Positive-depth Deligne–Lusztig induction for p-adic reductive groups

(j.w. w/ Charlotte Chan)

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Masao Oi

(National Taiwan Univ.)

§1. Introduction.

- F: a non-archimedean local field
  (i.e. a finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$ (p: a prime number)
  (p-adic number field)

- G: a connected reductive group / F
  (e.g. $G = GL_n, SL_n, Sp_{2n}, SO_n, U_n, G_2, \ldots$)

  $\rightsquigarrow$ $G(F)$: locally compact (profinite) group "p-adic reductive gp"

☆ We are interested in representation theory of $G(F)$.

Why?? ◎ $\mathbb{Q}_p, \mathbb{F}_p((t))$ vs $\mathbb{R}$: both are locally compact.
("local fields")

$\rightsquigarrow$ $G(F)$ should be parallel to $G(\mathbb{R})$.

◎ "local version" of automorphic representations.

⋮

$\{$irreducible smooth representations of $G(F)\}/\sim$

$\Big($ (π, V) a rep. of $G(F)$ is smooth $\underset{\text{def.}}{\Longleftrightarrow}$ For any $v \in V$, $\mathrm{Stab}_{G(F)}(v) \underset{\text{op}}{\subset} G(F)$ $\Big)$

$\bigcup$

$\{$irreducible supercuspidal repn's of $G(F)\}/\sim$

$\Big($ $\underset{\text{def.}}{\Longleftrightarrow}$ cannot obtained by "parabolic induction". $\Big)$

"building blocks"

Ultimate goal
Construct / Classify all them!

Conj (folklore).

(π, V) an irr. smooth rep. of $G(F)$ is supercuspidal

$\Longleftrightarrow$ π is of the form $\text{c-Ind}_K^{G(F)} \rho$ for some

- $K \subset G(F)$ open cpt-mod-center subgp.
- ρ a fin. dim. irr. rep. of K

Note (basic fact).

If $\left(\begin{array}{l} - K \subset G(F) \text{ open cpt-mod-center subgp.} \\ - \rho \text{ a fin.dim. irr. rep. of } K \end{array} \right.$

satisfies that $\pi := \text{c-Ind}_K^{G(F)} \rho$ is irred, then π is s.c.

But... finding such $\underbrace{(K, \rho)}_{\text{call "a s.c. type"}}$ is not easy at all!

Non-example

$\rho = \mathbb{1}$: triv. rep. of K.

$\rightsquigarrow \mathrm{End}_{G(F)}(\text{c-Ind}_K^{G(F)} \mathbb{1}) \simeq \mathrm{Hom}_K(\mathbb{1}, \mathrm{Res}_K^{G(F)} \text{c-Ind}_K^{G(F)} \mathbb{1})$.

$\simeq \underbrace{\{ f : K \backslash G(F) / K \to \mathbb{C} \ : \text{cpt. supp.} \}}_{\text{very big}}$.

Key: How to find a "nice" K & its irr. rep. ρ.

Today: Compare two different constructions of s.c. types.

① Algebraic construction by J.-K. Yu.

② Geometric construction by Deligne-Lusztig,

& Chan-Ivanov

§. Algebraic construction.

(a) "depth-zero case"

e.g. $\cdot\ G = GL_2 / F$.

$\cdot\ F = \mathbb{Q}_p \supset \mathcal{O}_F = \mathbb{Z}_p$ ring of integers.

$$\rightsquigarrow \quad G(F) \supset G(\mathcal{O}_F) \twoheadrightarrow G(\mathbb{F}_p).$$

$$\begin{pmatrix} \mathbb{Q}_p & \mathbb{Q}_p \\ \mathbb{Q}_p & \mathbb{Q}_p \end{pmatrix} \supset \begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ \mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix} \twoheadrightarrow \begin{pmatrix} \mathbb{F}_p & \mathbb{F}_p \\ \mathbb{F}_p & \mathbb{F}_p \end{pmatrix}$$

$\cdot$ Take an irred. cuspidal rep. of $GL_2(\mathbb{F}_p)$.

$\rightsquigarrow$ pull back to $GL_2(\mathbb{Z}_p)$.

$\rightsquigarrow$ extend to $\underset{\text{center of } GL_2(\mathbb{Q}_p)}{\underbrace{\mathbb{Q}_p^\times}} \cdot GL_2(\mathbb{Z}_p) =: K$. get a s.c. type

ρ !! (K, ρ)

☆ To generalize this construction to general G, we appeal to "Bruhat–Tits theory".

