0$". On the other hand, the statement of Theorem 9.4 is stronger than the equality $\langle R^G_{T\subset B}(\theta), R^G_{T'\subset B'}(\theta')\rangle = 0$. Thus both the assumption and the conclusion of Theorem 9.4 are stronger than (a consequence of) the inner product formula.

9.2. **Orthogonality relation for Green functions.** Recall that, for any connected reductive group G with center Z, the quotient group $G_{\mathrm{ad}} := G/Z$ is of adjoint type (i.e., a connected reductive group with trivial center). Moreover, it is not difficult to see the following.

- For any k-rational maximal torus T of G, its image T_{ad} in G_{ad} is a k-rational maximal torus of G_{ad}.
- The natural quotient map $G \to G_{\mathrm{ad}}$ induces a bijection $G^F_{\mathrm{unip}} \xrightarrow{1:1} G^F_{\mathrm{ad,unip}}$.

Lemma 9.6. *For any $u \in G^F_{\mathrm{unip}}$, we have $Q^G_T(u) = Q^{G_{\mathrm{ad}}}_{T_{\mathrm{ad}}}(\bar{u})$, where $\bar{u} \in G^F_{\mathrm{ad,unip}}$ is the image of u.*

Sketch of Proof. This follows from an alternative description of the Green function in terms of the variants of the Deligne–Lusztig varieties. A bit more precisely, the Green function Q^G_T can be also interpreted as the Lefschetz number of the action of G^F_{unip} on the variety " $X^G_{T\subset B}$" (see Week 5 notes). We can easily check that $X^G_{T\subset B}$ is canonically isomorphic to $X^{G_{\mathrm{ad}}}_{T_{\mathrm{ad}}\subset B_{\mathrm{ad}}}$, which implies that $Q^G_T(u) = Q^{G_{\mathrm{ad}}}_{T_{\mathrm{ad}}}(\bar{u})$. See [DL76, Definition 1.9] and its preceding remark for more details. □

Theorem 9.7 (Orthogonality relation for Green functions)**.** *Let T and T' be k-rational maximal tori of G. Let B and B' be Borel subgroup of G containing T and Q^G_T and $Q^G_{T'}$ associated Green functions. Then we have*

$$\frac{1}{|G^F|}\sum_{u\in G^F_{\mathrm{unip}}} Q^G_T(u)\cdot Q^G_{T'}(u) = \frac{|N_{G^F}(T,T')|}{|T^F|\cdot|T'^F|}.$$

Proof. The asserted identity is trivial if G is a torus. We handle the general case by the induction on $\dim G$. (Note that any 1-dimensional connected reductive group is a torus.) We also note that the desired identity does not change even if we replace G with G_{ad}. (The Green functions do not change by the previous lemma; all other numbers are multiplied by the same number.) Thus we may suppose that G is of adjoint type in the following.

Here, let us remember the proof of the inner product formula. For any characters θ of T^F and θ' of T'^F, we first computed $\langle R^G_{T\subset B}(\theta), R^G_{T'\subset B'}(\theta')\rangle$ as follows:

$$(*)\quad \langle R^G_{T\subset B}(\theta), R^G_{T'\subset B'}(\theta')\rangle = \frac{1}{|G^F|}\sum_{s\in G^F_{\mathrm{ss}}}\frac{1}{|(G^\circ_s)^F|^2}\sum_{\substack{x,y\in G^F\\ x^{-1}sx\in T^F\\ y^{-1}sy\in T'^F}}\theta(x^{-1}sx)\overline{\theta'(y^{-1}sy)}\sum_{u\in(G^\circ_s)^F_{\mathrm{unip}}} Q^{G^\circ_s}_{{}^xT}(u)\overline{Q^{G^\circ_s}_{{}^yT'}(u)}.$$

Then we utilized the orthogonality relation to rewrite this as follows:

$$= \cdots = \frac{1}{|G^F| \cdot |T^F|^2} \sum_{s \in G^F_{\mathrm{ss}}} \sum_{\substack{x \in G^F \\ n' \in N_{G^F}(T,T') \\ x^{-1}sx \in T^F}} \theta(x^{-1}sx)\overline{{}^{n'^{-1}}\theta'(x^{-1}sx)}$$

$$= \cdots = \sum_{w \in W_{G^F}(T,T')} \frac{1}{|T^F|} \sum_{t \in T^F} \theta(t)\overline{{}^{w^{-1}}\theta'(t)} = \cdots .$$

The point here is that, in the current setting, the same computation works for any $s \neq 1$. This is because, since G has trivial center, any nontrivial semisimple element s satisfies $\dim G_s^\circ < \dim G$, hence we can apply the induction hypothesis to G_s°. (Note that the condition that $s \neq 1$ is rephrased as the condition that $t \neq 1$ in the last sum.) On the other hand, for $s = 1$, the contribution to $(*)$ is simply given by

$$\frac{1}{|G^F|} \sum_{u \in G^F_{\mathrm{unip}}} Q^G_T(u)\overline{Q^G_{T'}(u)}.$$

Therefore, we see that $(*)$ is equal to

$$\underbrace{\frac{1}{|G^F|} \sum_{u \in G^F_{\mathrm{unip}}} Q^G_T(u)\overline{Q^G_{T'}(u)}}_{s=1} + \underbrace{\sum_{w \in W_{G^F}(T,T')} \frac{1}{|T^F|} \sum_{t \in T^F \smallsetminus \{1\}} \theta(t)\overline{{}^{w^{-1}}\theta'(t)}}_{s \neq 1}.$$

Here note that the second term for $s \neq 1$ can be computed as follows:

$$\begin{aligned}
&\sum_{w \in W_{G^F}(T,T')} \frac{1}{|T^F|} \sum_{t \in T^F \smallsetminus \{1\}} \theta(t)\overline{{}^{w^{-1}}\theta'(t)} \\
&= \sum_{w \in W_{G^F}(T,T')} \frac{1}{|T^F|} \sum_{t \in T^F} \theta(t)\overline{{}^{w^{-1}}\theta'(t)} - \frac{|W_{G^F}(T,T')|}{|T^F|} \\
&= |\{w \in W_{G^F}(T,T') \mid {}^w\theta = \theta'\}| - \frac{|W_{G^F}(T,T')|}{|T^F|}.
\end{aligned}$$

In other words, we obtained

$$\begin{aligned}
&\frac{1}{|G^F|} \sum_{u \in G^F_{\mathrm{unip}}} Q^G_T(u)\overline{Q^G_{T'}(u)} \\
&\qquad = \langle R^G_{T\subset B}(\theta), R^G_{T'\subset B'}(\theta')\rangle - |\{w \in W_{G^F}(T,T') \mid {}^w\theta = \theta'\}| + \frac{|W_{G^F}(T,T')|}{|T^F|}.
\end{aligned}$$

This shows the following:

> To obtain the orthogonality relation for Q^G_T and $Q^G_{T'}$, it is enough to find just one example of a pair (θ, θ') satisfying the inner product formula for $\langle R^G_{T\subset B}(\theta), R^G_{T'\subset B'}(\theta')\rangle$ (of course, in a way which is not based on the orthogonality relation)!

We first consider the case where either T^F or T'^F has a nontrivial character; we may assume that T^F has a nontrivial character θ. Note that, since the norm map for a torus is surjective (Lemma 9.1), θ cannot be geometrically conjugate to the trivial character of T'^F. Thus, by the disjointness theorem (Theorem 9.4), we have $\langle R^G_{T\subset B}(\theta), R^G_{T'\subset B'}(\mathbb{1})\rangle = 0$. On the other hand, obviously we have $|\{w \in W_{G^F}(T,T') \mid {}^w\theta = \mathbb{1}\}| = 0$.

We next consider the case where T^F and T'^F do not have a nontrivial character; this is equivalent to that $|T^F| = |T'^F| = 1$. In this case, q must be 2 and T and T' must be split over k. (We leave this for an exercise below.) This implies that T and T' are G^F-conjugate to a split k-rational maximal torus T_0 and also that $R^G_{T\subset B}(\mathbb{1}) \cong R^G_{T'\subset B'}(\mathbb{1}) \cong \mathrm{Ind}_{B_0^F}^{G^F} \mathbb{1}$ (we proved this in Week 6), where B_0 is a k-rational Borel subgroup of G containing T_0. Then we can check that

$$\langle R^G_{T\subset B}(\mathbb{1}), R^G_{T'\subset B'}(\mathbb{1})\rangle = \langle \mathrm{Ind}_{B_0^F}^{G^F} \mathbb{1}, \mathrm{Ind}_{B_0^F}^{G^F} \mathbb{1}\rangle = |W_{G^F}(T)|$$

(let me also leave this for an exercise!). On the other hand, obviously we have $|\{w \in W_{G^F}(T,T') \mid {}^w\mathbb{1} = \mathbb{1}\}| = |W_{G^F}(T)|$.

Therefore, in both cases, we found an example of a pair (θ, θ') satisfying the inner product formula. This completes the proof. □

Exercise 9.8. Let T be a k-rational maximal torus of a connected reductive group G over k. Prove that $|T^F| = 1$ only when $q = 2$ and T is split over k.

Hint: utilize the formula of $|T^F|$ in terms of the character group of T; see Week 5 notes.

Exercise 9.9. Prove that

$$\langle \mathrm{Ind}_{B_0^F}^{G^F} \mathbb{1}, \mathrm{Ind}_{B_0^F}^{G^F} \mathbb{1}\rangle = |W_{G^F}(T)|.$$

Hint: Recall that we proved this in the GL_2 case in Week 2. In fact, the same argument works; combine (1) Frobenius reciprocity, (2) Mackey decomposition formula, and (3) Bruhat decomposition.

9.3. **Steinberg representations.** Recall that, for $G = \mathrm{GL}_2$, the principal series representation $\mathrm{Ind}_B^G \mathbb{1}$ is the sum of two irreducible representations; the trivial representation and the Steinberg representations. (In this subsection, we temporarily omit the symbol "F" in the induced representations to make the notation lighter.) In fact, the notion of the Steinberg representation can be generalized to any connected reductive group over k.

Instead of explaining its definition in general, let us present an example of GL_3. Let $G := \mathrm{GL}_3$ and B be the upper-triangular Borel subgroup of G. We consider the principal series representation $\mathrm{Ind}_B^G \mathbb{1}$. Then, as in the GL_2 case, $\mathrm{Ind}_B^G \mathbb{1}$ contains the trivial representation. However, the different point is that $\mathrm{Ind}_B^G \mathbb{1}$ contains further more irreducible representations. To see this, let us consider the following subgroup:

$$P_{2,1} := \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \subset G.$$

Since B is contained in $P_{2,1}$, the associativity of the induction implies that

$$\mathrm{Ind}_B^G \mathbb{1} = \mathrm{Ind}_{P_{2,1}}^G (\mathrm{Ind}_B^{P_{2,1}} \mathbb{1}) \supset \mathrm{Ind}_{P_{2,1}}^G \mathbb{1}.$$

Then, how about subtracting $\mathrm{Ind}_{P_{2,1}}^G \mathbb{1}$ from $\mathrm{Ind}_B^G \mathbb{1}$? In fact, the remaining representation is still not irreducible! So let us also consider the following subgroup:

$$P_{1,2} := \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \subset G.$$

Then, for the same reason as above, we have $\mathrm{Ind}_B^G \mathbb{1} \supset \mathrm{Ind}_{P_{1,2}}^G \mathbb{1}$. In fact, $\mathrm{Ind}_{P_{1,2}}^G \mathbb{1}$ is a different subrepresentation from $\mathrm{Ind}_{P_{2,1}}^G \mathbb{1}$. So, how about subtracting both $\mathrm{Ind}_{P_{1,2}}^G \mathbb{1}$ and

$\mathrm{Ind}_{P_{2,1}}^{G}\mathbb{1}$ from $\mathrm{Ind}_{B}^{G}\mathbb{1}$? This also does not work because both $\mathrm{Ind}_{P_{1,2}}^{G}\mathbb{1}$ and $\mathrm{Ind}_{P_{2,1}}^{G}\mathbb{1}$ contains the trivial representation, but the multiplicity of the trivial representation in $\mathrm{Ind}_{B}^{G}\mathbb{1}$ is exactly one! In other words, the trivial representation is subtracted doubly. We shouldn't give up here; how about considering the following representation:

$$(\mathrm{Ind}_{B}^{G}\mathbb{1})-(\mathrm{Ind}_{P_{1,2}}^{G}\mathbb{1})-(\mathrm{Ind}_{P_{2,1}}^{G}\mathbb{1})+\mathbb{1}.$$

In fact, this gives an irreducible subrepresentation of $\mathrm{Ind}_{B}^{G}\mathbb{1}$! This is the definition of the Steinberg representation of $\mathrm{GL}_3(\mathbb{F}_q)$.

In general, the Steinberg representation is defined according to a similar idea. The subgroups $P_{1,2}$ and $P_{2,1}$ are examples of so-called *parabolic subgroups* of G. The idea is to consider a certain signed sum of the induced representations from all parabolic subgroups based on the "inclusion-exclusion principle" as in the GL_3 case. The Steinberg representation can be investigated independently of Deligne–Lusztig theory. The precise definition of the Steinberg representation of G^F (let us write St_G) and its basic properties are summarized in, for example, [Car85, Chapter 6].

Therefore, in this course, let us just believe the existence of the representation St_G of G^F satisfying the following properties.

Proposition 9.10 (Character formula for St_G)**.** *For any $g\in G^F$, we have*

$$\mathrm{St}_G(g)=\begin{cases}(-1)^{r_G-r_{G_s^\circ}}\cdot\mathrm{St}_{G_s^\circ}(1) & \text{if } g=s \text{ is semisimple,}\\ 0 & \text{otherwise.}\end{cases}$$

Here, for any connected reductive group G over k, we let r_G denote its k-split rank, i.e., the dimension of the maximal k-split torus of G.

Proposition 9.11 (Dimension formula)**.** *We let B_0 be a k-rational Borel subgroup of G with unipotent radical U_0. Then we have* $\mathrm{St}_G(1)=|U_0^F|$.

Exercise 9.12. Show that the above propositions in the case where $G=\mathrm{GL}_2$.

9.4. **Character formula for DL representations on semisimple elements.**

Theorem 9.13 (Dimension formula)**.** *We have*

$$R_T^G(1)=(-1)^{r_G-r_T}\cdot\frac{|G^F|}{|T^F|\cdot\mathrm{St}_G(1)}=(-1)^{r_G-r_T}\cdot|G^F/T^F|_{p'},$$

where $(-)_{p'}$ denotes the prime-to-p part.

Proof. In fact, $|U_0^F|$ is equal to the p-part of $|G^F|$. On the other hand, $|T^F|$ is prime-to-p for any k-rational maximal torus of G. Hence, by the dimension formula of the Steinberg representation, we have

$$|G^F/T^F|_{p'}=\frac{|G^F|}{|T^F|\cdot\mathrm{St}_G(1)}.$$

Thus our task is to show the first equality.

Recall that $R_T^G(1)=Q_T^G(1)$ by the definition of the Green function. By the same reasoning as in the proof of the orthogonality relation, we may assume that G is of adjoint type. Moreover, by induction on $\dim G$, we may assume that the identity holds for G_s° for any semisimple $s\neq 1$.

We first consider the case where T^F does not have a nontrivial character. Recall that, in this case, T must be a split maximal torus T_0. Thus we have

$$R_T^G(1) = \dim \operatorname{Ind}_{B_0^F}^{G^F} \mathbb{1} = \frac{|G^F|}{|B_0^F|} = \frac{|G^F|}{|T_0^F| \cdot |U_0^F|} = \frac{|G^F|}{|T^F| \cdot |\mathrm{St}_G(1)|}.$$

We next consider the case where T^F has a nontrivial character θ. Recall that the Steinberg representation St_G is contained in $R_{T_0}^G(\mathbb{1}) = \operatorname{Ind}_{B_0^F}^{G^F} \mathbb{1}$. Thus, by applying the disjointness theorem to $R_T^G(\theta)$ and $R_{T_0}^G(\mathbb{1})$, we get

$$\langle R_T^G(\theta), \mathrm{St}_G \rangle = 0.$$

On the other hand, we can also compute $\langle R_T^G(\theta), \mathrm{St}_G \rangle$ directly by using the Deligne–Lusztig character formula and the character formula for Steinberg representations as follows:

$$\begin{aligned}
\langle R_T^G(\theta), \mathrm{St}_G \rangle &= \frac{1}{|G^F|} \sum_{g \in G^F} R_T^G(\theta)(g) \cdot \overline{\mathrm{St}_G(g)} \\
&= \frac{1}{|G^F|} \sum_{s \in G_{\mathrm{ss}}^F} \sum_{u \in (G_s^\circ)_{\mathrm{unip}}^F} R_T^G(\theta)(su) \cdot \overline{\mathrm{St}_G(su)} \\
&= \frac{1}{|G^F|} \sum_{s \in G_{\mathrm{ss}}^F} R_T^G(\theta)(s) \cdot \overline{\mathrm{St}_G(s)} \\
&= \frac{1}{|G^F|} \sum_{s \in G_{\mathrm{ss}}^F} \frac{1}{|(G_s^\circ)^F|} \sum_{\substack{x \in G^F \\ x^{-1}sx \in T^F}} {}^x\theta(s) \cdot Q_{{}^xT}^{G_s^\circ}(1) \cdot (-1)^{r_G - r_{G_s^\circ}} \cdot \mathrm{St}_{G_s^\circ}(1).
\end{aligned}$$

