ON THE ENDOSCOPIC LIFTING OF SIMPLE SUPERCUSPIDAL REPRESENTATIONS OF CLASSICAL GROUPS

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1. Introduction: Endoscopy and the Local Langlands correspondence

This article is a summary of the author's talk at the RIMS workshop "Automorphic forms and related topics" on February 10, 2017. We report on some results on the endoscopic liftings of simple supercuspidal representations of classical groups. We first recall the local Langlands correspondence for classical groups, which is a background of the problems considered in this article.

For a connected reductive group $\mathbf{G}$ over a p-adic field F, we consider the set $\Pi(G)$ of equivalence classes of irreducible smooth representations of $G := \mathbf{G}(F)$ and the set $\Phi(G)$ of L-parameters of $\mathbf{G}$. Then the *conjectural local Langlands correspondence for* $\mathbf{G}$ predicts that there exists a natural map from $\Pi(G)$ to $\Phi(G)$ with finite fibers (L-packets). In other words there exists a natural partition of $\Pi(G)$ into L-packets parametrized by $\Phi(G)$:

$$\Pi(G) = \coprod_{\phi \in \Phi(G)} \Pi_{\phi}.$$

In the case of $\mathbf{G} = \mathrm{GL}_N$, this correspondence was established by Harris and Taylor in [HT01]. In this case, each L-packet Π_{ϕ} is a singleton and the naturality of the partition is formulated in terms of the local L-factors and ε-factors.

Recently Arthur established the local Langlands correspondence for quasi-split classical groups, namely symplectic or special orthogonal groups, in his book [Art13] (the case of unitary group was done by Mok in [Mok15]). In these cases, each L-packet is not necessarily a singleton, and the naturality of the partition is formulated via the *endoscopic character relation*.

We next recall what is the endoscopic character relation. Let us assume that a quasi-split classical group $\mathbf{G}$ is an *twisted endoscopic group* of GL_N. That is we have an involution θ of GL_N and an L-embedding ι from the L-group of $\mathbf{G}$ to that of GL_N such that the image of the dual group $\widehat{G}$ of $\mathbf{G}$ coincides with some $\hat{\theta}$-twisted centralizer in $\widehat{\mathrm{GL}_N} = \mathrm{GL}_N(\mathbb{C})$ (here $\hat{\theta}$ is the dual involution of θ). For example, the dual group of the odd special orthogonal group SO_{2n+1} is given by the symplectic group $\mathrm{Sp}_{2n}(\mathbb{C})$, and SO_{2n+1} is a twisted endoscopic group of GL_{2n} with respect to the natural embedding of $\mathrm{Sp}_{2n}(\mathbb{C})$ into $\mathrm{GL}_{2n}(\mathbb{C})$. Let ϕ be an L-parameter of $\mathbf{G}$. Then, since ϕ is a homomorphism from the local Langlands group $W_F \times \mathrm{SL}_2(\mathbb{C})$ to the L-group of $\mathbf{G}$, we get an L-parameter of GL_N by composing ϕ with ι:

$$\begin{array}{ccc} W_F \times \mathrm{SL}_2(\mathbb{C}) & \xrightarrow{\iota\circ\phi} & {}^{L}\mathrm{GL}_N \\ & \searrow^{\phi} & \uparrow \iota \\ & & {}^{L}\mathbf{G} \end{array}$$

Here W_F is the Weil group of F. Thus we get a pair of L-packets $\Pi_\phi \subset \Pi(G)$ and $\Pi_{\iota\circ\phi} \subset \Pi(\mathrm{GL}_N(F))$ which are related via the natural operation on the dual side. In this situation, we call the unique representation in $\Pi_{\iota\circ\phi}$ the *endoscopic lifting of* Π_ϕ from G to $\mathrm{GL}_N(F)$. Then the endoscopic character relation is an equality of characters of representations in these L-packets, and characterizes the endoscopic lifting representation-theoretically:

$$\Theta_{\pi,\theta}(g) = \sum_{h \mapsto g} \frac{D_G(h)^2}{D_{\mathrm{GL}_N,\theta}(g)^2} \Delta_{G,\mathrm{GL}_N}(h,g) \sum_{\pi_G \in \Pi_\phi} \Theta_{\pi_G}(h),$$

Here,

- π is the endoscopic lifting of Π_ϕ from G to $\mathrm{GL}_N(F)$,
- Θ_{π_G} (resp. $\Theta_{\pi,\theta}$) is the character of π_G (resp. the θ-twisted character of π),
- D_G (resp. $D_{\mathrm{GL}_N,\theta}$) is the Weyl-discriminant (resp. the θ-twisted Weyl-discriminants),
- Δ_{G,GL_N} is the Kottwitz-Shelstad transfer factor,
- g is a strongly θ-regular θ-semisimple element of $\mathrm{GL}_N(F)$, and
- the sum is over stable conjugacy classes of norms $h \in G$ of g.