- G : a conn. red. gp / F

$\rightsquigarrow$ $\mathcal{B}(G,F)$: Bruhat–Tits building of G. $\curvearrowleft G(F)$.
(a poly-simplicial set)

$$\begin{array}{l} \mathcal{B}(G,F) \\ \;\;\rotatebox{90}{$\in$}\; \\ \;\; x \\ \;\;\downarrow \\ G_{x,0} := \mathrm{Stab}_{G(F)}(x) \end{array}$$

$G_{x,0} := \mathrm{Stab}_{G(F)}(x)$: parahoric subgroup.
(x : vertex $\Leftrightarrow$ $G_{x,0}$ maximal)

$\triangledown$

$G_{x,r}$ $(r \in \mathbb{R}_{\geq 0})$: Moy–Prasad filtration

e.g. $G = GL_2$.

$$G_{x,0} = \begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ \mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix} \supsetneq \underset{\substack{\| \\ G_{x,0+}}}{G_{x,1}} = \begin{pmatrix} 1+p\mathbb{Z}_p & p\mathbb{Z}_p \\ p\mathbb{Z}_p & 1+p\mathbb{Z}_p \end{pmatrix} \supsetneq G_{x,2} = \begin{pmatrix} 1+p^2\mathbb{Z}_p & p^2\mathbb{Z}_p \\ p^2\mathbb{Z}_p & 1+p^2\mathbb{Z}_p \end{pmatrix} \rhd \cdots$$

- $G_{x,0}/G_{x,0+} = \mathbb{G}(\mathbb{F}_q)$ for a conn. red. gp. $\mathbb{G}/\mathbb{F}_q$. ← residue field of $\mathcal{O}_F$.
- Take an irr. cusp. rep. ρ of $\mathbb{G}(\mathbb{F}_q)$

$\rightsquigarrow$ extend to $\underset{\text{center}}{Z}(G(F)) \cdot G_{x,0} =: K$

$\rho^{!!}$

$\rightsquigarrow$ get a s.c. type (K, ρ).

A s.c type obtained in this way is called "depth 0 type".

There is an invariant for irred. smooth repn's of a p-adic red. gp. called "depth".

<u>Thm</u> (Moy-Prasad)

This construction gives & exhausts all depth 0 s.c repn's.

<u>Q</u>. positive depth ?? $\rightsquigarrow$ J.-K. Yu's construction.

④ general depth.

(Assume G: tamely ramified, $p \nmid |W_G|$.) Weyl gp.

e.g. $G = GL_n$
$p \nmid |\mathfrak{S}_n| = n!$
$\Leftrightarrow p > n$

<u>input</u> : $(\vec{G}, \vec{\phi}, x, \rho_0)$

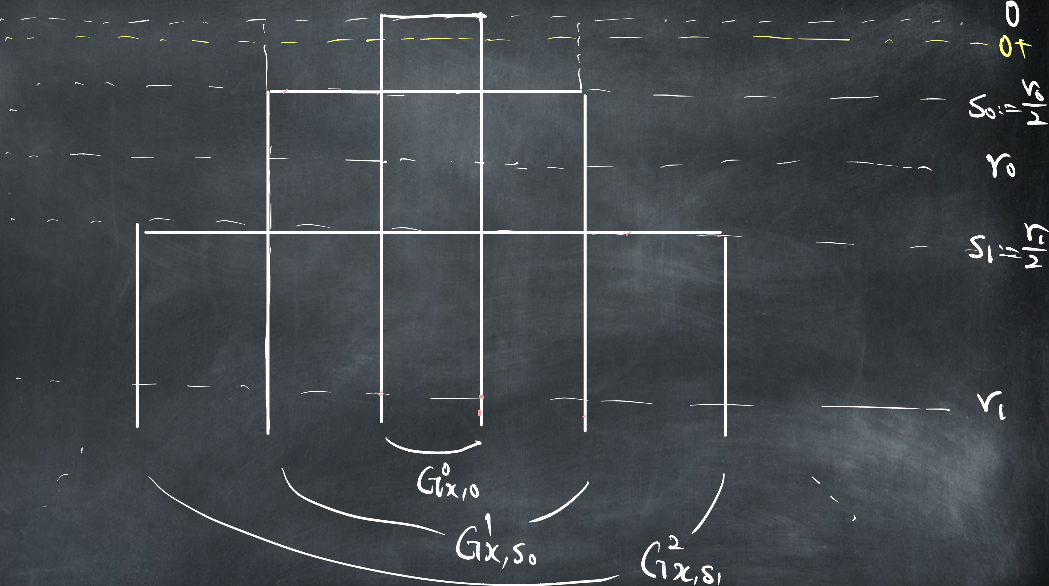
- $\vec{G} = (G^0 \subset G^1 \subset G^2 \subset \cdots \subset G^d = G)$ tame Levi subgps.
- $\vec{\phi} = (\phi_0, \phi_1, \ldots, \phi_d)$