The idea of the proof is similar to that of the orthogonality relation. We divide the above sum according to $s = 1$ or $s \neq 1$. For $s = 1$, the summand is $Q_T^G(1) \cdot \mathrm{St}_G(1)$. For $s \neq 1$, by the induction hypothesis, the summand is given by

$$\begin{aligned}
&\frac{1}{|(G_s^\circ)^F|} \sum_{\substack{x \in G^F \\ x^{-1}sx \in T^F}} {}^x\theta(s) \cdot Q_{{}^xT}^{G_s^\circ}(1) \cdot (-1)^{r_G - r_{G_s^\circ}} \cdot \mathrm{St}_{G_s^\circ}(1) \\
&= \frac{1}{|(G_s^\circ)^F|} \sum_{\substack{x \in G^F \\ x^{-1}sx \in T^F}} {}^x\theta(s) \cdot (-1)^{r_{G_s^\circ} - r_{{}^xT}} \cdot \frac{|(G_s^\circ)^F|}{|{}^xT^F| \cdot \mathrm{St}_{G_s^\circ}(1)} \cdot (-1)^{r_G - r_{G_s^\circ}} \cdot \mathrm{St}_{G_s^\circ}(1) \\
&= \frac{1}{|T^F|} \sum_{\substack{x \in G^F \\ x^{-1}sx \in T^F}} (-1)^{r_G - r_T} {}^x\theta(s).
\end{aligned}$$

Therefore, since $\langle R_T^G(\theta), \mathrm{St}_G \rangle = 0$, we get

$$\underbrace{Q_T^G(1) \cdot \mathrm{St}_G(1)}_{s=1} + \underbrace{\sum_{s \in G_{\mathrm{ss}}^F \smallsetminus \{1\}} \frac{(-1)^{r_G - r_T}}{|T^F|} \sum_{\substack{x \in G^F \\ x^{-1}sx \in T^F}} {}^x\theta(s)}_{s \neq 1} = 0.$$

By the same trick as in the proof of the orthogonality relation, the second term (the $s \neq 1$ part) on the left-hand side is equal to

$$\frac{|G^F|}{|T^F|} \cdot (-1)^{r_G - r_T} \sum_{t \in T^F \smallsetminus \{1\}} \theta(t) = -\frac{|G^F|}{|T^F|} \cdot (-1)^{r_G - r_T}$$

(we used that θ is a nontrivial character). Hence we get

$$Q_T^G(1) = (-1)^{r_G - r_T} \cdot \frac{|G^F|}{|T^F| \cdot \mathrm{St}_G(1)}.$$

□

Corollary 9.14 (Deligne–Lusztig character formula on semisimple elements)**.** *For any $s \in G^F_{\mathrm{ss}}$, we have*

$$R_T^G(\theta)(s) = \frac{(-1)^{r_{G_s^\circ} - r_T}}{|T^F| \cdot \mathrm{St}_{G_s^\circ}(1)} \sum_{\substack{x \in G^F \\ x^{-1}sx \in T^F}} {}^x\theta(s).$$

Proof. By the Deligne–Lusztig character formula and the dimension formula, we have

$$\begin{aligned} R_T^G(\theta)(s) &= \frac{1}{|(G_s^\circ)^F|} \sum_{\substack{x \in G^F \\ x^{-1}sx \in T^F}} {}^x\theta(s) \cdot Q_{{}^xT}^{G_s^\circ}(1) \\ &= \frac{1}{|(G_s^\circ)^F|} \sum_{\substack{x \in G^F \\ x^{-1}sx \in T^F}} {}^x\theta(s) \cdot (-1)^{r_{G_s^\circ} - r_T} \cdot \frac{|(G_s^\circ)^F|}{|{}^xT^F| \cdot \mathrm{St}_{G_s^\circ}(1)} \\ &= \frac{(-1)^{r_{G_s^\circ} - r_T}}{|T^F| \cdot \mathrm{St}_{G_s^\circ}(1)} \sum_{\substack{x \in G^F \\ x^{-1}sx \in T^F}} {}^x\theta(s). \end{aligned}$$

□

Exercise 9.15. Show that the above corollary implies that $R_T^G(\theta) \otimes \mathrm{St}_G \cong \mathrm{Ind}_{T^F}^{G^F} \theta$.
Hint: Use the Frobenius character formula for induced representations.

9.5. **Exhaustion theorem.** For any $s \in G^F$, we let $\mathbb{1}_{[s]}$ denote the characteristic function of the G^F-conjugacy class $G^F \cdot s := \{xsx^{-1} \mid x \in G^F\}$ of s, i.e., $\mathbb{1}_{[s]} \colon G^F \to \mathbb{C}$ is a class function such that

$$\mathbb{1}_{[s]}(g) = \begin{cases} 1 & g \in G^F \cdot s, \\ 0 & g \notin G^F \cdot s. \end{cases}$$

We write $\mathcal{T}_G$ for the set of k-rational maximal tori of G (literally, all such tori; not G^F-conjugacy classes). For any $T \in \mathcal{T}_G$, we write $(T^F)^\vee$ for the set of characters of T^F.

Proposition 9.16. *For any $s \in G^F_{\mathrm{ss}}$, we have*

$$\frac{1}{\mathrm{St}_G(s)} \sum_{s \in T \in \mathcal{T}} \sum_{\theta \in (T^F)^\vee} (-1)^{r_G - r_T} \cdot \theta(s)^{-1} \cdot R_T^G(\theta) = |(G^F)_s| \cdot \mathbb{1}_{[s]}.$$

Note that $(G^F)_s$ denotes the centralizer of s in G^F.

Proof. We put $\mu :=$ LHS and $\mu' :=$ RHS. To show that $\mu = \mu'$, it is enough to check that $\langle \mu - \mu', \mu - \mu' \rangle = 0$. For this, it suffices to show that all of $\langle \mu, \mu \rangle$, $\langle \mu, \mu' \rangle$, and $\langle \mu', \mu' \rangle$ are equal.

Let us first compute $\langle \mu, \mu \rangle$. By using the inner product formula, we get

$$\begin{aligned}\langle \mu, \mu \rangle &= \frac{1}{\mathrm{St}_G(s)^2} \sum_{\substack{s \in T \in \mathcal{T}_G \\ s \in T' \in \mathcal{T}_G}} \sum_{\substack{\theta \in (T^F)^\vee \\ \theta' \in (T'^F)^\vee}} \theta(s)^{-1} \overline{\theta'(s)^{-1}} \cdot \langle R_T^G(\theta), R_{T'}^G(\theta') \rangle \\ &= \frac{1}{\mathrm{St}_G(s)^2} \sum_{\substack{s \in T \in \mathcal{T}_G \\ s \in T' \in \mathcal{T}_G}} \sum_{\substack{\theta \in (T^F)^\vee \\ \theta' \in (T'^F)^\vee}} \theta(s)^{-1} \overline{\theta'(s)^{-1}} \cdot |\{w \in W_{G^F}(T, T') \mid \theta' = {}^w\theta\}|.\end{aligned}$$

Here, we change the index sets by noting the following bijection:

$$\begin{aligned}&\{((T, \theta), n) \in \mathcal{I} \times G^F \mid s \in T, s \in {}^nT\} \\ &\xrightarrow{1:1} \{((T, \theta), (T'\theta'), n) \in \mathcal{I} \times \mathcal{I} \times G^F \mid s \in T, s \in T', n \in N_{G^F}(T, T'), \theta' = {}^n\theta\} \\ &\qquad\qquad : ((T, \theta), n) \mapsto ((T, \theta), ({}^nT, {}^n\theta), n)\end{aligned}$$

where we put $\mathcal{I}$ to be the set of pairs (T, θ) of $T \in \mathcal{T}_G$ and $\theta \in (T^F)^\vee$. Then the above sum equals

$$\frac{1}{\mathrm{St}_G(s)^2 \cdot |T^F|} \sum_{\substack{((T,\theta),n) \in \mathcal{I} \times G^F \\ s \in T \\ s \in {}^nT}} \theta(s)^{-1} \cdot {}^n\theta(s).$$

We note that the sum of $\theta(s)^{-1n} \cdot \theta(s) = \theta(s^{-1}n^{-1}sn)$ over all characters θ of T^F is zero when $s^{-1}n^{-1}sn \neq 1$ and equal to $|T^F|$ when $s^{-1}n^{-1}sn = 1$ (equivalently, $n \in (G^F)_s$). Therefore, the above equals

$$\frac{1}{\mathrm{St}_G(s)^2 \cdot |T^F|} \sum_{\substack{(T,n) \in \mathcal{T}_G \times (G^F)_s \\ s \in T}} |T^F| = \frac{|(G^F)_s|}{\mathrm{St}_G(s)^2} \cdot |\{T \in \mathcal{T}_G \mid s \in T\}|.$$

We note that $\mathrm{St}_G(s) = (-1)^{r_G - r_{(G_s)^\circ}} \mathrm{St}_{G_s^\circ}(1)$ and also that $s \in T$ if and only if $T \subset G_s^\circ$ (hence $\{T \in \mathcal{T}_G \mid s \in T\}$ is nothing but $\mathcal{T}_{G_s^\circ}$). Then, by using the fact that $|\mathcal{T}_{G_s^\circ}| = \mathrm{St}_{G_s^\circ}(1)^2$ (see [Car85, Theorem 3.4.1]), we arrive at

$$\langle \mu, \mu \rangle = |(G^F)_s|.$$

Let us next compute $\langle \mu, \mu' \rangle$. We note that, for any class function $f \in C(G)$, we have $\langle f, \mathbb{1}_{[s]} \rangle = f(s)$. Indeed,

$$\langle f, \mathbb{1}_{[s]} \rangle = \frac{1}{|G^F|} \sum_{g \in G^F \cdot s} f(g) \cdot |(G^F)_s| = \frac{|G^F \cdot s| \cdot |(G^F)_s|}{|G^F|} f(s) = f(s).$$

Keeping this in mind, by using the Deligne–Lusztig character formula on semisimple elements, we get

$$\begin{aligned}
\langle \mu, \mu' \rangle &= \mu(s) \\
&= \frac{1}{\mathrm{St}_G(s)} \sum_{s \in T \in \mathcal{T}_G} \sum_{\theta \in (T^F)^\vee} (-1)^{r_G - r_T} \cdot \theta(s)^{-1} \cdot R_T^G(\theta)(s) \\
&= \frac{1}{\mathrm{St}_G(s)} \sum_{s \in T \in \mathcal{T}_G} \sum_{\theta \in (T^F)^\vee} (-1)^{r_G - r_T} \cdot \theta(s)^{-1} \cdot \frac{(-1)^{r_{G_s^\circ} - r_T}}{|T^F| \cdot \mathrm{St}_{G_s^\circ}(1)} \sum_{\substack{x \in G^F \\ x^{-1}sx \in T^F}} \theta^x(s) \\
&= \frac{1}{\mathrm{St}_G(s)} \sum_{s \in T \in \mathcal{T}_G} \sum_{\theta \in (T^F)^\vee} \sum_{\substack{x \in G^F \\ x^{-1}sx \in T^F}} \frac{(-1)^{r_G - r_{G_s^\circ}}}{|T^F| \cdot \mathrm{St}_{G_s^\circ}(1)} \theta(s^{-1}x^{-1}sx).
\end{aligned}$$

Here we carry out a similar argument to the previous computation; the sum of $\theta(s^{-1}x^{-1}sx)$ over all characters θ of T^F is zero when $s^{-1}x^{-1}sx = 1$ and equal to $|T^F|$ when $s^{-1}x^{-1}sx = 1$ (equivalently, $x \in (G^F)_s$). Hence the above equals

$$\frac{1}{\mathrm{St}_G(s)} \sum_{s \in T \in \mathcal{T}_G} \frac{(-1)^{r_G - r_{G_s^\circ}}}{\mathrm{St}_{G_s^\circ}(1)} \cdot |(G^F)_s|.$$

Again noting that the index set is equal to $\mathcal{T}_{G_s^\circ}$ and that $\mathrm{St}_G(s) = (-1)^{r_G - r_{(G_s)^\circ}} \mathrm{St}_{G_s^\circ}(1)$, we conclude

$$\langle \mu, \mu' \rangle = |(G^F)_s|$$

by using [Car85, Theorem 3.4.1].

Let us finally compute $\langle \mu', \mu' \rangle$:

$$\langle \mu', \mu' \rangle = \mu'(s) = |(G^F)_s|.$$

□

Corollary 9.17. *Let ρ be an irreducible representation of G^F. For any $s \in G_{\mathrm{ss}}^F$, we have*

$$\Theta_\rho(s) = \frac{1}{\mathrm{St}_G(s)} \sum_{s \in T \in \mathcal{T}_G} \sum_{\theta \in (T^F)^\vee} (-1)^{r_G - r_T} \cdot \theta(s)^{-1} \cdot \langle \rho, R_T^G(\theta) \rangle.$$

Proof. As noted in the proof of the previous proposition, we have $\Theta_\rho(s) = \langle \Theta_\rho, \mu' \rangle$ with the notation as there. By the proposition, we get $\Theta_\rho(s) = \langle \Theta_\rho, \mu \rangle$; this is nothing but the right-hand side of the asserted equality. □

Theorem 9.18 (Exhaustion theorem)**.** *For any irreducible representation ρ of G^F, there exists a k-rational maximal torus T of G and its character θ such that ρ is contained in $R_T^G(\theta)$.*

Proof. Apply the previous corollary to $s = 1$; then we get

$$\Theta_\rho(1) = \frac{1}{\mathrm{St}_G(1)} \sum_{T \in \mathcal{T}_G} \sum_{\theta \in (T^F)^\vee} (-1)^{r_G - r_T} \cdot \langle \rho, R_T^G(\theta) \rangle.$$

The left-hand side is the dimension of ρ, hence not zero. Thus the right-hand side is also not zero. In particular, $\langle \rho, R_T^G(\theta) \rangle$ must be nonzero for at least one (T, θ). □

10. Week 10: Proof of the orthogonality relation for Green functions

Recall that we proved the inner product formula for Deligne–Lusztig representations by assuming the following:

Theorem 10.1 (Disjointness theorem)**.** *Let T and T' be k-rational maximal tori of G. Suppose that characters θ of T^F and θ' of T'^F are not geometrically conjugate. Then $R^G_{T\subset B}(\theta)$ and $R^G_{T'\subset B'}(\theta')$ do not contain a common irreducible representation.*

The aim of this week is to prove the disjointness theorem.

10.1. **Preliminary reduction.** Before we prove the disjointness theorem, let us introduce some purely-algebraic lemmas. Recall that, for any representation (ρ, V) of G^F, its dual (*contragredient*) representation $(\rho^\vee, V^\vee)$ is defined by $V^\vee := \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ and

$$\langle \rho^\vee(g)(v^\vee), v\rangle = \langle v^\vee, \rho(g^{-1})(v)\rangle$$

for any $g \in G^F$, $v \in V$, $v^\vee \in V^\vee$.

Lemma 10.2. *For any representation ρ of G^F, we have $\Theta_{\pi^\vee}(g) = \Theta_\pi(g^{-1}) = \overline{\Theta_\pi(g)}$.*

Exercise 10.3. Prove Lemma 10.2.

Lemma 10.4. *We have $R^G_{T\subset B}(\theta)^\vee \cong R^G_{T\subset B}(\theta^{-1})$.*

Proof. By Lemma 10.2, to prove the assertion, it suffices to check that $\overline{R^G_{T\subset B}(\theta)(g)} = R^G_{T\subset B}(\theta^{-1})(g)$ for any $g \in G^F$. If we write $g = su$ for the Jordan decomposition of g, then, by the Deligne–Lustig character formula, we have

$$\begin{aligned}\overline{R^G_{T\subset B}(\theta)(g)} &= \frac{1}{|(G_s^\circ)^F|}\sum_{\substack{x\in G^F\\ x^{-1}sx\in T^F}} \overline{\theta(x^{-1}sx)}\cdot \overline{Q^{G_s^\circ}_{{}^xT}(u)}\\ &= \frac{1}{|(G_s^\circ)^F|}\sum_{\substack{x\in G^F\\ x^{-1}sx\in T^F}} \theta^{-1}(x^{-1}sx)\cdot Q^{G_s^\circ}_{{}^xT}(u) = R^G_{T\subset B}(\theta^{-1})(g).\end{aligned}$$

(Recall that the Green function is $\mathbb{Z}$-valued and that $\overline{\theta} = \theta^{-1}$). □

Lemma 10.5. *Let R and R' be representations of G^F. Then R and R' contain a common irreducible representation if and only if $R\otimes R'^\vee$ contains the trivial representation of G^F.*

Proof. Let us write $R = \sum_\rho n_\rho \rho$ and $R' = \sum_\rho n'_\rho \rho$. Here, note that $n_\rho, n'_\rho \in \mathbb{Z}_{\geq 0}$ since R and R' are "genuine" (not "virtual") representations of G^F. Then we have

$$R\otimes R'^\vee = \sum_{\rho,\rho'} n_\rho n'_{\rho'} \rho\otimes\rho'^\vee,$$

where ρ and ρ' run all (isomorphism classes of) irreducible representations of G^F. Note that $\rho\otimes\rho'^\vee$ contains $\mathbb{1}$ if and only if $\mathrm{Hom}_{G^F}(\mathbb{1}, \rho\otimes\rho'^\vee) \neq 0$. Since we have

$$\mathrm{Hom}_{G^F}(\mathbb{1}, \rho\otimes\rho'^\vee) \cong \mathrm{Hom}_{G^F}(\rho', \rho)$$

(so-called the Hom $-\otimes$ adjunction), it is furthermore equivalent to that $\rho\cong\rho'$ since ρ and ρ' are irreducible. Moreover, in this case, $\mathrm{Hom}_{G^F}(\rho', \rho)$ is 1-dimensional by Schur's lemma. In other words, $\rho\otimes\rho'^\vee$ contains $\mathbb{1}$ with multiplicity one. Therefore, the multiplicity of the trivial representation $\mathbb{1}$ in $R\otimes R'^\vee$ is given by $\sum_\rho n_\rho n'_\rho$. Since $n_\rho, n'_\rho \in \mathbb{Z}_{\geq 0}$, we have $\sum_\rho n_\rho n'_\rho \neq 0$ if and only if there exists ρ satisfying $n_\rho n'_\rho \neq 0$, i.e., both R and R' contains ρ. □

Now let us start to prove the disjointness theorem. Suppose that θ of T^F and θ' of T'^F are characters not geometrically conjugate. Our goal is to show that $R^G_{T\subset B}(\theta)$ and $R^G_{T'\subset B'}(\theta')$ have no common irreducible constituent. To show this, it is enough to show the following:

Proposition 10.6. *If θ of T^F and θ' of T'^F are characters not geometrically conjugate, then $H^i_c(\mathcal{X}^G_{T\subset B}, \overline{\mathbb{Q}}_\ell)[\theta^{-1}] \otimes H^j_c(\mathcal{X}^G_{T'\subset B'}, \overline{\mathbb{Q}}_\ell)[\theta']$ do not contain the trivial representation for any $i, j \in \mathbb{Z}_{\geq 0}$.*

Indeed, since we have

$$R^G_{T\subset B}(\theta^{-1}) \otimes R^G_{T'\subset B'}(\theta') \cong \sum_{i,j\in\mathbb{Z}_{\geq 0}} H^i_c(\mathcal{X}^G_{T\subset B}, \overline{\mathbb{Q}}_\ell)[\theta^{-1}] \otimes H^j_c(\mathcal{X}^G_{T'\subset B'}, \overline{\mathbb{Q}}_\ell)[\theta'],$$

Proposition 10.6 implies that $R^G_{T\subset B}(\theta^{-1}) \otimes R^G_{T'\subset B'}(\theta')$ do not contain the trivial representation. Then, by Lemmas 10.5 and 10.4, we see that $R^G_{T\subset B}(\theta)$ and $R^G_{T'\subset B'}(\theta')$ do not contain the same irreducible representation.