Since the characters of representations satisfy the linear independence, this equality characterizes the each L-packets of G.

Here we consider the following natural problem:

Describe the local Langlands correspondence for $\mathbf{G}$ explicitly.

Then, from the above formulation of the local Langlands correspondence for $\mathbf{G}$, we can divide this problem into the following two problems:

(1) For a given irreducible smooth representation $\pi_G \in \Pi(G)$, determine the finite subset (L-packet) of $\Pi(G)$ containing π_G and the representation π of $\mathrm{GL}_N(F)$ satisfying the endoscopic character relation.
(2) Determine the L-parameter corresponding to π.

Namely, we can divide the problem of explicit description of the local Langlands correspondence for $\mathbf{G}$ into the problems of explicit description of the endoscopic lifting from $\mathbf{G}$ to GL_N and the local Langlands correspondence for GL_N.

In this article, we report on some results on the first problem for *simple supercuspidal representations*, which were introduced by Gross-Reeder in [GR10], of quasi-split classical groups.

Notation. Let p be an odd prime number. We fix a p-adic field F. We denote its ring of integers, its maximal ideal, and its residue field by $\mathcal{O}$, $\mathfrak{p}$, and k, respectively. For $x \in \mathcal{O}$, $\bar{x}$ denotes the image of x in k. For an algebraic group $\mathbf{G}$ over F, we denote its F-rational points $\mathbf{G}(F)$ by G.

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2. Simple supercuspidal representations of classical groups

We recall the definition of simple supercuspidal representations of classical groups briefly. See [GR10] or [Oi16b] for the details of the arguments in this section.

We first take a quasi-split classical group $\mathbf{G}$ over F, that is a general linear group, an unitary group, a symplectic group, or a special orthogonal group. For simplicity, we assume that $\mathbf{G}$ is split. We fix an F-split maximal torus $\mathbf{T}$ of $\mathbf{G}$. Then it defines an apartment $\mathcal{A}(\mathbf{G}, \mathbf{T})$ of the Bruhat-Tits building of $\mathbf{G}$. By taking a fundamental alcove $\mathcal{C}$ of this apartment, we get the corresponding Iwahori subgroup I of G, which is a minimal parahoric subgroup of G. If we take a point $\mathbf{x}$ of the closure of $\mathcal{C}$, then we get a filtration of I by the Moy-Prasad theory. We take this point $\mathbf{x}$ to be the barycenter of the alcove $\mathcal{C}$, and denote the first two steps of the filtration by I^+ and I^{++}. Then we have an isomorphism

$$I^+/I^{++} \cong k^{\oplus l+1},$$

where l is the rank of $\mathbf{G}$. For an character χ of I^+, we say that χ is *affine generic* if χ satisfies the following two conditions:

- χ is trivial on I^{++}, and
- χ is not trivial on every summand k of I^+/I^{++}.

Let χ be an character of ZI^+ such that $\chi|_{I^+}$ is affine generic. Here Z is the F-valued points of the center $\mathbf{Z}$ of G. Then we define the normalizer $N_G(I^+; \chi)$ of χ as follows:

$$N_G(I^+; \chi) := \{n \in N_G(I^+) \mid \chi^n = \chi\}.$$

Here $N_G(I^+)$ is the normalizer of I^+ in G, and χ^n is the twist of χ via n defined by

$$\chi^n(x) := \chi(nxn^{-1}).$$

Now we can define simple supercuspidal representations of G. We have the following key proposition:

Proposition 2.1. (1) *We have a decomposition*

$$\text{c-Ind}_{ZI^+}^{G} \chi \cong \bigoplus_{\tilde{\chi}} \dim(\tilde{\chi}) \cdot \pi_{\tilde{\chi}}.$$