  $\phi_i : G^i(F) \to \mathbb{C}^\times$: character of depth r_i.

  $(0 < r_0 < \cdots < r_d)$.
- $x \in \mathcal{B}(G^0, F)$: a vertex

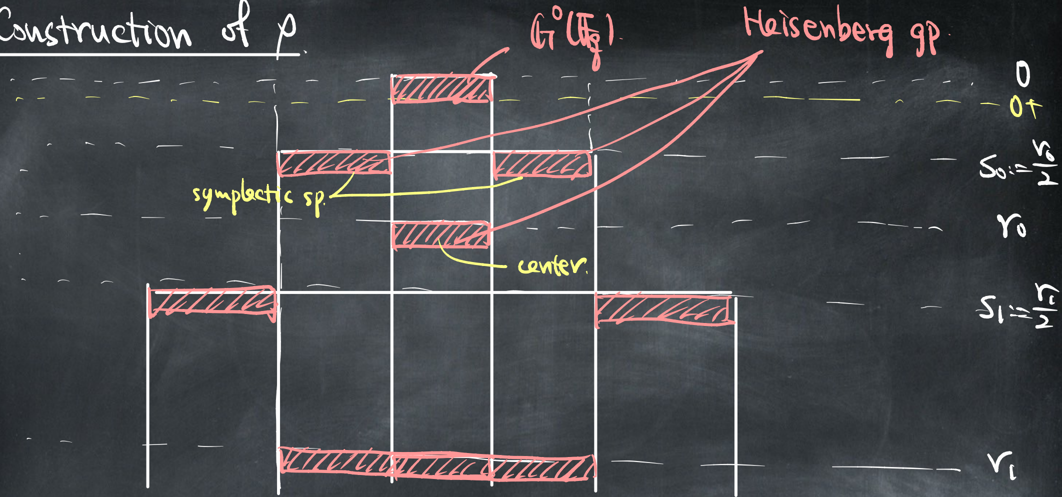
  $(\subset \mathcal{B}(G^i, F) \rightsquigarrow \{G^i_{x,r}\}_{r \in \mathbb{R}_{\geq 0}}$ MP filtn. for each i)
- ρ_0 : an irr. cusp. rep. of $G^0_{x,0} / G^0_{x,0+} \simeq \mathbb{G}^0(\mathbb{F}_q)$.

Construction of K



Yu's pyramid

Construction of ρ



ϕ_i : triv. on $G^i_{x,r_{i-1}+}$ but not on $G^i_{x,r_{i-1}}$

$\rightsquigarrow$ defines a nontriv. char. of the center of a finite Heisenberg gp.

Stone–von Neumann $\rightsquigarrow$ defines a unique irr. rep. of the i-th pyramid $=: \kappa_i$.

Define $\rho(\vec{G}, \vec{\Phi}, x, \rho_0) := \rho_0 \otimes \kappa_0 \otimes \kappa_1 \otimes \cdots \otimes \kappa_d$: rep. of the full pyramid K

§. Geometric construction

<u>input</u> :
- $T \subset G$ an unramified max. torus $/F$
- $x \in \mathcal{B}(T) \subset \mathcal{B}(G)$
- $\theta : T(F) \to \mathbb{C}^\times$: a character (depth r).