Remark 10.7. Here is a "dangerous bend". To show that $R^G_T(\theta)^\vee \cong R^G_T(\theta^{-1})$ in Lemma 10.4, we utilized the Deligne–Lusztig character formula; taking the alternating sum is crucially important for this. In other words, it could be possible that each individual $H^i_c(\mathcal{X}^G_{T\subset B}, \overline{\mathbb{Q}}_\ell)[\theta]^\vee$ is **not** isomorphic to $H^i_c(\mathcal{X}^G_{T\subset B}, \overline{\mathbb{Q}}_\ell)[\theta^{-1}]$. Therefore, we **cannot** discuss in the following way: [25]

> If $H^i_c(\mathcal{X}^G_{T\subset B}, \overline{\mathbb{Q}}_\ell)[\theta^{-1}] \otimes H^j_c(\mathcal{X}^G_{T'\subset B'}, \overline{\mathbb{Q}}_\ell)[\theta']$ do not contain the trivial representation, then $H^i_c(\mathcal{X}^G_{T\subset B}, \overline{\mathbb{Q}}_\ell)[\theta]$ and $H^j_c(\mathcal{X}^G_{T'\subset B'}, \overline{\mathbb{Q}}_\ell)[\theta']$ do not contain the same irreducible representation (this part is **wrong** for the above reason). Hence, in particular, $R^G_{T\subset B}(\theta)$ and $R^G_{T'\subset B'}(\theta')$ do not contain the trivial representation.

By the "Künneth formula", we have

$$H^k_c(\mathcal{X}^G_{T\subset B} \times \mathcal{X}^G_{T'\subset B'}, \overline{\mathbb{Q}}_\ell) \cong \bigoplus_{i+j=k} H^i_c(\mathcal{X}^G_{T\subset B}, \overline{\mathbb{Q}}_\ell) \otimes H^j_c(\mathcal{X}^G_{T'\subset B'}, \overline{\mathbb{Q}}_\ell)$$

(this is a general fact about ℓ-adic cohomology, which holds for any product $X_1 \times X_2$ of algebraic varieties X_1 and X_2; see [Car85, Property 7.1.9]). This isomorphism is $G^F \times T^F \times T'^F$-equivariant. Here, on the left-hand side, we consider the action of $G^F \times T^F \times T'^F$ on $\mathcal{X}^G_{T\subset B} \times \mathcal{X}^G_{T'\subset B'}$ given by $(g, t, t') \cdot (x, x') := (gxt, gx't')$. Therefore, we get

$$H^k_c(\mathcal{X}^G_{T\subset B} \times \mathcal{X}^G_{T'\subset B'}, \overline{\mathbb{Q}}_\ell)[\theta^{-1} \boxtimes \theta'] \cong \bigoplus_{i+j=k} H^i_c(\mathcal{X}^G_{T\subset B}, \overline{\mathbb{Q}}_\ell)[\theta^{-1}] \otimes H^j_c(\mathcal{X}^G_{T'\subset B'}, \overline{\mathbb{Q}}_\ell)[\theta'].$$

Hence, by putting $\boldsymbol{\theta} := \theta^{-1} \boxtimes \theta'$, it is enough to show that

$$H^k_c(\mathcal{X}^G_{T\subset B} \times \mathcal{X}^G_{T'\subset B'}, \overline{\mathbb{Q}}_\ell)[\boldsymbol{\theta}]$$

does not contain the trivial representation for any k, or equivalently,

$$H^k_c(\mathcal{X}^G_{T\subset B} \times \mathcal{X}^G_{T'\subset B'}, \overline{\mathbb{Q}}_\ell)^{G^F}[\boldsymbol{\theta}] = 0$$

for any k (the upper G^F denotes the G^F-invariant part).

Now we appeal to another fact on the ℓ-adic cohomology (see [Car85, Property 7.1.8]):

$$H^k_c(\mathcal{X}^G_{T\subset B} \times \mathcal{X}^G_{T'\subset B'}, \overline{\mathbb{Q}}_\ell)^{G^F} \cong H^k_c((\mathcal{X}^G_{T\subset B} \times \mathcal{X}^G_{T'\subset B'})/G^F, \overline{\mathbb{Q}}_\ell),$$

[25] I have to confess that I was enough stupid to try this at the beginning.

where $(\mathcal{X}^G_{T\subset B}\times\mathcal{X}^G_{T'\subset B'})/G^F$ denotes the quotient of $\mathcal{X}^G_{T\subset B}\times\mathcal{X}^G_{T'\subset B'}$ by the action of the finite group G^F (given by $g\cdot(x,x')=(gx,gx')$).

We summarize our discussion so far. The disjoint theorem for $R^G_{T\subset B}(\theta)$ and $R^G_{T'\subset B'}(\theta')$ is now reduced to the following:

Claim. If θ and θ' are characters of T^F and T'^F not geometrically conjugate, then

$$H^k_c((\mathcal{X}^G_{T\subset B}\times\mathcal{X}^G_{T'\subset B'})/G^F,\overline{\mathbb{Q}}_\ell)[\boldsymbol{\theta}]=0$$

for any $k\in\mathbb{Z}_{\geq 0}$, where we put $\boldsymbol{\theta}:=\theta^{-1}\boxtimes\theta'$.

10.2. **Reformulation of geometric conjugacy.** Let $\mathbb{Z}_{(p)}$ be the localization of $\mathbb{Z}$ with respect to the prime ideal (p), i.e.,

$$\mathbb{Z}_{(p)}:=\{a/b\in\mathbb{Q}\mid a,b\in\mathbb{Z},\,p\nmid b\}\subset\mathbb{Q}.$$

Note that the groups $\overline{\mathbb{F}}_p^\times$ and $\mathbb{Z}_{(p)}/\mathbb{Z}$ are isomorphic. A naive explanation of this fact is as follows. Recall that, for any $n\in\mathbb{Z}_{>0}$, $\mathbb{F}_{p^n}$ is generated over $\mathbb{F}_p$ by the solutions to the equation $x^{p^n}-x=0$. Hence $\mathbb{F}_{p^n}^\times$ is a subset of $\overline{\mathbb{F}}_p^\times$ consisting of the solutions to $x^{p^n-1}-1=0$, i.e., the subset of (p^n-1)-th roots of unity. Thus, if we fix its generator ζ_{p^n-1}, then we can define an isomorphism

$$\mathbb{F}_{p^n}^\times\xrightarrow{\cong}\tfrac{1}{p^n-1}\mathbb{Z}/\mathbb{Z}\colon\zeta_{p^n-1}^k\mapsto k.$$

Since $\overline{\mathbb{F}}_p=\bigcup_{n\in\mathbb{Z}_{>0}}\mathbb{F}_{p^n}$, by choosing the generators ζ_{p^n-1} in a "coherent way", we can extend the above isomorphism to

$$\overline{\mathbb{F}}_p^\times\xrightarrow{\cong}\varinjlim_{n\in\mathbb{Z}_{>0}}\tfrac{1}{p^n-1}\mathbb{Z}/\mathbb{Z}.$$

The right-hand side is nothing but $\mathbb{Z}_{(p)}/\mathbb{Z}$ (note that any prime-to-p positive integer divides p^n-1 for some $n\in\mathbb{Z}_{>0}$).

As we can see from this construction, we do **not** have a canonical choice of an isomorphism $\overline{\mathbb{F}}_p^\times\cong\mathbb{Z}_{(p)}/\mathbb{Z}$. In the following, let us fix such an isomorphism.

Now let T be a k-rational maximal torus of a connected reductive group G over k. Recall that its cocharacter group $X_*(T)=\mathrm{Hom}(\mathbb{G}_{\mathrm{m}},T)$ has an action of the Frobenius F, which is given by $\gamma\mapsto F\circ\gamma$. We write $X_*(T)_{(p)}:=X_*(T)\otimes_{\mathbb{Z}}\mathbb{Z}_{(p)}$. Let us consider the following short exact sequence:

$$0\to\mathbb{Z}\to\mathbb{Z}_{(p)}\to\mathbb{Z}_{(p)}/\mathbb{Z}\to 0.$$

Since $X_*(T)$ is a free $\mathbb{Z}$-module, this induces

$$0\to X_*(T)\to X_*(T)_{(p)}\to X_*(T)\otimes_{\mathbb{Z}}(\mathbb{Z}_{(p)}/\mathbb{Z})\to 0.$$

Since the Frobenius action preserves each term, we get a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X_*(T) & \longrightarrow & X_*(T)_{(p)} & \longrightarrow & X_*(T)\otimes_{\mathbb{Z}}(\mathbb{Z}_{(p)}/\mathbb{Z}) & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle F-1} & & \downarrow{\scriptstyle F-1} & & \downarrow{\scriptstyle F-1} & & \\
0 & \longrightarrow & X_*(T) & \longrightarrow & X_*(T)_{(p)} & \longrightarrow & X_*(T)\otimes_{\mathbb{Z}}(\mathbb{Z}_{(p)}/\mathbb{Z}) & \longrightarrow & 0.
\end{array}$$

Therefore, by applying the snake lemma, we get an exact sequence

$$\begin{aligned}\mathrm{Ker}(F-1\mid X_*(T)_{(p)})&\to\mathrm{Ker}(F-1\mid X_*(T)\otimes_{\mathbb{Z}}(\mathbb{Z}_{(p)}/\mathbb{Z}))\\&\to\mathrm{Cok}(F-1\mid X_*(T))\to\mathrm{Cok}(F-1\mid X_*(T)_{(p)}).\end{aligned}$$

Lemma 10.8. *The kernel of the endomorphism $F-1$ of $X_*(T)\otimes_{\mathbb{Z}}(\mathbb{Z}_{(p)}/\mathbb{Z})$ is isomorphic to T^F.*

Proof. Recall that we have fixed an isomorphism $\overline{\mathbb{F}}_q^\times \cong \mathbb{Z}_{(p)}/\mathbb{Z}$, hence we have $X_*(T)\otimes_{\mathbb{Z}}(\mathbb{Z}_{(p)}/\mathbb{Z}) \cong X_*(T)\otimes_{\mathbb{Z}}\overline{\mathbb{F}}_q^\times$. We consider the following map:

$$X_*(T)\otimes_{\mathbb{Z}}\overline{\mathbb{F}}_q^\times \to T(\overline{\mathbb{F}}_q)=T\colon \gamma\otimes x\mapsto \gamma(x).$$

Then this is a well-defined homomorphism, which is consistent with the Frobenius actions on the both sides. Moreover, this is a bijection (for example, we can easily check it by fixing an isomorphism $T\cong\mathbb{G}_{\mathrm{m}}^r$). Hence the kernel of the endomorphism $F-1$ of $X_*(T)\otimes_{\mathbb{Z}}\overline{\mathbb{F}}_q^\times$ is identified with T^F on the right-hand side. □

Lemma 10.9. *The endomorphism $F-1$ of $X_*(T)_{(p)}$ is an isomorphism. In particular, the connecting homomorphism*

$$T^F\to \mathrm{Cok}(F-1\mid X_*(T)) = X_*(T)/(F-1)X_*(T).$$

constructed above is an isomorpshim.

Proof. Note that $X_*(T)_{(p)}$ is contained in $X_*(T)_{\mathbb{Q}} := X_*(T)\otimes_{\mathbb{Z}}\mathbb{Q}$. To show that $F-1$ is an isomorphism, it is enough to check that the determinant of $F-1$ is a prime-to-p integer. (Then, the inverse matrix to $F-1$, which is taken in $X_*(T)_{\mathbb{Q}}$, has its entries in $X_*(T)_{(p)}$).

Recall (from Week 5) that the endomorphism F of $X_*(T)_{\mathbb{Q}}$ is equal to qF_0, where q denotes the q-multiplication map and F_0 is an endomorphism of $X_*(T)_{\mathbb{Q}}$ of finite order. This means that $\det(F-1)$ is expressed as $\prod_{i=1}^r(q\zeta_i-1)$, where $r=\dim T$ and ζ_i is a root of unity. Let $K:=\mathbb{Q}(\zeta_i\mid i=1,\ldots,r)$; then each $q\zeta_i-1$ belongs to the ring of intergers $\mathcal{O}_K$ of K. It suffices to check that $q\zeta_i-1$ is not contained in $p\mathcal{O}_K$, but this is clear because $q\zeta_i-1$ is equivalent to -1 modulo $p\mathcal{O}_K$. □

We have obtained an identification

$$X_*(T)/(F-1)X_*(T)\cong T^F.$$

In particular, if a character θ of T^F is given, then we can regard it as a character of $X_*(T)$.

Proposition 10.10. *Let T and T' be k-rational maximal tori of G. Let θ and θ' be characters of T^F and T'^F. Then θ and θ' are geometrically conjugate if and only if there exists $g\in G$ such that $T'={}^gT$ and the induced map $\mathrm{Int}(g)\colon X_*(T)\cong X_*(T')$ transfers θ to θ'.*

The proof of this proposition is not difficult, but we omit; see [Car85, Propositions 4.1.2 and 4.1.3]. When the latter condition of the above proposition is satisfied, let us say "the characters of $X_*(T)$ and $X_*(T')$ induced by θ and θ' are geometrically conjugate".

10.3. **Structure of the quotient of Deligne–Lusztig varieties.** Let us investigate the structure of the quotient variety $(\mathcal{X}^G_{T\subset B}\times\mathcal{X}^G_{T'\subset B'})/G^F$. We write $\mathcal{S}$ for this quotient variety. We put

$$\mathcal{S}' := \{(u,u',z)\in F(U)\times F(U')\times G\mid uF(z)=zu'\}.$$

Proposition 10.11. *The following map is bijective and $T^F\times T'^F$-equivariant:*

$$\varphi\colon\mathcal{S}\to\mathcal{S}'\colon (x,x')\mapsto (x^{-1}F(x), x'^{-1}F(x'), x^{-1}x').$$

Here, $T^F\times T'^F$ acts on the left-hand side by $(t,t')\cdot(x,x')=(xt,xt')$ and on the right-hand side by $(t,t')\cdot(u,u',z)=(t^{-1}ut, t'^{-1}u't', t^{-1}zt')$. Furthermore, this bijection is an isomorphism of algebraic varieties.

Proof. The well-definedness of the map can be easily checked by recalling the definition of the Deligne–Lusztig variety:

$$\mathcal{X}^G_{T\subset B} := \{x \in G \mid x^{-1}F(x) \in F(U)\}.$$

The equivariance is also clear.

Let us check the injectivity of the map. Suppose that $(x, x'), (y, y') \in \mathcal{X}^G_{T\subset B} \times \mathcal{X}^G_{T'\subset B'}$ map to the same element, i.e,

$$(x^{-1}F(x), x'^{-1}F(x'), x^{-1}x') = (y^{-1}F(y), y'^{-1}F(y'), y^{-1}y').$$

By comparing the first entries, we see that $yx^{-1} \in G^F$; in other words, there exists an element $g \in G^F$ satisfying $y = gx$. Similarly, by comparing the second entries, there exists an element $g' \in G^F$ satisfying $y' = g'x'$. Finally, by looking at the third entries, we obtain $g = g'$. This means that (x, x') and (y, y') are in the same G^F-orbit.

Let us next check the surjectivity. Suppose that $(u, u', z) \in \mathcal{S}$, i.e., $u \in F(U)$, $u' \in F(U')$, $z \in G$ satisfy $uF(z) = zu'$. By applying Lang's theorem to u and u', we can find an element $x, x' \in G$ satisfying $x^{-1}F(x) = u$ and $x'^{-1}F(x') = u'$, respectively. Note that then $xzx'^{-1} \in G^F$. Indeed, we have

$$F(xzx'^{-1}) = F(x)F(z)F(x')^{-1} = (xu) \cdot (u^{-1}zu') \cdot (x'u')^{-1} = xzx'^{-1}.$$

Hence, if we put $g := xzx'^{-1} \in G^F$, then we have $\varphi(x, gx') = (u, u', z)$.