Here the sum is over the set of irreducible representations of $N_G(I^+; \chi)$ containing χ (namely, irreducible constituents of $\text{c-Ind}_{ZI^+}^{N_G(I^+;\chi)} \chi$*), and* $\pi_{\tilde{\chi}} := \text{c-Ind}_{N_G(I^+;\chi)}^{G}(\tilde{\chi})$. *Moreover, each $\pi_{\tilde{\chi}}$ is irreducible, hence supercuspidal.*

(2) *For an another pair $(\chi', \tilde{\chi}')$, $\pi_{\tilde{\chi}} \cong \pi_{\tilde{\chi}'}$ if and only if $\chi^n \cong \chi'$ and $\tilde{\chi}^n \cong (\tilde{\chi}')^n$ for some $n \in N_G(I^+)$.*

We call the irreducible supercuspidal representations of G obtained in this way *simple supercuspidal representations.*

By computing the normalizer $N_G(I^+)$ of I^+, we can describe the set of equivalence classes of simple supercuspidal representations explicitly. For example, in the case of GL_N, we can compute an Iwahori subgroup and the set of simple supercuspidal representations as follows:

Example 2.2 (the case of $\mathbf{G} = \mathrm{GL}_N$). We take $\mathbf{T}$ to be the diagonal maximal torus, and choose the fundamental alcove $\mathcal{C}$ contained in the chamber corresponding to the upper-triangular Borel subgroup. Then the corresponding Iwahori subgroup and its filtration are given by

$$I = \begin{pmatrix} \mathcal{O}^\times & & \mathcal{O} \\ & \ddots & \\ \mathfrak{p} & & \mathcal{O}^\times \end{pmatrix}, \ I^+ = \begin{pmatrix} 1+\mathfrak{p} & & \mathcal{O} \\ & \ddots & \\ \mathfrak{p} & & 1+\mathfrak{p} \end{pmatrix}, \text{ and}$$

$$I^{++} = \begin{pmatrix} 1+\mathfrak{p} & \mathfrak{p} & & \mathcal{O} \\ & \ddots & \ddots & \\ & \mathfrak{p} & \ddots & \mathfrak{p} \\ \mathfrak{p}^2 & & & 1+\mathfrak{p} \end{pmatrix}.$$

The normalizer of I in G is given by

$$Z_G I\langle\varphi_a\rangle,$$

for any $a \in k^\times$, where Z_G is the center of G and φ_a is a matrix defined as follows:

$$\begin{pmatrix} 0 & I_{N-1} \\ \varpi a & 0 \end{pmatrix}.$$

Here I_{N-1} is the unit matrix of size $N-1$ and ϖ is a uniformizer of F. Note that we have $\varphi^N = \varpi a$. Then the set of equivalence classes of simple supercuspidal representations of G is parametrized by the set $\widehat{k^\times} \times k^\times \times \mathbb{C}^\times$. To be more precise, for $(\omega, a, \zeta) \in \widehat{k^\times} \times k^\times \times \mathbb{C}^\times$, we define a character $\tilde{\chi}_{(\omega,a,\zeta)}$ of $ZI^+\langle\varphi_{a^{-1}}\rangle$ by

$$\begin{aligned} \tilde{\chi}_{(\omega,a,\zeta)}(z) &:= \omega(z) \text{ for } z \in k^\times = \mathbf{Z}(k) \subset Z, \\ \tilde{\chi}_{(\omega,a,\zeta)}(x) &:= \psi(\overline{x_{12}} + \cdots + \overline{x_{N,N-1}} + a\overline{\varpi^{-1}x_{N1}}) \text{ for } x = (x_{ij})_{ij} \in I^+, \text{ and} \\ \tilde{\chi}_{(\omega,a,\zeta)}(\varphi_a) &:= \zeta. \end{aligned}$$

Here we fixed a non-trivial additive character ψ of k. Then the representation $\pi_{(\omega,a,\zeta)} := \text{c-Ind}_{ZI^+\langle\varphi_{a-1}\rangle}^G \tilde{\chi}_{(\omega,a,\zeta)}$ is a simple supercuspidal representation, and we can check that every simple supercuspidal representation of $\mathrm{GL}_N(F)$ is equivalent to $\pi_{(\omega,a,\zeta)}$ for a unique $(\omega, a, \zeta) \in \widehat{k^\times} \times k^\times \times \mathbb{C}^\times$.