- Choose $T \subset B = T \cdot U \subset G$ a Borel subgp. — may not be $/F$
- Define $\mathbb{T}_r \subset \mathbb{B}_r = \mathbb{T}_r \cdot \mathbb{U}_r \subset \mathbb{G}_r$ alg. gps $/ \overline{\mathbb{F}}_q$

  s.t. $\mathbb{T}_r(\mathbb{F}_q) \simeq T_0 / T_{r+}$, $\mathbb{G}_r(\mathbb{F}_q) \simeq G_{x,0}/G_{x,r+}$.
- Put $X^{\mathbb{G}_r}_{\mathbb{T}_r} := \{ g \in \mathbb{G}_r \mid g^{-1} \mathrm{Frob}(g) \in \mathbb{U}_r \}$. $\curvearrowleft$ $\mathbb{G}_r(\mathbb{F}_q) \times \mathbb{T}_r(\mathbb{F}_q)$

$$\rightsquigarrow \rho^G_{(T,\theta)} := \sum_{i \geq 0} (-1)^i H^i_c(X^{\mathbb{G}_r}_{\mathbb{T}_r}, \overline{\mathbb{Q}}_\ell)[\theta]$$

positive-depth Deligne-Lusztig induction.

(fix $\mathbb{C} \simeq \overline{\mathbb{Q}}_\ell$)

Summary so far

	Algebraic	Geometric
input	$(\vec{G}, \vec{\phi}, x, \rho_0)$	(T, x, θ).
output K		
ρ	$\rho_0 \otimes K_0 \otimes \cdots \otimes K_{d-1}$	$\sum_i (-1)^i H^i_c(X, \bar{\mathbb{Q}}_\ell)[\theta]$.

(K, ρ) : s.c. type.

§ Main Thm

- $(\vec{G}, \vec{\phi}, x, \rho_0)$: Yu's input datum.

  ⤷ By classical Deligne–Lusztig theory,

  $\left\{\begin{array}{l} \cdot\ \exists\, \mathbb{T} \subset \mathbb{G}^0 \text{ max. torus } / \mathbb{F}_q \\ \cdot\ \exists\, \phi_{-1} : \mathbb{T}(\mathbb{F}_q) \to \mathbb{C}^\times \end{array}\right.$ ⟶ "essentially unique".

  s.t. $\langle \rho^{\mathbb{G}^0}_{(\mathbb{T}, \phi_{-1})}, \rho_0 \rangle \neq 0$.

$\rightsquigarrow \exists\, T < G^0 / F$ s.t. $T_0 / T_{0+} \simeq \mathbb{T}(\mathbb{F}_q)$.

$\rightsquigarrow$ put $\theta := \prod_{i=-1}^{d} \phi_i|_{T(F)}$

Fact (Kaletha).
Suppose $\rho^{\mathbb{G}^0}_{(\mathbb{T}, \phi_{-1})}$ is irr. (up to sign). $\left(\overset{\text{def}}{\iff} \rho_0 : \text{regular} \overset{\text{def}}{\iff} \theta : \text{reg.}\right)$
Then $(\vec{G}, \vec{\phi}, x, \rho_0)$ can be recovered by (T, θ).

Main Thm

Let $(\vec{G},\vec{\phi},x,\rho_0)$ be an input datum of Yu's constr. ($\mapsto (T,\theta)$)

Suppose: ρ_0 is regular & T is unramified.

Then $\mathrm{Ind}_{\mathrm{stuff}}^{G_{x,0}} \rho(\vec{G},\vec{\phi},x,\rho_0) \simeq \underbrace{(-1)^{\circledast}}_{\text{an explicit sign (depends on } (\vec{G},\vec{\phi},x,\rho_0))} \rho^{G}_{(T,\theta)}$ if $q \gg 0$.

$\tilde{\rho}^{G}_{(T,\theta)}$

Non-regular case

"fiber" of $(\vec{G},\vec{\phi},x,\rho_0) \mapsto (T,\theta)$ is given by

irred. constituents of $\rho^{G^0}_{(\mathbb{T},\phi_{-1})}$.

Write $\rho^{G^0}_{(\mathbb{T},\phi_{-1})} \simeq \bigoplus_i \underbrace{\rho_{0,i}}_{\text{irr.}}^{\oplus n_i}$ $n_i \in \mathbb{Z}$ ← ρ_0 is one of $\rho_{0,i}$'s.

& Define $\tilde{\rho}^{G}_{(T,\theta)} := \bigoplus_i \mathrm{Ind}_{\mathrm{stuff}}^{G_{x,0}} \rho(\vec{G},\vec{\phi},x,\rho_{0,i})^{\oplus n_i}$

$\rightsquigarrow$ Main Thm also holds for non-reg. data.

§ Outline of prf.