To show that this bijection is in fact an isomorphism of algebraic varieties, we need more about algebraic geometry. We do not explain the details in this course; please see [Car85, Proof of Theorem 7.3.8, 221-222 pages]. □

By this proposition, our task is furthermore reduced to show the vanishing of $H^i_c(\mathcal{S}', \overline{\mathbb{Q}}_\ell)[\boldsymbol{\theta}]$ for each $i \in \mathbb{Z}_{\geq 0}$. The idea of computing the cohomology of $\mathcal{S}'$ is to divide $\mathcal{S}'$ into "cells", where the cohomologies are more computable. The key is the following general fact, which is a generalization of the decomposition $\mathrm{GL}_2 = B \sqcup B(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})B$ used in Week2:

Theorem 10.12 (Bruhat decomposition)**.** *We have the following disjoint union decomposition:*

$$G = \bigsqcup_{w\in W_G(T)} B\dot{w}B,$$

where $\dot{w} \in N_G(T)$ is any representative of $w \in W_G(T)$. Here, each $B\dot{w}B$ is locally closed and equal to $UT\dot{w}U_w$, where $U_w := U \cap w^{-1}\overline{U}w$.[26] *Moreover, for any $w' \in W_G(T)$, the union $\bigsqcup_{w\leq w'} B\dot{w}B$ is closed, where "$\leq$" denotes the "Bruhat order" on the Weyl group.*

Let us first rewrite the Bruhat decomposition in a way more useful for our purpose. Recall that B be a Borel subgroup of G containing T with unipotent radical U. Since T is k-rational, $F^{-1}(B)$ is also a Borel subgroup of G containing T; its unipotent radical is given by $F^{-1}(U)$. The same statement holds for $B' = T'U'$. We fix $g \in G$ satisfying ${}^gT' = T$ and ${}^gF^{-1}(B') = F^{-1}(B)$ (hence ${}^gF^{-1}(U') = F^{-1}(U)$). For each $w \in W_G(T)$, we fix its representative $\dot{w} \in N_G(T)$ and put

$$G_w := (U \cap {}^w\overline{U})T\dot{w}gU'.$$

Lemma 10.13. *We have $G = \bigsqcup_{w\in W_G(T)} G_w$. Moreover, each G_w is locally closed in G and satisfies the same closure relation as the Bruhat decomposition $G = \bigsqcup_{w\in W_G(T)} B\dot{w}B$.*

[26]The symbol $\overline{U}$ denotes the unipotent radical of the "opposite" Borel subgroup $\overline{B}$. You can just think of it as a generalization of the lower-triangular Borel subgroup of GL_n.

Proof. By the Bruhat decomposition, we have

$$G = \bigsqcup_{w \in W_G(T)} UT\dot{w}U_w = \bigsqcup_{w \in W_G(T)} UT\dot{w}(U \cap w^{-1}(\overline{U})w)$$

By inverting the both side, we get

$$G = \bigsqcup_{w \in W_G(T)} (U \cap w^{-1}(\overline{U})w)\dot{w}^{-1}TU = \bigsqcup_{w \in W_G(T)} (U \cap {}^{w}\overline{U})\dot{w}TU.$$

(Here, in the second equality, we replaced w with w^{-1}.) Since we have ${}^{g}U' = U$, we get

$$G = \bigsqcup_{w \in W_G(T)} (U \cap {}^{w}\overline{U})\dot{w}T^{g}U'.$$

By multiplying both sides by g from the right, we get

$$G = \bigsqcup_{w \in W_G(T)} (U \cap {}^{w}\overline{U})\dot{w}TgU' = \bigsqcup_{w \in W_G(T)} G_w$$

(note that $T\dot{w} = \dot{w}T$).

The assertion on the topology follows from by the above proof (we just rewrote each cell). □

Recall that

$$\mathcal{S}' := \{(u, u', z) \in F(U) \times F(U') \times G \mid uF(z) = zu'\}.$$

For each $w \in W$, we put

$$\mathcal{S}'_w := \{(u, u', z) \in F(U) \times F(U') \times G_w \mid uF(z) = zu'\}.$$

Then we obviously have $\mathcal{S}' = \bigsqcup_{w \in W_G(T)} \mathcal{S}'_w$ and each cell $\mathcal{S}'_w$ is locally closed in $\mathcal{S}'$. Moreover, it can be easily checked that each G_w is stable under the left T-multiplication and the right T'-multiplication. This implies that $\mathcal{S}'_w$ is stable under the action of $T^F \times T'^F$ on $\mathcal{S}'$. Therefore, by a property of ℓ-adic cohomology (see [Car85, Property 7.1.6]), we have the following:

> If $H^i_c(\mathcal{S}'_w, \overline{\mathbb{Q}}_\ell)[\boldsymbol{\theta}] = 0$ for each $i \in \mathbb{Z}_{\geq 0}$ and $w \in W_G(T)$, then we have $H^i_c(\mathcal{S}', \overline{\mathbb{Q}}_\ell)[\boldsymbol{\theta}]$ for each $i \in \mathbb{Z}_{\geq 0}$.

Note that, by a property of the Bruhat decomposition, the natural product map

$$(U \cap {}^{w}\overline{U}) \times T\dot{w}g \times U' \to (U \cap {}^{w}\overline{U})T\dot{w}gU' =: G_w$$

is bijective Thus we have

$$\mathcal{S}'_w = \{(u, u', v, a, v') \in F(U) \times F(U') \times (U \cap {}^{w}\overline{U}) \times T\dot{w}g \times U' \mid uF(vav') = vav'u'\}.$$

We finally introduce the following variety for each $w \in W_G(T)$:

$$\mathcal{S}''_w := \{(\xi, \xi', v, a, v') \in F(U) \times F(U') \times (U \cap {}^{w}\overline{U}) \times T\dot{w}g \times U' \mid \xi F(a) = vav'\xi'\}.$$

Then it is easy to verify that the map

$$(u, u', v, a, v') \mapsto (uF(v), u'F(v')^{-1}, v, a, v')$$

gives an isomorphism of varieties $\mathcal{S}'_w \cong \mathcal{S}''_w$. Moreover, under this isomorphism, the action of $T^F \times T'^F$ on $\mathcal{S}'_w$ is transformed into an action on $\mathcal{S}''_w$ given by

$$(t, t') \cdot (\xi, \xi', v, a, v') = (t^{-1}\xi t, t'^{-1}\xi' t', t^{-1}vt, t^{-1}at', t'^{-1}v't').$$

Let us summarize our discussion so far. Now the proof of the disjointness theorem is reduced to the following:

Claim. If θ and θ' are characters of T^F and T'^F not geometrically conjugate, then

$$H_c^k(\mathcal{S}''_w, \overline{\mathbb{Q}}_\ell)[\boldsymbol{\theta}] = 0$$

for any $k \in \mathbb{Z}_{\geq 0}$ and $w \in W_G(T)$, where we put $\boldsymbol{\theta} := \theta^{-1} \boxtimes \theta'$.

10.4. **Proof of the disjointness theorem.** We introduce a subgroup H_w of $T \times T'$ as follows:

$$H_w := \{(t, t') \in T \times T' \mid F(t')t'^{-1} = F(\dot{w}g)^{-1}(F(t)t^{-1})F(\dot{w}g)\}.$$

Thus is a closed subgroup of $T \times T'$ contains $T^F \times T'^F$. The crucially important property of this subgroup is the following:

Lemma 10.14. *The action of $T^F \times T'^F$ on $\mathcal{S}''_w$ extends to an action of H_w which is given by the same formula.*

Proof. For any $(t, t') \in H_w$ and $(\xi, \xi', v, a, v') \in \mathcal{S}''_w$, let us check that $(t, t') \cdot (\xi, \xi', v, a, v') = (t^{-1}\xi t, t'^{-1}\xi' t', t^{-1}vt, t^{-1}at', t'^{-1}v't')$ belongs to $\mathcal{S}''_w$. Recall that

$$(t, t') \cdot (\xi, \xi', v, a, v') = (t^{-1}\xi t, t'^{-1}\xi' t', t^{-1}vt, t^{-1}at', t'^{-1}v't').$$

Thus the right-hand side of the defining equation of $\mathcal{S}''_w$ (i.e., "$\xi F(a)$") is given by

$$(t^{-1}\xi t) \cdot F(t^{-1}at') = t^{-1}\xi t F(t)^{-1}F(a)F(t').$$

On the other hand, the left-hand side of the defining equation of $\mathcal{S}''_w$ (i.e., "$vav'\xi'$") is given by

$$(t^{-1}vt) \cdot (t^{-1}at') \cdot (t'^{-1}v't') \cdot (t'^{-1}\xi' t') = t^{-1}vav'\xi' t' = t^{-1}\xi F(a)t'$$

(we used the defining equation of $\mathcal{S}''_w$ in the second equality). Hence these coincide if and only if we have

$$tF(t)^{-1}F(a)F(t') = F(a)t'.$$

By putting $a = s\dot{w}g$ for some $s \in T$, this is equivalent to

$$tF(t)^{-1}F(\dot{w}g)F(t') = F(\dot{w}g)t'$$

(we used that $F(s)$ commutes with $tF(t)^{-1}$). This is nothing but the defining equation of H_w. ☐

Proposition 10.15. *Let X be an algebraic variety with an action of a connected algebraic group H. Then the action of H on $H_c^i(X, \overline{\mathbb{Q}}_\ell)$ is trivial.*

By this proposition, the action of H_w° on $H_c^i(\mathcal{S}''_w, \overline{\mathbb{Q}}_\ell)$ is trivial. In particular, the action of $(T^F \times T'^F) \cap H_w^\circ$ on $H_c^i(\mathcal{S}''_w, \overline{\mathbb{Q}}_\ell)$ is trivial.

Now let us complete the proof of the disjointness theorem. We write $\tilde{\theta}$ and $\tilde{\theta}'$ for the characters of $X_*(T)$ and $X_*(T')$ induced by θ and θ', respectively. By the characterization of the geometric conjugacy, our task is to show the following:

Claim. Suppose that

$$H_c^i(\mathcal{S}''_w, \overline{\mathbb{Q}}_\ell)[\boldsymbol{\theta}] \neq 0$$

for some $i \in \mathbb{Z}_{\geq 0}$ and $w \in W_G(T)$, where we put $\boldsymbol{\theta} := \theta^{-1} \boxtimes \theta'$. Then $\tilde{\theta}$ and $\tilde{\theta}'$ are geometrically conjugate.

We suppose that $H^i_c(\mathcal{S}', \overline{\mathbb{Q}}_\ell)[\boldsymbol{\theta}] \neq 0$. Then, since $(T^F \times T'^F) \cap H^\circ_w$ acts on this space trivially, we have that $\boldsymbol{\theta} = \theta^{-1} \boxtimes \theta'$ is trivial on $(T^F \times T'^F) \cap H^\circ_w$.

We define a group homomorphism

$$\phi\colon T \times T' \to T; \quad (t, t') \mapsto F(\dot{w}g)t'F(\dot{w}g)^{-1}t.$$

We consider the "Lang map" of $T \times T'$ (note that this is a group homomorphism since $T \times T'$ is abelian):

$$L\colon T \times T' \to T \times T'; \quad (t, t') \mapsto (F(t)t^{-1}, F(t')t'^{-1}).$$

Then, by definition, we see that $H_w \subset T \times T'$ is nothing but the kernel of $\phi \circ L$.

We look at the maps on cocharacter groups induced by ϕ and L.

Lemma 10.16. *Let S be a k-rational subtorus of T. Let $X_*(T) \twoheadrightarrow X_*(T)/(F-1)X_*(T) \cong T^F$ be the surjective homomorphism constructed above. Then the image of $X_*(T) \cap (F-1)X_*(S)_{p'}$ is contained in $T^F \cap S$.*

Exercise 10.17. Prove this lemma. Hint: Go back to the construction of the identification $X_*(T)/(F-1)X_*(T) \cong T^F$ in Section 10.2 (the connecting homomorphism of the snake lemma).

We apply this lemma to $H^\circ_w \subset T \times T'$. Then we see that, under the homomorphism

$$X_*(T) \oplus X_*(T') \to T^F \times T'^F,$$

the subgroup $(X_*(T) \oplus X_*(T')) \cap (F-1)X_*(H^\circ_w)_{(p)}$ is mapped into $(T^F \times T'^F) \cap H^\circ_w$. In other words, the character $(\tilde{\theta}^{-1}, \tilde{\theta}')$ of $X_*(T) \oplus X_*(T')$ is trivial on $(X_*(T) \oplus X_*(T')) \cap (F-1)X_*(H^\circ_w)_{(p)}$.

Lemma 10.18. *We put $M := \mathrm{Ker}(\phi\colon X_*(T) \oplus X_*(T') \to X_*(T))$. Then M is contained in the kernel of $(\tilde{\theta}^{-1}, \tilde{\theta}')$.*

Proof. Let $m \in M$. Since $(X_*(T) \oplus X_*(T'))/(F-1)(X_*(T) \oplus X_*(T'))$ is isomorphic to $T^F \times T'^F$, its order is finite and prime-to-p. Thus there exists a prime-to-p integer $n \in \mathbb{Z}$ such that $nm = (F-1)\xi$ for some $\xi \in X_*(T) \oplus X_*(T')$. As $(F-1)\xi = nm \in M$, we have that $\xi \in \mathrm{Ker}(\phi \circ L) = X_*(H_w) = X_*(H^\circ_w)$. Hence m belongs to $(F-1)X_*(H^\circ_w)_{(p)}$, which means that m lies in the kernel of $(\tilde{\theta}^{-1}, \tilde{\theta}')$ by the remark in the paragraph above Lemma. □

Let $\gamma \in X_*(T)$. Then, by the definition of M, $(\gamma, \mathrm{Int}(F(\dot{w}g)) \circ \gamma) \in X_*(T) \oplus X_*(T')$ belongs to M. Hence, by the above lemma, $(\tilde{\theta}^{-1}, \tilde{\theta}')$ maps $(\gamma, \mathrm{Int}(F(\dot{w}g)) \circ \gamma)$ to 1. In other words, we have

$$\tilde{\theta}^{-1}(\gamma) \cdot \tilde{\theta}'(\mathrm{Int}(F(\dot{w}g)) \circ \gamma) = 1.$$

Equivalently, we have

$$\tilde{\theta}(\gamma) = \tilde{\theta}'(\mathrm{Int}(F(\dot{w}g)) \circ \gamma).$$

This means that the characters $\tilde{\theta}$ and $\tilde{\theta}'$ are geometrically conjugate.

11. Week 11: Cuspidal representations

Recall that, in Week 2, we investigated cuspidal representations of $\mathrm{GL}_2(\mathbb{F}_q)$. We first defined principal representations of $\mathrm{GL}_2(\mathbb{F}_q)$ by considering the induction from Borel subgroups, and then defined the cuspidality. The aim of this week is to first generalize the notion of the cuspidality to any finite group of Lie type and investigate it from the viewpoint of Deligne–Lusztig theory.

11.1. Parabolic subgroups.

Let G be a connected reductive group over $k = \mathbb{F}_q$.

Proposition/Definition 11.1.

(1) Let P be a k-rational closed subgroup of G. We say that P is a *k-rational parabolic subgroup* of G if $P_{\overline{k}}$ contains a Borel subgroup of $G_{\overline{k}}$.

(2) For any k-rational parabolic subgroup P of G, there exists a k-rational connected reductive subgroup L of P such that P is the semi-direct product $P = L \ltimes U_P$, where U_P is the unipotent radical of P. We call such an L a *k-rational Levi subgroup* of P. We call the decomposition $P = L \ltimes U_P$ a *Levi decomposition*.

Remark 11.2.

(1) By definition, G and any k-rational Borel subgroup of G are obviously parabolic subgroups; these are maximal/minimal parabolic subgroups.

(2) Note that a Levi subgroup of a given parabolic subgroup is not unique in general.

Example 11.3. Let $G = \mathrm{GL}_3$.

(1) We put

$$P := \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \subset G.$$

Then this is a k-rational parabolic subgroup of G. The unipotent radical of P is given by

$$U_P = \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \subset P.$$

Hence, for example, a Levi subgroup of P can be taken to be

$$\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix} \subset P.$$

(2) We put

$$P' := \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \subset G.$$

Then this is a k-rational parabolic subgroup of G. The unipotent radical of P' is given by

$$U_{P'} = \begin{pmatrix} 1 & * & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \subset P'.$$

Hence, for example, a Levi subgroup of P' can be taken to be

$$\begin{pmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \subset P'.$$

Recall that there always exists a k-rational Borel subgroup of G since $k = \mathbb{F}_q$; let us fix such a B_0. We call a k-rational parabolic subgroup P *standard parabolic* if P contains B_0.

Fact 11.4. *Any k-rational parabolic subgroup of G is $G(k)$-conjugate to a k-rational standard parabolic subgroup of G.*

The above definition of a parabolic subgroup is too abstract. So let us also introduce a concrete description of (standard) parabolic subgroups. In the following (in this subsection), we assume that G is split for simplicity. But the theory does not change even when G is non-split.

Recall that reductive groups are classified by root data "$(X, R, X^\vee, R^\vee)$". Let us first review how $(X, R, X^\vee, R^\vee)$ is associated to G (Week 4). We let T_0 be a split k-rational maximal torus of G contained in B_0. Then X and $X^\vee$ are defined to be $X^*(T_0)$ and $X_*(T_0)$. The sets R and $R^\vee$ are finite subsets of X and $X^\vee$; these are called the sets of roots and coroots. An element $\alpha \in X$ belongs to R if and only if there exists a closed subgroup U_α of G such that

- U_α is isomorphic to $\mathbb{G}_a$ (fix $\iota\colon \mathbb{G}_a \cong U_\alpha$), and
- U_α is normalized by T_0-conjugation and satisfies
$$t \cdot \iota(x) \cdot t^{-1} = \iota(\alpha(t) \cdot x)$$
for any $t \in T_0$ and $x \in \mathbb{G}_a$.

Let us call a root $\alpha \in R$ a *positive root* if its associated root subgroup U_α is contained in the unipotent radical U_0 of the fixed Borel subgroup B_0. We write R_+ for the subset of R of positive roots. We put $R_- := -R_+$ and call an element of R_- a *negative root*. Note that R_- is also a subset of R since we have $-R = R$.