In a similar way to this example, we can compute sets of representatives of simple supercuspidal representations of quasi-split classical groups, and parametrize them by triples consisting of

(1) a central character ω,
(2) an "equivalence class" of an affine generic character χ on I^+, and
(3) images of the normalizer of χ.

Moreover, as in the above example, the set of (2) is in fact exhausted by affine generic characters whose only one or two components of $k^{\oplus l+1}(\cong I^+/I^{++})$ are twisted by a non-zero element of $k^\times$, and we can parametrize them by $k^\times$ or $\mu_2 \times k^\times$. By a case-by-case computation, we get the following table:

Remark 2.3. In the above parametrization of simple supercuspidal representations, we have to fix some non-canonical data. For example, in the case of GL_N, in order to parametrize the set of equivalence classes of affine generic characters of I^+, we have to fix a uniformizer ϖ of F and a non-trivial additive character ψ of k. In the case of the unitary group $\mathrm{U}_{E/F}(N)$ attached to an unramified quadratic extension E/F, we have to fix a trace-zero element of

TABLE 1. Parametrizing sets and the depth of simple supercuspidal representations of classical groups

group	(1)	(2)	(3)	depth^{-1}
GL_N	$\widehat{k^\times}$	$k^\times$	$\mathbb{C}^\times$	N
unramified $\mathrm{U}_{E/F}(N)$	$\widehat{\mathrm{U}_{k_E/k}(1)}$	$k^\times$	1	N
SO_{2n+1}	1	$k^\times$	μ_2	$2n$
Sp_{2n}	$\widehat{\mu_2}$	$\mu_2 \times k^\times$	1	$2n$
split SO_{2n}	$\widehat{\mu_2}$	$\mu_2 \times k^\times$	μ_2	$2n-2$
unramified SO_{2n}	$\widehat{\mu_2}$	$\mu_2 \times k^\times$	μ_2	$2n-2$
ramified SO_{2n}	$\widehat{\mu_2}$	$k^\times$	1	$2n$

the residue field k_E of E in addition to ϖ and ψ. Thus the above parametrizations are non-canonical and depend on such data.

Remark 2.4. We can characterize the simple supercuspidal representations via the *depth* of admissible representations. For an admissible representation π of G, we can define the depth of π, which is a non-negative rational number, by using the Moy-Prasad theory. Then we can check that an irreducible admissible representation π of G is simple supercuspidal if and only if π has the minimal positive depth. In the case of split connected reductive group $\mathbf{G}$, the minimal positive depth is given by the inverse of the Coxeter number of $\mathbf{G}$. For example, in the case of GL_N, it is $\frac{1}{N}$.

3. Main results

First we explain the endoscopic groups which we consider in this article. We put

$$J_N := \begin{pmatrix} & & & 1 \\ & & -1 & \\ & \iddots & & \\ (-1)^{N-1} & & & \end{pmatrix}.$$

We treat the endoscopic groups of the following four types:

(1) $(\mathbf{G}, \mathbf{H}) = (\mathrm{GL}_{2n}, \mathrm{SO}_{2n+1})$**:** Let θ be an automorphism of GL_{2n} over F defined by $\theta(g) = J_{2n}\,{}^t g^{-1} J_{2n}^{-1}$. Then SO_{2n+1} is an endoscopic group for $(\mathrm{GL}_{2n}, \theta)$ with respect to a natural embedding of L-groups:

$${}^L\mathbf{H} = \mathrm{Sp}_{2n}(\mathbb{C}) \times W_F \hookrightarrow \mathrm{GL}_{2n}(\mathbb{C}) \times W_F = {}^L\mathbf{G}.$$