$G_r(\mathbb{F}_q) = G_{x,0}/G_{x,r+}$

Want to compare the virtual repn's $\underbrace{\tilde{\rho}^G_{(T,\theta)}}_{\text{algebraic}}$ & $\underbrace{\rho^G_{(T,\theta)}}_{\text{geometric}}$ of $G_{x,0}$

<u>Idea</u>: Compare trace characters $\Theta_{\tilde{\rho}^G_{(T,\theta)}}$ & $\Theta_{\rho^G_{(T,\theta)}}$.

Put $\tilde{Q}^G_{(T,\theta)} := \Theta_{\tilde{\rho}^G_{(T,\theta)}}|_{G_r(\mathbb{F}_q)\,\text{unip.}}$ "Green function".

unipotent elements.

<u>Thm</u> (1) For any $\gamma \in G_r(\mathbb{F}_q)$ w/ Jordan decomposition $\gamma = \underbrace{s}_{\text{s.s.}} \cdot \underbrace{u}_{\text{unip.}}$, $u \in Z_G(s)^\circ$

$$\Theta_{\tilde{\rho}^G_{(T,\theta)}}(\gamma) = (-1)^{\circledast} \cdot |Z_{G_r}(s)(\mathbb{F}_q)|^{-1} \cdot \sum_{\substack{x \in G_r(\mathbb{F}_q) \\ xsx^{-1} \in T_r}} \theta^x(s) \cdot \tilde{Q}^{Z_G(s)^\circ}_{(T^x,\theta^x)}(u)$$

(2) $\tilde{Q}^G_{(T,\theta)}$ depends only on $\theta|_{T_{0+}}$

Comments on proof

- (A) geometric : imitate Deligne-Lusztig's argument. (but more involved)
- (B) algebraic : re-examine an explicit character formula of Adler-DeBacker-Spice.

Naively, the most direct approach should be

1. Compare Q & $\tilde{Q}$.
2. Then compare Θ_ρ & $\Theta_{\tilde{\rho}}$ using Thm.

But, our proof does NOT discuss in this way.

Key observation. (def) $Z_{G_\gamma}(s) = T_\gamma \iff Z_G(s) = T$.

If $\gamma \in T_\gamma(\mathbb{F}_q) \subset G_\gamma(\mathbb{F}_q)$ is regular semisimple,

Thm (1) gives $\Theta_{\tilde{\rho}^G_{(T,\theta)}}(\gamma) = (-1)^{\circledast} \cdot \sum_{w \in W_{G_\gamma}(T_\gamma)} \theta^w(\gamma)$ ... $\circledast$

Note: the regular semisimple locus is Zariski dense.

$\rightsquigarrow$ **Naive (too optimistic) expectation.**

$\tilde{\rho}^G_{(T,\theta)}$ is characterized by $\circledast$ if $q \to \infty$.

Prop (Henniart's trick)

If $q \gg 0$, then there exists at most 1 irr. rep. of $G_\gamma(\mathbb{F}_q)$ satisfying the character formula $\circledast$ on reg. s.s. elements.

proof only uses Cauchy-Schwarz!

proof of Main Result ($\tilde{\rho}^G_{(T,\theta)}$ vs $(-1)^{\circledast}\cdot\rho^G_{(T,\theta)}$).

- regular case: Both $\tilde{\rho}$ & ρ are irred. (up to sign).

  $\rightsquigarrow$ Thm (char. formula) & Prop (Henniart) imply $\tilde{\rho} \simeq (-1)^{\circledast}\cdot\rho$.

- non-reg. case.

  [**Key**: If $q \gg 0$, $\exists$ reg. $\theta' : T(F) \to \mathbb{C}^\times$ s.t. $\theta'|_{T(F)_{0+}} \equiv \theta|_{T(F)_{0+}}$.

$$\rightsquigarrow Q^G_{(T,\theta)} \underset{\text{Thm (2)}}{=} Q^G_{(T,\theta')} \underset{\text{reg. case}}{=} (-1)^{\circledast}\cdot\tilde{Q}^G_{(T,\theta')} \underset{\text{Thm (2)}}{=} (-1)^{\circledast}\cdot\tilde{Q}^G_{(T,\theta)}$$

By Thm (1), we also get $\Theta_{\rho^G_{(T,\theta)}} \equiv (-1)^{\circledast}\cdot\Theta_{\tilde{\rho}^G_{(T,\theta)}}$ //

Thank you!

謝謝!