Fact 11.5. *(1) We have $R = R_+ \sqcup R_-$.*
(2) There exists a unique subset $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ of R_+ such that any positive root is uniquely written as a $\mathbb{Z}_{\geq 0}$-linear combination of $\alpha_1, \ldots, \alpha_l$; $\alpha = \sum_{i=1}^{l} n_i \alpha_i$ ($n_i \in \mathbb{Z}_{\geq 0}$).

We call Δ the set of *simple roots*. Note that, by this fact and the definition of R_-, any negative root is uniquely written as a $\mathbb{Z}_{\leq 0}$-linear combination of simple roots.

Remark 11.6. Recall that, the construction of root datum $(X, R, X^\vee, R^\vee)$ depends on the choice of T_0, but does not on B_0. On the other hand, the notions of a positive root and a simple root depends on B_0.

Now let I be any subset of Δ. We consider the following subset R_I of R:
$$R_I := \Bigl\{\alpha = \sum_{i=1}^{l} n_i \alpha_i \in R \Bigm| n_i \geq 0 \text{ if } i \notin I\Bigr\}.$$
We define a k-rational closed subgroup P_I of G by
$$P_I := \langle T_0, U_\alpha \mid \alpha \in R_I \rangle.$$
For example:

- When $I = \Delta$, we have $R_\Delta = R$ and $P_\Delta = \langle T_0, U_\alpha \mid \alpha \in R \rangle = G$.
- When $I = \emptyset$, we have $R_\emptyset = R_+$ and $P_\emptyset = \langle T_0, U_\alpha \mid \alpha \in R_+ \rangle = B_0$.

In particular, in general, P_I is a k-rational closed subgroup of G containing B_0, hence a standard parabolic subgroup.

Fact 11.7. *The above construction gives an order-preserving bijection*

$$\{I \subset \Delta\} \xrightarrow{1:1} \{k\text{-rational standard parabolic subgroups of } G\} \colon I \mapsto P_I.$$

Moreover, each P_I is equipped with a natural Levi subgroup (we call the "standard Levi subgroup") L_I, which is given by

$$L_I = \langle T_0, U_\alpha \mid \alpha \in R_I^0 \rangle,$$

where

$$R_I^0 := \Big\{ \alpha = \sum_{i=1}^{l} n_i \alpha_i \in R \,\Big|\, n_i = 0 \text{ if } i \notin I \Big\}.$$

Example 11.8. Let $G = \mathrm{GL}_3$. Let T_0 be its diagonal maximal torus. As usual, we choose B_0 to be the upper-triangular one.

$$B_0 := \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix},$$

then both P and P' contains B_0, hence are standard. The set R of roots is given by

$$R = \{\pm(e_1 - e_2), \pm(e_2 - e_3), \pm(e_1 - e_3)\}.$$

The corresponding root subgroups are as follows:

$$U_{e_1-e_2} = \begin{pmatrix} 1 & * & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad U_{e_2-e_1} = \begin{pmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$U_{e_2-e_3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}, \quad U_{e_3-e_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & * & 1 \end{pmatrix},$$

$$U_{e_1-e_3} = \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad U_{e_3-e_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ * & 0 & 1 \end{pmatrix}.$$

Thus the positive roots are $e_1 - e_2$, $e_2 - e_3$, $e_1 - e_3$. The negative roots are $e_2 - e_1$, $e_3 - e_2$, $e_3 - e_1$. The set of simple roots Δ in this case is given by $\{e_1 - e_2, e_2 - e_3\}$ (indeed, we have $e_1 - e_3 = (e_1 - e_2) + (e_2 - e_3)$). We can check that the standard parabolic subgroups corresponding to subsets of Δ are as follows:

$$P_\Delta = \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix},$$

$$P_{\{e_1-e_2\}} = \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix}, \quad P_{\{e_2-e_3\}} = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix},$$

$$P_\emptyset = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}.$$

11.2. **Parabolic induction.** Let P be a k-rational parabolic subgroup of G. Let L be a k-rational Levi subgroup of P and U_P be the unipotent radical of P; hence we have $P = L \ltimes U_P$. Note that, as U_P is connected, we can check that $(P/U_P)^F \cong P^F/U_P^F$ by the usual argument via Lang's theorem. In particular, we have a canonical surjection (quotient)

$$P^F \twoheadrightarrow P^F/U_P^F \cong (P/U_P)^F \cong L^F.$$

Definition 11.9. Suppose that σ is a representation of L^F. By pulling back σ to P^F via $P^F \twoheadrightarrow L^F$, we regard σ as a representation of P^F. We call its induction $\mathrm{Ind}_{P^F}^{G^F} \sigma$ to G^F the *parabolic induction* of σ.

Example 11.10. Recall that a Borel subgroup is a minimal parabolic subgroup. Thus let us take P to be B_0. In this case, a Levi subgroup of B_0 can be taken to be T_0. The parabolic induction of a 1-dimensional representation ($\mathrm{Ind}_{B_0^F}^{G^F} \chi$ for a character $\chi\colon B_0^F \to \mathbb{C}^\times$) is nothing but the principal series representation, which was introduced before.

Definition 11.11. Let ρ be a representation of G^F. We say that ρ is *cuspidal* if there does not exist a pair (P, σ) of a proper k-rational parabolic subgroup $P \subsetneq G$ with a Levi L and a representation σ of L^F such that $\langle \rho, \mathrm{Ind}_{P^F}^{G^F} \sigma \rangle \neq 0$.

We explain why the cuspidal representations are so important. Suppose that ρ is a **non**-cuspidal irreducible representation of G^F. Then, by definition, there exists a pair $(P_1 \subsetneq G, \sigma_1)$ such that ρ is contained in $\mathrm{Ind}_{P_1^F}^{G^F} \sigma_1$. We may assume that such a σ_1 is irreducible. Let us consider what will happen if σ_1 is not a cuspidal representation of L_1^F. Then, again by definition, σ_1 is contained in $\mathrm{Ind}_{P_2^F}^{L_1^F} \sigma_2$ for some proper parabolic subgroup $P_2 \subsetneq L_1$ with a Levi L_2 and an irreducible representation σ_2 of L_2^F. We can continue this procedure, but not forever because there cannot exist an infinite chain of proper parabolic subgroups. In other words, eventually we arrive at a pair (P, σ), where σ is a cuspidal irreducible representation of L^F.

Exercise 11.12. Prove the associativity of the parabolic induction.

Proposition 11.13. *Let ρ be a representation of G^F. The following are equivalent:*

(1) ρ is cuspidal;

(2) for any k-rational parabolic subgroup P with a k-rational Levi L, we have $\langle \rho, \mathrm{Ind}_{U_P^F}^{G^F} \mathbb{1} \rangle = 0$.

Proof. Note that

$$\mathrm{Ind}_{U_P^F}^{G^F} \mathbb{1} \cong \mathrm{Ind}_{P^F}^{G^F}(\mathrm{Ind}_{U_P^F}^{P^F} \mathbb{1}) \cong \bigoplus_{\sigma \in \mathrm{Irr}(L^F)} \mathrm{Ind}_{P^F}^{G^F} \sigma,$$

where the sum is over all irreducible representations of L^F (we used that $P^F/U_P^F \cong L^F$). Therefore, we have $\langle \rho, \mathrm{Ind}_{U_P^F}^{G^F} \mathbb{1} \rangle = 0$ if and only if $\langle \rho, \mathrm{Ind}_{P^F}^{G^F} \sigma \rangle = 0$ for all irreducible representations σ of L^F. □

11.3. **DL's cuspidality criterion.** Suppose that T is a k-rational maximal torus of G contained in a k-rational parabolic subgroup P of G. Let L be a Levi subgroup of P and U_P the unipotent radical of P. Then, under the map $P \twoheadrightarrow P/U_P \cong L$, T is mapped to a k-rational maximal torus of L isomorphically (the kernel of the map is $T \cap U_P$, which is semisimple and unipotent, hence trivial). Let us again write T for the k-rational maximal torus of L determined in this way.

Proposition 11.14. *For any character $\theta\colon T^F \to \mathbb{C}^\times$, we have* $\mathrm{Ind}_{P^F}^{G^F}(R_T^L(\theta)) \cong R_T^G(\theta)$.

Proof. Let us fix a Borel subgroup B of G such that $T \subset B \subset P$. Note that we can always find such a Borel subgroup. (Indeed, by definition, P contains a Borel subgroup of G, say B'. Let T' be any maximal torus contained in B'. Then, as any two maximal tori of a connected linear algebraic group are conjugate, T and T' are P-conjugate, say $T = pT'p^{-1}$. By putting $B := pB'p^{-1}$, we get a desired Borel.) Let U be the unipotent radical of B.

We introduce a set $\mathcal{P}$ as follows:

$$\mathcal{P} := \{P' \subset G \mid P' \text{ is a parabolic subgroup of } G \text{ which is } G^F\text{-conjugate to } P\}.$$

Note that we have a bijection $G^F/P^F \xrightarrow{1:1} \mathcal{P}$ given by $y \mapsto yPy^{-1}$ (here, we use a fact that, for any parabolic subgroup P, its normalizer group $N_G(P)$ is P itself). Recall that the Deligne–Lusztig variety $\mathcal{X}_{T\subset B}^G$ is defined by

$$\mathcal{X}_{T\subset B}^G := \{x \in G \mid x^{-1}F(x) \in F(U)\}.$$

For each $P' \in \mathcal{P}$, we define a subvariety $\mathcal{X}_{T\subset B}^G(P')$ of $\mathcal{X}_{T\subset B}^G$ by

$$\mathcal{X}_{T\subset B}^G(P') := \{x \in G \mid x^{-1}F(x) \in F(U),\, xPx^{-1} = P'\}.$$

Claim. We have $\mathcal{X}_{T\subset B}^G = \bigsqcup_{P'\in\mathcal{P}} \mathcal{X}_{T\subset B}^G(P')$.

Proof of Claim. The union on the right-hand side is obviously contained in the left-hand side and also disjoint. Thus it is enough to check the converse inclusion. Let $x \in \mathcal{X}$. Then our task is to show that there exists $P' \in \mathcal{P}$ satisfying $xPx^{-1} = P'$. In other words, it suffices to show that there exists $y \in G^F$ satisfying $xPx^{-1} = yPy^{-1}$. Since we have $x^{-1}F(x) \in F(U) \subset F(B) \subset F(P) = P$, we have an element $z \in P$ such that $x^{-1}F(x) = z$. By applying Lang's lemma to $z \in P$, we can find an element $p \in P$ satisfying $x^{-1}F(x) = p^{-1}F(p)$, or equivalently, $xp^{-1} \in G^F$. Then we have $xPx^{-1} = (xp^{-1})P(xp^{-1})^{-1}$. So y can be taken to be xp^{-1}. □

Here, we appeal to a general fact that $B_L := L \cap B$ is a Borel subgroup of L with unipotent radical $L \cap U$. Thus it makes sense to talk about the Deligne–Lusztig variety $\mathcal{X}_{T\subset B_L}^L$ associated to $T \subset B_L \subset L$.

Now let us suppose that $x \in \mathcal{X}_{T\subset B}^G(P')$, where $P' = yPy^{-1}$ with $y \in G^F$. Then, since $xPx^{-1} = P' = yPy^{-1}$, we have $y^{-1}x \in N_G(P) = P$. If we again write $y^{-1}x$ for the image of yx^{-1} in L under the map $P \to P/U_P \cong L$, then we have $(y^{-1}x)^{-1}F(yx^{-1}) = x^{-1}F(x) \in F(U)$, hence $(y^{-1}x)^{-1}F(yx^{-1}) \in L \cap F(U) = F(L \cap U)$. In other words, yx^{-1} belongs to $\mathcal{X}_{T\subset B_L}^L$. Thus we obtain a morphism

$$\mathcal{X}_{T\subset B}^G(P') \to \mathcal{X}_{T\subset B_L}^L\colon x \mapsto y^{-1}x,$$

which is an isomorphism whose inverse is simply given by $yx \leftarrow x$.

Therefore, in summary, we get a decomposition

$$\mathcal{X}_{T\subset B}^G = \bigsqcup_{P'\in\mathcal{P}} \mathcal{X}_{T\subset B}^G(P') = \bigsqcup_{y\in G^F/P^F} y\mathcal{X}_{T\subset B_L}^L.$$

It is not difficult to check that this decomposition implies that the representation of G^F realized on $H_c^i(\mathcal{X}_{T\subset B}^G, \overline{\mathbb{Q}}_\ell)$ is nothing but the induced representation of the representation of P^F realized on $H_c^i(\mathcal{X}_{T\subset B_L}^L, \overline{\mathbb{Q}}_\ell)$ (through the map $P^F \to L^F$). By also noting that the above decomposition is equivariant with respect to the right T^F-translation action, we conclude that $R_T^G(\theta) \cong \mathrm{Ind}_{P^F}^{G^F}(R_T^L(\theta))$. □

Definition 11.15. We say that a k-rational maximal torus T of G is *elliptic* if T is not contained in any proper k-rational parabolic subgroup of G.

Corollary 11.16. *If the Deligne–Lusztig representation $R_T^G(\theta)$ contains a cuspidal irreducible constituent, then T must be elliptic.*

Proof. Let us suppose that T is not elliptic, hence there exists a proper k-rational parabolic subgroup P with a k-rational Levi subgroup L. Let ρ be any irreducible representation contained in $R_T^G(\theta)$. Then, by the previous proposition, we have $R_T^G(\theta) \cong \mathrm{Ind}_{P^F}^{G^F} R_T^L(\theta)$. This means that there exists an irreducible representation ρ_L of L^F contained in $R_T^L(\theta)$ such that ρ is contained in $\mathrm{Ind}_{P^F}^{G^F} \rho_L$. Hence ρ is not cuspidal. □

Then, how about the converse statement? In fact, when the Deligne–Lusztig representation is irreducible (recall that we call such representation "regular"), the situation is understandable:

Proposition 11.17. *Suppose that S is an elliptic k-rational maximal torus of G. If $\eta\colon S^F \to \mathbb{C}^\times$ is a regular character, then $(-1)^{r_G-r_S} R_S^G(\eta)$ is an irreducible cuspidal representation of G^F.*

Proof. Recall that, in the proof of the exhaustion theorem, we established a formula

$$\frac{1}{\mathrm{St}_G(s)} \sum_{s\in T\in\mathcal{T}_G} \sum_{\theta\in(T^F)^\vee} (-1)^{r_G-r_T}\cdot\theta(s)^{-1}\cdot R_T^G(\theta) = |(G^F)_s|\cdot \mathbb{1}_{[s]}$$

for any $s \in G_{\mathrm{ss}}^F$. In particular, when $s=1$, we get

$$\frac{1}{\mathrm{St}_G(1)} \sum_{T\in\mathcal{T}_G} \sum_{\theta\in(T^F)^\vee} (-1)^{r_G-r_T}\cdot R_T^G(\theta) = |G^F|\cdot \mathbb{1}_{\{1\}}.$$

Note that we have $|G^F|\cdot\mathbb{1}_{\{1\}} = \mathrm{Ind}_{\{1\}}^{G^F}\mathbb{1}$. We utilize this formula for any k-rational Levi subgroup L of a k-rational parabolic subgroup P:

$$\frac{1}{\mathrm{St}_L(1)} \sum_{T\in\mathcal{T}_L} \sum_{\theta\in(T^F)^\vee} (-1)^{r_L-r_T}\cdot R_T^L(\theta) = \mathrm{Ind}_{\{1\}}^{L^F}\mathbb{1}.$$

We apply the parabolic induction from P^F to G^F to the both sides. Since we have $\mathrm{Ind}_{P^F}^{G^F}(\mathrm{Ind}_{\{1\}}^{L^F}\mathbb{1}) \cong \mathrm{Ind}_{U_P^F}^{G^F}\mathbb{1}$ (Exercise), the previous proposition implies that

$$\frac{1}{\mathrm{St}_L(1)} \sum_{T\in\mathcal{T}_L} \sum_{\theta\in(T^F)^\vee} (-1)^{r_L-r_T}\cdot R_T^G(\theta) = \mathrm{Ind}_{U_P^F}^{G^F}\mathbb{1}.$$

Since

- there are $|U_P^F|$-many lifts of a k-rational maximal torus T of L to a k-rational maximal torus of P,
- $\mathrm{St}_G(1) = \mathrm{St}_L(1)\cdot|U_P^F|$,
- $r_G = r_L$ (this follows from that L is a k-rational Levi),

we get

$$\frac{1}{\mathrm{St}_G(1)} \sum_{T\in\mathcal{T}_P} \sum_{\theta\in(T^F)^\vee} (-1)^{r_G-r_T}\cdot R_T^G(\theta) = \mathrm{Ind}_{U_P^F}^{G^F}\mathbb{1}.$$

Now we prove the cuspidality of the irreducible representation $(-1)^{r_G-r_S}R_S^G(\eta)$. (Recall that the irreducibility follows from the regularity of η and the dimension formula.) Our task is to show that, for any proper k-rational parabolic subgroup P of G, we have

$$\langle(-1)^{r_G-r_S}R_S^G(\eta), \mathrm{Ind}_{U_P^F}^{G_F}\mathbb{1}\rangle = 0.$$

By using the previous decomposition, we have

$$\langle(-1)^{r_G-r_S}R_S^G(\eta), \mathrm{Ind}_{U_P^F}^{G_F}\mathbb{1}\rangle = \frac{1}{\mathrm{St}_G(1)}\sum_{T\in\mathcal{T}_P}\sum_{\theta\in(T^F)^\vee}(-1)^{r_T-r_S}\cdot\langle R_S^G(\eta), R_T^G(\theta)\rangle.$$

By the inner product formula, each summand is given by

$$|\{w\in W_{G^F}(S,T) \mid {}^w\eta=\theta\}|.$$

However, by the assumption that S is elliptic, S cannot conjugate to any k-rational maximal torus T of P; in particular, this summand is zero. □

12. Week 12: Unipotent representations and Lusztig's Jordan decomposition

12.1. **Langlands dual and geometric conjugacy.** Let G be a connected reductive group over $k = \mathbb{F}_q$ as usual (F denotes its geometric Frobenius endomorphism). For simplicity, in the following discussion, we assume that G is split.