(2) $(\mathbf{G}, \mathbf{H}) = (\mathrm{Res}_{E/F}\mathrm{GL}_N, \mathrm{U}_{E/F}(N))$**:** Let E/F be an unramified quadratic extension of p-adic fields. Let θ be an automorphism of $\mathrm{Res}_{E/F}\mathrm{GL}_N$ over F defined by $\theta(g) = J_N\,{}^t c(g)^{-1} J_N^{-1}$. Here c is the Galois conjugation of the quadratic extension E/F. Then the unitary group $\mathrm{U}_{E/F}(N)$ is an endoscopic group for $(\mathrm{Res}_{E/F}\mathrm{GL}_N, \theta)$ with respect to the following embedding of L-groups:

$$\begin{aligned} {}^L\mathbf{H} = \mathrm{GL}_N(\mathbb{C}) \rtimes W_F &\hookrightarrow \big(\mathrm{GL}_N(\mathbb{C}) \times \mathrm{GL}_N(\mathbb{C})\big) \rtimes W_F = {}^L\mathbf{G} \\ g \rtimes \sigma &\mapsto (g, J_N\,{}^t g^{-1} J_N^{-1}) \rtimes \sigma. \end{aligned}$$

(3) $(\mathbf{G},\mathbf{H}) = (\mathrm{GL}_{2n+1}, \mathrm{Sp}_{2n})$**:** Let θ be an automorphism of GL_{2n+1} over F defined by $\theta(g) = J_{2n+1}\,{}^t g^{-1} J_{2n+1}^{-1}$. Then Sp_{2n} is an endoscopic group for $(\mathrm{GL}_{2n+1}, \theta)$ with respect to a natural embedding of L-groups:

$$ {}^L\mathbf{H} = \mathrm{SO}_{2n+1}(\mathbb{C}) \times W_F \hookrightarrow \mathrm{GL}_{2n+1}(\mathbb{C}) \times W_F = {}^L\mathbf{G}. $$

(4) $(\mathbf{G},\mathbf{H}) = (\mathrm{GL}_{2n}, \textbf{ramified}\,\mathrm{SO}_{2n})$**:** Let E/F be a ramified quadratic extension of p-adic fields. Let θ be an automorphism of GL_{2n} over F defined by $\theta(g) = J_{2n}\,{}^t g^{-1} J_{2n}^{-1}$. Then the non-split quasi-split even special orthogonal group $\mathrm{SO}_{2n,E}$ corresponding to E/F is an endoscopic group for $(\mathrm{GL}_{2n}, \theta)$ with respect to the following embedding of L-groups:

$$ \begin{aligned} {}^L\mathbf{H} = \mathrm{SO}_{2n}(\mathbb{C}) \rtimes W_F &\hookrightarrow \mathrm{GL}_{2n}(\mathbb{C}) \times W_F = {}^L\mathbf{G} \\ g \rtimes 1 &\mapsto g \rtimes 1, \\ 1 \rtimes \sigma &\mapsto \begin{cases} 1 \rtimes \sigma & \text{if } \sigma \in W_E, \\ w \rtimes \sigma & \text{otherwise.} \end{cases} \end{aligned} $$

Here, w is the following element:

$$ w := \begin{pmatrix} I_{n-1} & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & I_{n-1} \end{pmatrix}. $$

Now we state our main results.

Theorem 3.1. *Let $(\mathbf{G},\mathbf{H})$ be a pair of connected reductive groups over F which is a one of the above four types. Let π_H be a simple supercuspidal representation of H, ϕ_H the corresponding L-parameter (thus its L-packet Π_{ϕ_H} contained π_H), and π_G be the endoscopic lifting of Π_{ϕ_H} to G.*

(1) *In the case of (1), the L-packet Π_{ϕ_H} is a singleton and π_G is again simple supercuspidal. Moreover, if π_H corresponds to $(1, a, \zeta)$ in the sense of the parametrization in Table 1, then π_G corresponds to $(1, 2a, \zeta)$.*

(2) *In the case of (2), the L-packet Π_{ϕ_H} is a singleton and π_G is again simple supercuspidal. Moreover, if π_H corresponds to $(\omega, a, 1)$ in the sense of the parametrization in Table 1, then π_G corresponds to*

$$ \begin{cases} (\omega, a, -\omega(-1)) & \text{if } N \text{ is even} \\ (\omega, a\epsilon, \omega(-1)) & \text{if } N \text{ is odd,} \end{cases} $$

where ϵ is the fixed trace-zero element of the residue field of E used in the parametrization in Table 1.