Recall that split connected reductive groups over k are classified by root data. Let $(X, R, X^\vee, R^\vee)$ the root datum determined by G (if we take a k-rational split maximal torus T_0 of G, then X and $X^\vee$ can be taken to be $X^*(T_0)$ and $X_*(T_0)$, respectively). We note that the swapped quadruple $(X^\vee, R^\vee, X, R)$ also satisfies the axioms of a root datum. We call this root datum the *dual root datum* of $(X, R, X^\vee, R^\vee)$. Again by the classification theorem of reductive groups, there exists a split connected reductive group over k whose root datum is given by $(X^\vee, R^\vee, X, R)$. We call this reductive group the *Langlands dual group* of G. Let $\hat{G}$ denote it (we use the same symbol "F" for the geometric Frobenius of $\hat{G}$). Hence, if we take a k-rational split maximal torus $\hat{T}_0$ of $\hat{G}$, then we have $X^\vee \cong X^*(\hat{T}_0)$ and $X \cong X_*(\hat{T}_0)$.

Remark 12.1. (1) The Dynkin diagram of $\hat{G}$ is the dual diagram of that of G in the sense that the underlying diagram is the same and the directions of arrows are reversed. In particular, among A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2, only B_n and C_n are swapped under taking the dual; all other diagrams are self-dual.

(2) The Langlands dual group $\hat{G}$ is simply-connected (resp. adjoint) if and only if G is adjoint (resp. simply-connected).

type of G	type A_{n-1}			type B_n	
G	GL_n	SL_n	PGL_n	Spin_{2n+1}	SO_{2n+1}
$\hat{G}$	GL_n	PGL_n	SL_n	PSp_{2n}	Sp_{2n}
type of $\hat{G}$	type A_{n-1}			type C_n	

type of G	type C_n		type D_n		
G	Sp_{2n}	PSp_{2n}	Spin_{2n}	SO_{2n}	PSO_{2n}
$\hat{G}$	SO_{2n+1}	Spin_{2n+1}	PSO_{2n}	SO_{2n}	Spin_{2n}
type of $\hat{G}$	type B_n		type D_n		

Now we reinterpret the notion of the geometric conjugacy in terms of the Langlands dual group. Recall that G^F-conjugacy classes of k-rational maximal tori of G are classified by the conjugacy classes of $W_0 := W_G(T_0)$. Let T be a k-rational maximal torus of G whose conjugacy class is represented by $w \in W_0$. In fact, the Weyl group of the Langlands dual group $\hat{W}_0 := W_{\hat{G}}(\hat{T}_0)$ is isomorphic to W_0. Thus, by regarding w as an element of $\hat{W}_0$, we can find a k-rational maximal torus $\hat{T}$ of $\hat{G}_0$ whose conjugacy class is represented by $w \in \hat{W}_0$.

We note that $X_*(T_0) \cong X^\vee \cong X^*(\hat{T}_0)$. This isomorphism is equivariant with respect to the action of the Frobenius (in fact, since we are assuming that G is split, the Frobenius actions on $X_*(T_0)$ and $X^*(\hat{T}_0)$ are trivial). Since any maximal tori are conjugate, by fixing $g \in G$ such that $T = {}^gT_0$, we obtain an isomorphism $X_*(T_0) \cong X_*(T)$ (given by the pull-back via g-conjugation). Similarly, we also have an isomorphism $X^*(\hat{T}_0) \cong X^*(\hat{T})$. Consequently, we obtain

$$X_*(T) \cong X_*(T_0) \cong X^\vee \cong X^*(\hat{T}_0) \cong X^*(\hat{T}).$$

By chasing the above construction of $\hat{T}$ carefully, we can check the following:

> we may find $\hat{T}$ such that the resulting isomorphism $X_*(T) \cong X^*(\hat{T})$ is equivariant with respect to the Frobenius actions.

Now recall that we have an isomorphism

$$T^F \cong X_*(T)/(F-1)X_*(T).$$

(Week 10). In fact, we also have

$$(T^F)^\vee \cong X^*(T)/(F-1)X^*(T),$$

where $(T^F)^\vee := \mathrm{Hom}(T^F, \mathbb{C}^\times)$ (see [Car85, Proposition 3.2.3]). Therefore, by also using the previous Frobenius-equivariant identification $X_*(T) \cong X^*(\hat{T})$, we finally obtain an identification

$$(T^F)^\vee \cong X^*(T)/(F-1)X^*(T) \cong X_*(\hat{T})/(F-1)X_*(\hat{T}) \cong \hat{T}^F.$$

Hence, any character of T^F can be regarded as an element of $\hat{T}^F \subset \hat{G}^F$.

Let us summarize our discussion. We put $\mathcal{T}_G$ to be the set of k-rational maximal tori of G. We put $\mathcal{I}_G$ to be the set of pairs (T,θ) such that $T \in \mathcal{T}_G$ and $\theta \in (T^F)^\vee$. Similarly, we put $\mathcal{J}_{\hat{G}}$ to be the set of pairs $(\hat{T}, s)$ such that $\hat{T} \in \mathcal{T}_{\hat{G}}$ and $s \in \hat{T}^F$. We constructed an element $(\hat{T}, s) \in \mathcal{J}_{\hat{G}}$ from a pair $(T,\theta) \in \mathcal{I}_G$.

Note that both sets $\mathcal{I}_G$ and $\mathcal{J}_{\hat{G}}$ are equipped with the actions of G^F and $\hat{G}^F$ by conjugation, respectively. We denote the sets of their G^F-conjugacy classes by the symbol $\mathcal{I}_G/{\sim_{G^F}}$ and $\mathcal{J}_G/{\sim_{\hat{G}^F}}$.

On the other hand, we also have an equivalence relation on $\mathcal{I}_G$ given by $(T_1,\theta_1) \sim (T_2,\theta_2)$ if and only if $R^G_{T_1}(\theta_1)$ and $R^G_{T_2}(\theta_2)$ contains a common irreducible constituent.

Theorem 12.2. *The previous construction induces the following diagram*

$$\begin{array}{ccc} \mathcal{I}_G/{\sim_{G^F}} & \xrightarrow{1:1} & \mathcal{J}_G/{\sim_{\hat{G}^F}} \\ \downarrow & & \downarrow \\ \mathcal{I}_G/{\sim} & \xrightarrow{1:1} & \hat{G}^F_{\mathrm{ss}}/{\sim_{\hat{G}^F}} \end{array} \qquad \begin{array}{ccc} (T,\theta) & \longmapsto & (\hat{T}, s) \\ \downarrow & & \downarrow \\ (T,\theta) & \longmapsto & s \end{array}$$

Proof. We omit the proof; see, for example, [GM20, Corollary 2.5.14]. □

12.2. **Lusztig's Jordan decomposition.**

Definition 12.3. Let $s \in \hat{G}^F_{\mathrm{ss}}$. We let $\mathcal{E}(G^F, s)$ be the set of isomorphism classes of irreducible representations ρ of G^F such that $\langle \rho, R^G_T(\theta)\rangle$ for some $(T,\theta) \in \mathcal{I}_G$ whose G^F-conjugacy class (associated as in the previous section) corresponds to s. We call the set $\mathcal{E}(G^F, s)$ the *Lustig series of irreducible representations* associated to $s \in \hat{G}^F_{\mathrm{ss}}$.

Remark 12.4. Recall that we say an irreducible representation ρ of G^F is unipotent if there exists a k-rational maximal torus T of G satisfying $\langle \rho, R^G_T(\mathbb{1})\rangle$. Then the associated semisimple element of G^F is 1. Hence, $\mathcal{E}(G^F, 1)$ is nothing but the set of irreducible unipotent representations of G^F.

Let us write $\mathrm{Irr}(G^F)$ for the set of isomorphism classes of irreducible representations of G^F.

Theorem 12.5. *We have a decomposition*

$$\mathrm{Irr}(G^F) = \bigsqcup_{s \in \hat{G}^F_{\mathrm{ss}}/\sim} \mathcal{E}(G^F, s),$$

where the sum is over $\hat{G}^F$-conjugacy classes of semisimple elements of $\hat{G}^F$.

Proof. We first utilize the exhaustion theorem. The exhaustion theorem tells us that, for any $\rho \in \mathrm{Irr}(G)$, we can find a pair $(T, \theta) \in \mathcal{I}_G$ such that the associated Deligne–Lusztig representation $R^G_T(\theta)$ contains ρ, i.e., $\langle \rho, R^G_T(\theta)\rangle \neq 0$. Hence, by putting $s \in \hat{G}^F_{\mathrm{ss}}$ to be an element corresponding to (T, θ), we have $\rho \in \mathcal{E}(G^F, s)$. In other words, we get $\mathrm{Irr}(G^F) = \bigcup_{s \in \hat{G}^F_{\mathrm{ss}}} \mathcal{E}(G^F, s)$. Moreover, by definition, $\mathcal{E}(G^F, s)$ depends only on the G^F-conjugacy class of s. Hence $\mathrm{Irr}(G^F) = \bigcup_{s \in \hat{G}^F_{\mathrm{ss}}/\sim} \mathcal{E}(G^F, s)$.

We next use the disjointness theorem. Suppose that $\mathcal{E}(G^F, s_1)$ and $\mathcal{E}(G^F, s_2)$ has nonempty intersection ($s_1, s_2 \in G^F_{\mathrm{ss}}$); let ρ be any element of $\mathcal{E}(G^F, s_1) \cap \mathcal{E}(G^F, s_2)$. Then there exists $(T_i, \theta_i) \in \mathcal{I}_G$ whose geometric conjugacy class corresponds to the G^F-conjugacy class of s_i for each $i = 1, 2$. By the disjointness theorem, the geometric conjugacy classes of (T_2, θ_1) and (T_2, θ_2) must coincide. In other words, G^F-conjugacy classes of s_1 and s_2 are the same. □

By the above theorem, to classify the irreducible representations of G^F, it is enough to determine $\mathcal{E}(G^F, s)$ for each $s \in G^F_{\mathrm{ss}}$.

Theorem 12.6 (Lusztig)**.** *Suppose that the center of G is connencted. Then, for each $s \in G^F_{\mathrm{ss}}$, there exists a bijection*

$$\mathcal{E}(G^F, s) \xrightarrow{1:1} \mathcal{E}(G^F_s, 1) \colon \rho \mapsto \rho_0$$

such that, for any $(T, \theta) \in \mathcal{I}_{G_s} \subset \mathcal{I}_G$ which corresponds to s, we have

$$(-1)^{r_G} \langle \rho, R^G_T(\theta)\rangle_{G^F} = (-1)^{r_{G_s}} \langle \rho_0, R^{G_s}_T(\theta)\rangle_{G^F_s}.$$

In particular, by combining this theorem with the previous one, we get

$$\mathrm{Irr}(G^F) \cong \bigsqcup_{s \in \hat{G}^F_{\mathrm{ss}}/\sim} \mathcal{E}(G^F_s, 1).$$

This decomposition is called Lusztig's Jordan decomposition. By Lusztig's Jordan decomposition, in order to classify irreducible representations of G^F, we are reduced to classify all irreducible unipotent representations of G^F and its smaller reductive subgroups.

Here let us compare Lusztig's Jordan decomposition with the normal Jordan decomposition:

$$G^F = \bigsqcup_{s \in G^F_{\mathrm{ss}}} (G^F_s)_{\mathrm{unip}},$$

which induces a decomposition of the rational conjugacy classes:

$$G^F/{\sim_{G^F}} = \bigsqcup_{s \in G^F_{\mathrm{ss}}/\sim_{G^F}} (G^F_s)_{\mathrm{unip}}/{\sim_{G^F_s}}.$$

(Here, we are still assuming that the center of G is connected. In fact, this implies that the centralizer group $Z_G(s)$ of any element $s \in G^F_{\mathrm{ss}}$ is connected.)

Recall that, for any finite group G, the number of the isomorphism classes of irreducible representations of G is equal to the number of the G^F-conjugacy classes of G. Then, does this suggests that there is an explicit relationship (in particular, a bijection) between them? In general, the answer is NO (although sometimes it is possible; for example, when $G = \mathfrak{S}_n$, both the sets of irreducible representations and conjugacy classes are parametrized by Young diagrams.) Nevertheless, we can often find parallel phenomena in these two different worlds; the phenomena on representations and conjugacy classes are often referred to as *spectral* and *geometric* counterparts of the group theory of G, respectively. In this sense, Lusztig's Jordan decomposition can be thought of as a spectral analogue of the usual Jordan decomposition.

12.3. **Representations of Weyl groups.** In Lusztig's classification of irreducible unipotent representations of G^F, irreducible representations of the Weyl group W_0 play a crucial rule. Here we introduce some ingredients needed to state Lusztig's results.

Recall that the dimension of $\mathrm{End}_{G^F}(\mathrm{Ind}_{B^F}^{G^F} \mathbb{1})$ is given by $|W_0|$. In fact, we furthermore have that $\mathrm{End}_{G^F}(\mathrm{Ind}_{B^F}^{G^F} \mathbb{1})$ and $\mathbb{C}[W_0]$ are isomorphic as $\mathbb{C}$-algebras. This implies that the irreducible representations of G^F contained in $\mathrm{Ind}_{B^F}^{G^F} \mathbb{1}$ bijectively correspond to irreducible representations of W_0. Let ρ_χ denote the irreducible constituent of $\mathrm{Ind}_{B^F}^{G^F} \mathbb{1}$ corresponding to $\chi \in \mathrm{Irr}(W_0)$.

By the theory of *Iwahori–Hecke algebra*, we can explicitly describe the dimension of ρ_χ as a polynomial in q (the cardinality of $k = \mathbb{F}_q$). We let $d_\chi(t) \in \mathbb{Q}[t]$ be the polynomial obtained by replacing q in the explicit dimension formula of ρ_χ with "t", which is a formal variable. We call this polynomial *generic degree* or *formal dimension* of $\chi \in \mathrm{Irr}(W_0)$. We define a non-negative integer $a_\chi \in \mathbb{Z}_{\geq 0}$ to be the greatest integer such that t^{a_χ} divides $d_\chi(t)$.

On the other hand, we introduce the *coinvariant ring* $R(W_0)$ of W_0 in the following way. Let S be the symmetric algebra associated to the real vector space $X^*(T_0)_\mathbb{R}$. Since $X^*(T_0)$ has an action of W_0, this is a graded $\mathbb{R}$-algebra equipped with an action of W_0. Let J_+ be the ideal of S generated by all W-invariant homogeneous vectors of positive degree. Then we define $R(W_0) := S/J_+$. It is known that $R(W_0)$ is a finite-dimensional graded algebra $R(W_0) = \bigoplus_{i\geq 0} R_i$ such that each R_i has an action of W_0. We define a non-negative integer $b_\chi \in \mathbb{Z}_{\geq 0}$ for $\chi \in \mathrm{Irr}(W_0)$ to be the smallest integer such that R_{b_χ} contains χ as a representation of W_0.

Proposition/Definition 12.7. In general, it is known that we have $a_\chi \leq b_\chi$. We say that $\chi \in \mathrm{Irr}(W_0)$ is *special* when $a_\chi = b_\chi$.

12.4. **Unipotent representations.** Let us still keep assuming that G is split. Again recall that the G^F-conjugacy classes of k-rational maximal tori of G are parametrized by the conjugacy classes of the Weyl group W_0. Now our aim is to classify all irreducible unipotent representations of G. In other words, we want to determine the irreducible decompositions of $R_{T_w}^G(\mathbb{1})$ for $w \in W_0$, where T_w denotes any k-rational maximal torus of G^F corresponding to w.

For any $\chi \in \mathrm{Irr}(W_0)$, we define a virtual representation R_χ of G^F by

$$R_\chi := \frac{1}{|W_0|} \sum_{w\in W_0} \Theta_\chi(w) \cdot R_{T_w}^G(\mathbb{1}).$$

Then determining the irreducible decompositions of $R_{T_w}^G(\mathbb{1})$ for $w \in W_0$ is equivalent to determining the irreducible decompositions of R_χ for $\chi \in \mathrm{Irr}(W_0)$. Indeed, suppose that we know "all" about R_χ for any $\chi \in \mathrm{Irr}(W_0)$. Then we can extract the information of $R_{T_{w_0}}^G(\mathbb{1})$

for a given $w_0 \in W_0$ in the following way:

$$\sum_{\chi \in \mathrm{Irr}(W_0)} R_\chi \cdot \overline{\Theta_\chi(w_0)} = \sum_{\chi \in \mathrm{Irr}(W_0)} \frac{1}{|W_0|} \sum_{w \in W_0} \Theta_\chi(w) \cdot R^G_{T_w}(\mathbb{1}) \cdot \overline{\Theta_\chi(w_0)}$$
$$= \sum_{w \in W_0} \frac{1}{|W_0|} \sum_{\chi \in \mathrm{Irr}(W_0)} \Theta_\chi(w) \cdot \overline{\Theta_\chi(w_0)} \cdot R^G_{T_w}(\mathbb{1}) = R^G_{T_{w_0}}(\mathbb{1}).$$

Here, in the last equality, we used the fact that

$$\sum_{\chi \in \mathrm{Irr}(W_0)} \Theta_\chi(w) \cdot \overline{\Theta_\chi(w_0)} = \begin{cases} \frac{|W_0|}{|W_0 \cdot w_0|} & \text{if } w \text{ is conjugate to } w_0, \\ 0 & \text{otherwise,} \end{cases}$$

where $W_0 \cdot w_0$ denotes the conjugacy class of w_0 (the orthogonality relation of irreducible characters of a finite group; for example, see [Ser77, Chapter 2, Proposition 7]).

For any finite group Γ, we put

$$\mathcal{M}(\Gamma) := \{(x, \sigma) \mid x \in \Gamma/{\sim_\Gamma}, \sigma \in \mathrm{Irr}(\Gamma_x)\},$$

where $\Gamma/{\sim_\Gamma}$ is the set of conjugacy classes and $\Gamma_x := Z_\Gamma(x)$. We define a pairing $\{-,-\} \colon \mathcal{M}(\Gamma) \times \mathcal{M}(\Gamma) \to \mathbb{C}$ by

$$\{(x,\sigma),(y,\tau)\} := \sum_{\substack{g \in \Gamma \\ xgyg^{-1} = gyg^{-1}x}} |\Gamma_x|^{-1} \cdot |\Gamma_y|^{-1} \cdot \Theta_\sigma(gyg^{-1}) \cdot \overline{\Theta_\tau(g^{-1}xg)}.$$

For any function $f \colon \mathcal{M}(\Gamma) \to \mathbb{C}$, we define a function $\hat{f} \colon \mathcal{M}(\Gamma) \to \mathbb{C}$ by

$$\hat{f}((y,\tau)) := \sum_{(x,\sigma) \in \mathcal{M}(\Gamma)} \{(x,\sigma),(y,\tau)\} \cdot f((x,\sigma)).$$

We call the function $\hat{f}$ the *non-abelian Fourier transform* of f.

Now we explain Lusztig's result. For each family $\mathcal{F} \subset \mathrm{Irr}(W_0)$, Lusztig constructed a finite group $\Gamma_\mathcal{F}$ equipped with an embedding $\mathcal{F} \subset \mathcal{M}(\Gamma_\mathcal{F})$. We define

$$X(W_0) := \bigsqcup_{\mathcal{F}} \mathcal{M}(\Gamma_\mathcal{F}),$$

where the sum is over all families of $\mathrm{Irr}(W_0)$. For each $\chi \in \mathcal{F}$, we let z_χ denote its image in $\mathcal{M}(\Gamma_\mathcal{F}) \subset X(W_0)$. Recall that each $\mathcal{M}(\Gamma_\mathcal{F})$ is equipped with a pairing $\{-,-\}$. We extend them to $X(W_0)$ in an obvious way, i.e., for any distinct families $\mathcal{F} \neq \mathcal{F}'$, the extended pairing $\{-,-\}$ is zero on $\mathcal{M}(\Gamma_\mathcal{F}) \times \mathcal{M}(\Gamma_{\mathcal{F}'})$.

Theorem 12.8. *There exists a bijection*

$$X(W_0) \to \mathcal{E}(G^F, 1) \colon z \mapsto \rho_z$$

satisfying

$$R_\chi = \sum_{z' \in X(W_0)} \{z', z_\chi\} \cdot \rho_{z'}.$$

Remark 12.9. (1) The above theorem says that, in particular, the number of irreducible unipotent representations of G^F is independent of q. It is governed by the Weyl group W_0, which is only determined by G.

(2) In fact, when G is of type E_7 or E_8, we have to modify the definition of the pairing $\{-,-\}$ a bit for some particular families $\mathcal{F}$ called *exceptional families.*

(3) When G is simple, only possibilities of a finite group $\Gamma_{\mathcal{F}}$ for a family $\mathcal{F}$ are $(\mathbb{Z}/2\mathbb{Z})^m$ (for some $m \in \mathbb{Z}_{>0}$), $\mathfrak{S}_3$, $\mathfrak{S}_4$, $\mathfrak{S}_5$.

(4) By noting the above description of R_χ, we define a virtual representation R_z for any $z \in X(W_0)$ by

$$R_z = \sum_{z' \in X(W_0)} \{z', z\} \cdot \rho_{z'}.$$

This virtual representation (or its character) is called an *almost character* of G^F.

(5) By looking at the book [Lus84] (or also [Car85, Sections 13.8 and 13.9]), we can find tables of all irreducible unipotent representations of G^F.

13. Week 13: Example session

13.1. **Algebraic characterization of regular Deligne–Lusztig representations.** In this course, we have studied Deligne–Lusztig's construction of a virtual representation $R_T^G(\theta)$, which is critically based on very deep geometric discussions. The motivating problem we want to discuss here is the following:

Q1. Is there a purely-algebraic characterization of $R_T^G(\theta)$?

Let us recall the Deligne–Lusztig character formula:

Theorem 13.1 (Deligne–Lusztig character formula)**.** *Let $g \in G^F$ with Jordan decomposition $g = su$. Then we have*

$$R_T^G(\theta)(g) = \frac{1}{|(G_s^\circ)^F|} \sum_{\substack{x \in G^F \\ x^{-1}sx \in T^F}} \theta(x^{-1}sx) \cdot Q_{{}^xT}^{G_s^\circ}(u).$$

Since any virtual representation is uniquely determined by its character, we can think of this formula as the characterization of the Deligne–Lusztig virtual representation $R_T^G(\theta)$. However, the right-hand side contains the Green functions $Q_{{}^xT}^{G_s^\circ}$. Remember that it is the restriction of the character of $R_{{}^xT}^{G_s^\circ}(\mathbb{1})$ to the set of unipotent elements; so its definition unavoidably depends on geometry.

But then, how about looking at the character values only on regular semisimple elements? Recall the following (an easy consequence of the Deligne–Lusztig character formula):

Corollary 13.2. *Suppose that $s \in G_{\mathrm{rs}}^F$ (the set of regular semisimple elements of G^F).*

1. *If s is not conjugate to any element of T^F, then we have $R_T^G(\theta)(s) = 0$.*
2. *If s is conjugate to an element of T^F (suppose that $s \in T^F$), then we have*

$$R_T^G(\theta)(s) = \sum_{w \in W_{G^F}(T)} {}^w\theta(s),$$

where $W_{G^F}(T) := N_{G^F}(T)/T^F$.

The right-hand side of this formula only consists of purely algebraic quantities! So we next come up with the following question:

Q2. Is the above character formula on G_{rs}^F enough to characterize $R_T^G(\theta)$?

In general, to determine a given representation from its character, we have to look at all its character values. However, sometimes (depending on a group and a representation), it is possible to determine a given representation by only looking at its character values on some special elements. For example, recall that $\mathfrak{S}_3$ has two 1-dimensional irreducible representations and only 2-dimensional representation. This means that, to distinguish the 2-dimensional irreducible representation from the others, it is only enough to look at their character values at 1! This example is maybe too stupid, but in any case we can hope that we could give an affirmative answer to Q2 in some cases.

Indeed, we can find the following "reasonable" answer[27]:

[27] This result is due to Charlotte Chan and I (joint work), which is based on a preceding work of Guy Henniart.

Theorem 13.3. *Let $\theta\colon T^F \to \mathbb{C}^\times$ be a regular character, i.e., $\{w \in W_{G^F}(T) \mid {}^w\theta = \theta\} = \{1\}$. Suppose that the following inequality holds:*

$$\frac{|T^F|}{|T^F \smallsetminus T^F_{\mathrm{rs}}|} > 2 \cdot |W_{G^F}(T)|.$$

Then $R^G_T(\theta)$ is the unique irreducible representation (up to sign) such that, for any $s \in G^F_{\mathrm{rs}}$,

$$(*) \qquad R^G_T(\theta) = \begin{cases} 0 & \text{if } s \text{ is not conjugate to elements of } T^F, \\ \sum_{w \in W_{G^F}(T)} {}^w\theta(s) & \text{if } s \in T^F. \end{cases}$$

Here, the subscript "rs" denotes the subset of regular semisimple elements.

Before we proceed, let us give some comments. First, the inequality in the assumption basically says that we have "many" regular semisimple elements. Thus the intuitive meaning of this theorem is that "if we have sufficiently many regular semisimple elements, the Deligne–Lusztig character formula on regular semisimple elements is enough to determine a regular Deligne–Lustig representation". Because this inequality is first considered in the work of Henniart for $G = \mathrm{GL}_n$, let us call it the *Henniart inequality*.

Second, recall that $|T^F|$ can be described by looking at the characteristic polynomial of a Weyl element which defines the k-rational maximal torus T. In fact, it is also possible to determine $|T^F_{\mathrm{rs}}|$ as long as G and the Weyl element are explicitly specified. Thus, in principle, we can explicate the Henniart inequality. In particular, we can show that the Henniart inequality always holds whenever q is sufficiently large; we will present some examples later.

Now let us prove the above theorem. In the following, we put $G_{\mathrm{nrs}} := G \smallsetminus G_{\mathrm{rs}}$ and $T_{\mathrm{nrs}} := T \smallsetminus T_{\mathrm{rs}}$. According to the disjoint union decomposition $G^F = G^F_{\mathrm{rs}} \sqcup G^F_{\mathrm{nrs}}$, we divide the inner product $\langle -, - \rangle$ on the space of class function on G^F as follows:

$$\langle f_1, f_2 \rangle_\bullet := \frac{1}{|G^F|} \sum_{g \in G^F_\bullet} f_1(g) \cdot \overline{f_2(g)},$$

where $\bullet \in \{\mathrm{rs}, \mathrm{nrs}\}$. Hence we have $\langle f_1, f_2 \rangle = \langle f_1, f_2 \rangle_{\mathrm{rs}} + \langle f_1, f_2 \rangle_{\mathrm{nrs}}$.

Proof. Suppose that ρ is another irreducible virtual representation of G^F satisfying the same character formula as $R^G_T(\theta)$ on G^F_{rs}. We put $R := R^G_T(\theta)$ Then our task is to show that $\langle \rho, R \rangle \neq 0$.

We have

$$\langle \rho, \rho \rangle = \langle \rho, \rho \rangle_{\mathrm{rs}} + \langle \rho, \rho \rangle_{\mathrm{nrs}}$$

and

$$\langle R, R \rangle = \langle R, R \rangle_{\mathrm{rs}} + \langle R, R \rangle_{\mathrm{nrs}}.$$

Since both ρ and R are irreducible (the latter is due to that θ is regular), we have $\langle \rho, \rho \rangle = \langle R, R \rangle = 1$. On the other hand, by the assumption on ρ, we also have $\langle \rho, \rho \rangle_{\mathrm{rs}} = \langle R, R \rangle_{\mathrm{rs}}$. Hence we get $\langle \rho, \rho \rangle_{\mathrm{nrs}} = \langle R, R \rangle_{\mathrm{nrs}}$. Let us put

$$X := \langle \rho, \rho \rangle_{\mathrm{rs}} = \langle R, R \rangle_{\mathrm{rs}}, \quad Y := \langle \rho, \rho \rangle_{\mathrm{nrs}} = \langle R, R \rangle_{\mathrm{nrs}}$$

(thus X and Y are non-negative numbers satisfying $X + Y = 1$).

We have

$$\langle \rho, R \rangle = \langle \rho, R \rangle_{\mathrm{rs}} + \langle \rho, R \rangle_{\mathrm{nrs}}.$$

Again by the assumption on ρ, we have $\langle \rho, R \rangle_{\mathrm{rs}} = X$. On the other hand, by the Cauchy–Schwarz inequality, we have

$$|\langle \rho, R \rangle_{\mathrm{nrs}}| \leq \langle \rho, \rho \rangle^{\frac{1}{2}}_{\mathrm{nrs}} \cdot \langle R, R \rangle^{\frac{1}{2}}_{\mathrm{nrs}} = Y.$$

Therefore, if we have $X > Y$, then $\langle \rho, R \rangle$ cannot be equal to 0. Since $X + Y = 1$, the condition $X > Y$ is equivalent to that $X > \frac{1}{2}$.

Let us evaluate X:

$$X = \langle R, R \rangle_{\mathrm{rs}} = \frac{1}{|G^F|} \sum_{g \in G^F_{\mathrm{rs}}} R^G_T(\theta)(g) \cdot \overline{R^G_T(\theta)(g)}.$$

By the regular semisimple Deligne–Lusztig character formula, $R^G_T(\theta)(g) = 0$ for any $g \in G^F_{\mathrm{rs}}$ which is not conjugate to an element of T^F. Note that we have

$$\begin{aligned}(G^F/N_{G^F}(T)) \times T^F_{\mathrm{rs}} &\xrightarrow{1:1} \{g \in G^F_{\mathrm{rs}} \mid g \text{ is conjugate to an element of } T^F\} \\ (x, t) &\mapsto xtx^{-1}.\end{aligned}$$

Thus, again by using the regular semisimple Deligne–Lusztig character formula, we have

$$\begin{aligned} X &= \frac{1}{|G^F|} \sum_{x \in G^F/N_{G^F}(T)} \sum_{t \in T^F_{\mathrm{rs}}} R^G_T(\theta)(xtx^{-1}) \cdot \overline{R^G_T(\theta)(xtx^{-1})} \\ &= \frac{1}{|G^F|} \sum_{x \in G^F/N_{G^F}(T)} \sum_{t \in T^F_{\mathrm{rs}}} R^G_T(\theta)(t) \cdot \overline{R^G_T(\theta)(t)} \\ &= \frac{1}{|N_{G^F}(T)|} \sum_{t \in T^F_{\mathrm{rs}}} \sum_{w, w' \in W_{G^F}(T)} {}^w\theta(t) \cdot \overline{{}^{w'}\theta(t)}. \end{aligned}$$

By noting that $T^F_{\mathrm{rs}} = T^F - T^F_{\mathrm{nrs}}$, we get

$$X = \frac{1}{|N_{G^F}(T)|} \sum_{w, w' \in W_{G^F}(T)} \left(\sum_{t \in T^F} {}^w\theta(t) \cdot \overline{{}^{w'}\theta(t)} - \sum_{t \in T^F_{\mathrm{nrs}}} {}^w\theta(t) \cdot \overline{{}^{w'}\theta(t)} \right).$$

Here, since θ is regular, the orthogonality relation of characters implies that

$$\sum_{t \in T^F} {}^w\theta(t) \cdot \overline{{}^{w'}\theta(t)} = \begin{cases} |T^F| & \text{if } w = w', \\ 0 & \text{otherwise.} \end{cases}$$

Thus we get

$$\begin{aligned} X &= \frac{1}{|N_{G^F}(T)|} \cdot |W_{G^F}(T)| \cdot |T^F| - \frac{1}{|N_{G^F}(T)|} \cdot \sum_{w, w' \in W_{G^F}(T)} \sum_{t \in T^F_{\mathrm{nrs}}} {}^w\theta(t) \cdot \overline{{}^{w'}\theta(t)} \\ &= 1 - \frac{1}{|N_{G^F}(T)|} \cdot \sum_{w, w' \in W_{G^F}(T)} \sum_{t \in T^F_{\mathrm{nrs}}} {}^w\theta(t) \cdot \overline{{}^{w'}\theta(t)}. \end{aligned}$$

Hence, the triangle inequality implies that

$$X \geq 1 - \frac{1}{|N_{G^F}(T)|} \cdot |W_{G^F}(T)|^2 \cdot |T^F_{\mathrm{nrs}}| = 1 - \frac{|T^F_{\mathrm{nrs}}|}{|T^F|} \cdot |W_{G^F}(T)|.$$

Note that the Henniart inequality is equivalent to that

$$\frac{|T^F_{\mathrm{nrs}}|}{|T^F|} \cdot |W_{G^F}(T)| < \frac{1}{2}.$$

Hence, if the Henniart inequality holds, we obtain $X > \frac{1}{2}$. □

13.2. **Henniart inequality for Coxeter tori of exceptional groups.** As mentioned above, as long as the group G and its maximal torus T are specified, it is possible to explicate the Henniart inequality. For example, for any split adjoint simple group of exceptional type, the Henniart inequality for a k-rational maximal torus S of "Coxeter type"[28] is as in the following table:

Table 2. Henniart inequalities for Coxeter tori of exceptional groups

G	$\lvert S^F \rvert$	$\lvert S^F_{\mathrm{nrs}} \rvert$	condition on q
E_6	$(q^4-q^2+1)(q^2+q+1)$	q^2+q+1	$q>2$
E_7	$(q^6-q^3+1)(q+1)$	$\begin{cases} 3(q+1) & q\equiv -1 \mod 3 \\ q+1 & q\not\equiv -1 \mod 3 \end{cases}$	$\begin{cases} q>2 \\ q\text{: any} \end{cases}$
E_8	$q^8+q^7-q^5-q^4-q^3+q+1$	1	q: any
F_4	q^4-q^2+1	1	$q>2$
G_2	q^2-q+1	$\begin{cases} 3 & q\equiv -1 \mod 3 \\ 1 & q\not\equiv -1 \mod 3 \end{cases}$	$\begin{cases} q>6 \\ q>3 \end{cases}$

Therefore, only the cases which do not satisfy the Henniart inequality are

- G is of type E_6, $q=2$;
- G is of type F_4, $q=2$;
- G is of type G_2, $q=2,3,5$.

13.3. **The case of $G_2(\mathbb{F}_3)$.** In the following, let us investigate what is happening in the case where $G=G_2$ over $\mathbb{F}_3$. In fact, in this case, our characterization theorem for regular Deligne–Lusztig representations does not hold!

First, again recall that G^F-conjugacy classes of k-rational maximal tori of G are parametrized by the conjugacy classes in the absolute Weyl group W_0 of G. The group G_2 has 6 conjugacy classes; they are named "$\varnothing$", "A_1", "$\tilde{A}_1$", "$A_1\times\tilde{A}_1$", "A_2", and "G_2" (see [Car72, Table 7]). For any such conjugacy class Γ, let us write T_Γ for a k-rational maximal torus corresponding to Γ. Then the orders of T_Γ^F and $W_{G^F}(T_\Gamma)$ are given as follows (see also [Car72, Table 3 and Lemma 26]):

Table 3. Maximal tori of $G_2(\mathbb{F}_q)$

Γ	$\lvert T_\Gamma^F \rvert$	$\lvert T_\Gamma^F \rvert$ $(q=3)$	$\lvert W_{G^F}(T_\Gamma) \rvert$	split rank
$\varnothing$	$(q-1)^2$	4	12	2 (split)
A_1	$(q-1)(q+1)$	8	4	1
$\tilde{A}_1$	$(q-1)(q+1)$	8	4	1
$A_1\times\tilde{A}_1$	$(q+1)^2$	16	12	0 (elliptic)
A_2	q^2+q+1	13	6	0 (elliptic)
G_2	q^2-q+1	7	6	0 (elliptic, Coxeter)

These are actually contained in GAP3. To see it, first put:

```
gap> W:=CoxeterGroup("G",2);
```

[28] The Weyl group has a particular conjugacy class consisting of elements called "Coxeter elements". The maximal torus S corresponds to the Coxeter conjugacy class.

Then, by putting

```
gap> CharTable(W).classnames;
```

GAP3 gives the following output:

```
[ "A0", "~A1", "A1", "G2", "A2", "A1+~A1" ]
```

Moreover, the following gives the list of rational maximal tori corresponding to the above conjugacy classes:

```
gap> Twistings(W,[]);
```

```
[ (q-1)^2, (q-1)(q+1), (q-1)(q+1), (q^2-q+1), (q^2+q+1), (q+1)^2 ]
```

Note that "G_2" is the conjugacy class of Coxeter elements. Hence T_{G_2} is our maximal torus S. We can check that S^F has a non-regular semisimple element other than unit if and only if $q \equiv -1 \pmod 3$; in this case, the number of non-regular semisimple elements is 3. Also note that the rational Weyl group $W_{G^F}(S)$ is cyclic of order 6.

From now on, we focus on the case where $k = \mathbb{F}_3$.

The group $G_2(\mathbb{F}_3)$ has 23 conjugacy classes, hence has 23 irreducible representations. Table 4 is the list of 23 conjugacy classes; if a conjugacy class has name "nx", then it means that the order of any representative of the class is given by n. The last column of Table 4 expresses which tori contain semisimple elements within the conjugacy classes.

TABLE 4. Conjugacy classes of $G_2(\mathbb{F}_3)$

conjugacy class	order	order of centralizer	type	tori
1a	1	4245696	unit	all
2a	2	576	ss.	$\varnothing, A_1, \tilde{A}_1, A_1 \times \tilde{A}_1$
3a	3	5832	unip.	–
3b	3	5832	unip.	–
3c	3	729	unip.	–
3d	3	162	unip.	–
3e	3	162	unip.	–
4a	4	96	ss.	$\tilde{A}_1, A_1 \times \tilde{A}_1$
4b	4	96	ss.	$A_1, A_1 \times \tilde{A}_1$
6a	6	72	–	–
6b	6	72	–	–
6c	6	18	–	–
6d	6	18	–	–
7a	7	7	reg. ss.	G_2
8a	8	8	reg. ss.	$\tilde{A}_1$
8b	8	8	reg. ss.	A_1
9a	9	27	unip.	–
9b	9	27	unip.	–
9c	9	27	unip.	–
12a	12	12	–	–
12b	12	12	–	–
13a	13	13	reg. ss.	A_2
13b	13	13	reg. ss.	A_2

The character table of $G_2(\mathbb{F}_3)$ is as in Table 7. The 23 irreducible representations are named “Xn” in the decreasing order according to their dimensions. This table is cited from GAP3 ([S$^+$97]); if you are familiar with GAP3, Table 7 can be output just by typing:

```
>gap DisplayCharTable( CharTable( "G2(3)" ) );
```

(see `https://webusers.imj-prg.fr/~jean.michel/gap3/htm/chap049.htm#SECT037` for the details). In the following, we write X_n for the irreducible representation Xn.

We remark that among the 23 irreducible representations, the unipotent representations are

$$X_1, X_2, X_3, X_4, X_5, X_7, X_8, X_9, X_{10}, X_{21}$$

(X_2, X_3, X_4, and X_5 are cuspidal unipotent representations). This can be also seen by using GAP3:

```
gap> Display(UnipotentCharacters(CoxeterGroup("G",2)));
```

In the GAP3 output, the above unipotent representations are expressed as `phi{1,0}`, `G2[1]`, `G2[E3]`, `G2[E3^2]`, `G2[-1]`, `phi{1,3}'`, `phi{1,3}''`, `phi{2,1}`, `phi{2,2}`, `phi{1,6}`. (See `https://webusers.imj-prg.fr/~jean.michel/gap3/htm/chap098.htm` and also [Lus84, 372 page].)

TABLE 5. Unipotent representations of $G_2(\mathbb{F}_q)$

GAP3 label	dimension	label ($q=3$)	dim ($q=3$)	label (Lusztig)
`phi{1,0}`	1	X_1	1	–
`phi{1,6}`	q^6	X_{21}	729	–
`phi{1,3}'`	$q\phi_3(q)\phi_6(q)/3$	X_7	91	$(1,r)$
`phi{1,3}''`	$q\phi_3(q)\phi_6(q)/3$	X_8	91	$(g_3,1)$
`phi{2,1}`	$q\phi_2^2(q)\phi_3(q)/6$	X_9	104	$(1,1)$
`phi{2,2}`	$q\phi_2^2(q)\phi_6(q)/2$	X_{10}	168	$(g_2,1)$
`G2[-1]`	$q\phi_1^2(q)\phi_3(q)/2$	X_5	78	(g_2,ε)
`G2[1]`	$q\phi_1^2(q)\phi_6(q)/6$	X_2	14	$(1,\varepsilon)$
`G2[E3]`	$q\phi_1^2(q)\phi_2^2(q)/3$	X_3	64	(g_3,θ)
`G2[E3^2]`	$q\phi_1^2(q)\phi_2^2(q)/3$	X_4	64	(g_3,θ^2)

$\phi_1(q) = q-1$, $\phi_2(q) = q+1$, $\phi_3(q) = q^2+q+1$, $\phi_6(q) = q^2-q+1$.

We also remark that each unipotent representation is realized in $R^G_{T_\Gamma}(\mathbb{1})$ as in Table 6. To see this via GAP3, type the following:

```
gap> DeligneLusztigCharacter(CoxeterGroup("G",2),n);
```

Here, `n` means the nth conjugacy class of the Weyl group of G_2, where the conjugacy classes are arranged in the following order (`gap> PrintRec(CoxeterGroup("G",2));`):

$$\varnothing, \tilde{A}_1, A_1, G_2, A_2, A_1 \times \tilde{A}_1.$$

Now we discuss a counterexample to our characterization theorem for regular Deligne–Lusztig representations. We have $S^F \cong \mathbb{Z}/7\mathbb{Z}$ and $S^F_{\mathrm{nrs}} = \{1\}$. Moreover, we can check that $W_{G^F}(S)$ acts on the set of regular semisimple elements of S^F simply-transitively. Thus we see that there exists only one regular character θ of S^F up to conjugation.

By the dimension formula of Deligne–Lusztig representations, we have

$$\dim R^G_S(\theta) = \frac{|G^F|}{\dim \mathrm{St}_G \cdot |S^F|}.$$

Table 6. Unipotent Deligne–Lusztig representations $G_2(\mathbb{F}_q)$

Γ	$R^G_{T_\Gamma}(\mathbb{1})$
$\varnothing$	$X_1 + X_7 + X_8 + 2X_9 + 2X_{10} + X_{21}$
A_1	$X_1 - X_7 + X_8 - X_{21}$
$\tilde{A}_1$	$X_1 + X_7 - X_8 - X_{21}$
$A_1 \times \tilde{A}_1$	$X_1 - 2X_2 - 2X_5 - X_7 - X_8 + X_{21}$
A_2	$X_1 + X_2 - X_3 - X_4 - X_{10} + X_{21}$
G_2	$X_1 + X_3 + X_4 + X_5 - X_9 + X_{21}$

Note that $r_G = 2$ and $r_S = 0$, hence the sign appearing in the dimension formula is trivial. In other words, $R^G_S(\theta)$ is a genuine representation. Since we have

- $|G^F| = q^6 \cdot (q^2 - 1) \cdot (q^6 - 1)$ (see [Car85, Section 2.9]),
- $\dim \mathrm{St}_G = q^6$ (see [Car85, Proposition 6.4.4]),
- $|S^F| = q^2 - q + 1$,

we have

$$\dim R^G_S(\theta) = (q-1)^2 \cdot (q+1)^2 \cdot (q^2 + q + 1) = 832.$$

Thus we conclude that $R^G_S(\theta)$ is the irreducible representation X_{23}.

By the above description of the group S^F and the action of $W_{G^F}(S)$ on S^F, we see that

$$\sum_{w \in W_{G^F}(S)} \theta^w(s) = \sum_{i=1}^{6} \zeta_7^i = -1$$

for any regular semisimple element $s \in S^F$, where ζ_7 is a primitive 7th root of unity. Therefore, our characterization theorem in this case is asking whether an irreducible virtual representation of G^F such that

- $\Theta_\rho(s) = 0$ if the conjugacy class of s is one of "8a", "8b", "13a", and "13b" (see Table 4) and
- $\Theta_\rho(s) = \pm 1$ if the conjugacy class of s is "7a" (see Table 4)

is necessarily equal to $\pm R^G_S(\theta)$ or not.

By looking at the character table (Table 7), we can easily find that X_5 and X_9 satisfy these assumptions!

TABLE 7. Character table of $G_2(\mathbb{F}_3)$

	1a	2a	3a	3b	3c	3d	3e	4a	4b	6a	6b	6c	6d	7a	8a	8b	9a	9b	9c	12a	12b	13a	13b
X1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
X2	14	−2	5	5	−4	2	−1	2	2	1	1	−2	1	·	·	·	2	−1	−1	−1	−1	1	1
X3	64	·	−8	−8	1	4	−2	·	·	·	·	·	·	1	·	·	1	α	β	·	·	−1	−1
X4	64	·	−8	−8	1	4	−2	·	·	·	·	·	·	1	·	·	1	β	α	·	·	−1	−1
X5	78	−2	−3	−3	−3	−3	6	2	2	1	1	1	−2	1	·	·	·	·	·	−1	−1	·	·
X6	91	−5	10	10	10	1	1	3	3	−2	−2	1	1	·	−1	−1	1	1	1	·	·	·	·
X7	91	3	−8	19	1	4	−2	3	−1	·	3	·	·	·	1	−1	−2	1	1	·	−1	·	·
X8	91	3	19	−8	1	4	−2	−1	3	3	·	·	·	·	−1	1	−2	1	1	−1	·	·	·
X9	104	8	14	14	5	2	−1	·	·	2	2	2	−1	−1	·	·	2	−1	−1	·	·	·	·
X10	168	8	6	6	6	−3	6	·	·	2	2	−1	2	·	·	·	·	·	·	·	·	−1	−1
X11	182	6	20	−7	−7	2	2	2	2	·	−3	·	·	·	·	·	−1	−1	−1	2	−1	·	·
X12	182	6	−7	20	−7	2	2	2	2	−3	·	·	·	·	·	·	−1	−1	−1	−1	2	·	·
X13	273	−7	30	3	3	3	3	−3	1	2	−1	−1	−1	·	1	−1	·	·	·	·	1	·	·
X14	273	−7	3	30	3	3	3	1	−3	−1	2	−1	−1	·	−1	1	·	·	·	1	·	·	·
X15	448	·	16	16	−11	−2	−2	·	·	·	·	·	·	·	·	·	1	1	1	·	·	γ	δ
X16	448	·	16	16	−11	−2	−2	·	·	·	·	·	·	·	·	·	1	1	1	·	·	δ	γ
X17	546	2	−21	6	6	−3	−3	−2	6	−1	2	−1	−1	·	·	·	·	·	·	1	·	·	·
X18	546	2	6	−21	6	−3	−3	6	−2	2	−1	−1	−1	·	·	·	·	·	·	·	1	·	·
X19	728	−8	26	−28	−1	−1	−1	·	·	−2	4	1	1	·	·	·	−1	−1	−1	·	·	·	·
X20	728	−8	−28	26	−1	−1	−1	·	·	4	−2	1	1	·	·	·	−1	−1	−1	·	·	·	·
X21	729	9	·	·	·	·	·	−3	−3	·	·	·	·	1	−1	−1	·	·	·	·	·	1	1
X22	819	3	9	9	9	·	·	−1	−1	−3	−3	·	·	·	1	1	·	·	·	−1	−1	·	·
X23	832	·	−32	−32	−5	4	4	·	·	·	·	·	·	−1	·	·	1	1	1	·	·	·	·

every dot denotes 0, $\alpha := \frac{-1+3\sqrt{-3}}{2}$, $\beta := \frac{-1-3\sqrt{-3}}{2}$, $\gamma := \frac{-1+\sqrt{13}}{2}$, $\delta := \frac{-1-\sqrt{13}}{2}$.

13.4. **Unipotent representations.** One may notice that the above counterexample is given by unipotent representations. In fact, this is not an accident.

Let G be a connected reductive group over $k = \mathbb{F}_q$, T a k-rational maximal torus of G, and θ a regular character of T^F. We suppose that ρ is an irreducible representation of G^F having the same regular semisimple character values as $R_T^G(\theta)$.

Lemma 13.4. *Suppose that there exists a character $\theta' \colon T^F \to \mathbb{C}^\times$ such that*

(i) θ and θ' are not $W_{G^F}(T)$-conjugate, and

(ii) $\theta|_{T^F_{\mathrm{nrs}}} \equiv \theta'|_{T^F_{\mathrm{nrs}}}$.

Then we have either $\langle \rho, R_T^G(\theta)\rangle \neq 0$ or $\langle \rho, R_T^G(\theta')\rangle \neq 0$.

Proof. By the regularity of θ, we have

$$1 = \langle R_T^G(\theta), R_T^G(\theta)\rangle = \langle R_T^G(\theta), R_T^G(\theta)\rangle_{\mathrm{rs}} + \langle R_T^G(\theta), R_T^G(\theta)\rangle_{\mathrm{nrs}}. \tag{1}$$

On the other hand, by the assumption (i) and the inner product formula, we have

$$0 = \langle R_T^G(\theta), R_T^G(\theta')\rangle = \langle R_T^G(\theta), R_T^G(\theta')\rangle_{\mathrm{rs}} + \langle R_T^G(\theta), R_T^G(\theta')\rangle_{\mathrm{nrs}}. \tag{2}$$

Recall that, by the Deligne–Lusztig character formula

$$R_T^G(\theta)(g) = \frac{1}{|(G_s^\circ)^F|} \sum_{\substack{x \in G^F \\ x^{-1}sx \in T^F}} \theta(x^{-1}sx) \cdot Q_{{}^xT}^{G_s^\circ}(u).$$

we see that the character of $R_T^G(\theta)$ on G^F_{nrs} depends only on $\theta|_{T^F_{\mathrm{nrs}}}$. Thus assumption (ii) implies that $R_T^G(\theta)$ equals $R_T^G(\theta')$ on G^F_{nrs}. In particular, we have $\langle R_T^G(\theta), R_T^G(\theta)\rangle_{\mathrm{nrs}} = \langle R_T^G(\theta), R_T^G(\theta')\rangle_{\mathrm{nrs}}$. Thus, by the equalities (1) and (2), we get $\langle R_T^G(\theta), R_T^G(\theta)\rangle_{\mathrm{rs}} \neq \langle R_T^G(\theta), R_T^G(\theta')\rangle_{\mathrm{rs}}$.

We next look at the following two equalities:

$$\langle \rho, R_T^G(\theta)\rangle = \langle \rho, R_T^G(\theta)\rangle_{\mathrm{rs}} + \langle \rho, R_T^G(\theta)\rangle_{\mathrm{nrs}}, \tag{3}$$

$$\langle \rho, R_T^G(\theta')\rangle = \langle \rho, R_T^G(\theta')\rangle_{\mathrm{rs}} + \langle \rho, R_T^G(\theta')\rangle_{\mathrm{nrs}}. \tag{4}$$

Again by the same observation as above, we have $\langle \rho, R_T^G(\theta)\rangle_{\mathrm{nrs}} = \langle \rho, R_T^G(\theta')\rangle_{\mathrm{nrs}}$. Moreover, by the assumption on ρ, we have $\langle \rho, R_T^G(\theta)\rangle_{\mathrm{rs}} = \langle R_T^G(\theta), R_T^G(\theta)\rangle_{\mathrm{rs}}$ and $\langle \rho, R_T^G(\theta')\rangle_{\mathrm{rs}} = \langle R_T^G(\theta), R_T^G(\theta')\rangle_{\mathrm{rs}}$. Since we obtained $\langle R_T^G(\theta), R_T^G(\theta)\rangle_{\mathrm{rs}} \neq \langle R_T^G(\theta), R_T^G(\theta')\rangle_{\mathrm{rs}}$ in the previous paragraph, we have $\langle \rho, R_T^G(\theta)\rangle_{\mathrm{rs}} \neq \langle \rho, R_T^G(\theta')\rangle_{\mathrm{rs}}$. Therefore, by combining these equalities with (3) and (4), we get $\langle \rho, R_T^G(\theta)\rangle \neq \langle \rho, R_T^G(\theta')\rangle$. In particular, at least one of $\langle \rho, R_T^G(\theta)\rangle$ and $\langle \rho, R_T^G(\theta')\rangle$ is not zero. □

Note that Lemma 13.4 has the following immediate consequence (choose θ' to be the trivial character $\mathbb{1}$ of T^F):

Lemma 13.5. *If $\theta|_{T^F_{\mathrm{nrs}}} \equiv \mathbb{1}$, then we have either $\langle \rho, R_T^G(\theta)\rangle \neq 0$ or $\langle \rho, R_T^G(\mathbb{1})\rangle \neq 0$.*

Hence we get the following theorem (note that this result requires NO assumption on q):

Theorem 13.6. *Suppose that θ is a regular character of T^F whose restriction to T^F_{nrs} is trivial. Suppose that ρ is an irreducible representation of G^F equipped with a sign ε such that, for any regular semisimple element $g \in G^F$,*

$$\Theta_\rho(g) = \varepsilon \cdot \Theta_{R_T^G(\theta)}(g).$$

If ρ is not unipotent, then we necessarily have $\rho \cong \varepsilon R_T^G(\theta)$.

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