(3) *In the case of (3), the L-packet Π_{ϕ_H} consists of the adjoint orbit of π_H. The order of this L-packet is 2, and its endoscopic lifting π_G is an irreducible tempered representation of G given by*

$$ \mathrm{Ind}_{P_{2n,1}}^{G} \pi \boxtimes \omega_\pi, $$

where $P_{2n,1}$ is the F-valued points of a parabolic subgroup of GL_{2n+1} *whose Levi subgroup is given by* $\mathrm{GL}_{2n} \times \mathrm{GL}_1$*,* π *is a simple supercuspidal representation of* GL_{2n}*, and* ω_π *is the central character of* π*.*

(4) *In the case of* (4)*, the L-packet* Π_{ϕ_H} *is a singleton and* π_G *is again simple supercuspidal.*

Remark 3.2.
(1) The result in the case of (1) was also obtained by Adrian in [Adr15] under the assumption that $p \geq (e+2)(2n+1)$, where e is the absolute ramification index of F. Thus our result (1) is new for $2 < p < (e+2)(2n+1)$.
(2) The L-embedding considered in the case of (2) is called the *standard base change embedding*, and there exists another embedding called the *twisted base change embedding* from ${}^L\mathbf{H}$ to ${}^L\mathbf{G}$. For this embedding we have analogous results (see [Oi16b] for details).
(3) In the cases of (3) and (4), we can determine the correspondence of simple supercuspidal representations explicitly as in (1) and (2). This computation is in progress now.
(4) By the works of Bushnell-Henniart ([BH05]) and Imai-Tsushima ([IT15]), we have an explicit description of L-parameters of simple supercuspidal representations of GL_N. Thus combining it with the above theorem, we get an explicit description of the L-parameters of simple supercuspidal representations of classical groups of the above types.

Finally we comment on a rough outline of the proof of the above theorem. We show the above statements by case-by-case arguments.

(1), (2): The key point of the proof in these cases is to start from a simple supercuspidal representation of G, not H. To show the assertions directly, we first have to determine the structure of the L-packet containing π_H. However, if we start from a simple supercuspidal representation π_G of G of the form in Theorem (1) or (2), we can check easily that it is the endoscopic lifting of an L-packet of H which is a singleton consisting of a supercuspidal representation. Namely, we can avoid the difficulty of determining the structure of the L-packet.

We write π'_H for the supercuspidal representation of H which is "descended" from a simple supercuspidal representation π_G of G. Then our task is to show that this representation π'_H is simple supercuspidal and determine its parameter (in the sense of Table 1). These are done by investigating the endoscopic character relation. Since we can write the twisted characters of simple supercuspidal representations of G explicitly in terms of the *Kloosterman sums*, we get an description of the characters of π'_H via Kloosterman sums through the endoscopic character relation between π_G and π'_H. Then, by using elementary properties of Kloosterman sums, we can recover the simple supercuspidality of π'_H from its characters.

(3): In this case, we can not apply the above argument because we do not have a way to compute the twisted characters of representations which are parabolically induced from non-θ-stable parabolic subgroups. Thus we start from π_H. Our first task it to determine the structure of the L-packet Π_{ϕ_H}. To do this, we consider the standard endoscopy of H. By using the standard endoscopic character relation between H and its endoscopic groups, we can bound the depth of representations in Π_{ϕ_H} and show that every representation in Π_{ϕ_H} is either depth 0 supercuspidal or simple supercuspidal. Then the statement on the structure of Π_{ϕ_H} follows from the uniqueness of a generic representation and the constancy of formal degrees of representations in an L-packet.

Next we have to determine the endoscopic lifting to G. Since the order of the L-packet Π_{ϕ_H} is 2, Π_{ϕ_H} is the endoscopic lift of an L-packet Π'_{ϕ_H} of an endoscopic group of H. We can check that this endoscopic group is in fact a ramified even special orthogonal group $\mathbf{H}'$. Since it is known that this endoscopic lifting from H' to H is compatible with the θ-correspondence, we can conclude that Π'_{ϕ_H} consists of a single simple supercuspidal representation of H' by using the depth-preservation theorem for the θ-correspondence ([Pan02]). Thus our problem is reduced to the case of (4).

(4): From the arguments in the case of (3), we already know the structure of the L-packet containing a simple supercuspidal representation of H. Thus we can show the claim by the same method as in the cases of (1) and (2).

References

[Adr15] M. Adrian, *On the Langlands parameter of a simple supercuspidal representation: odd orthogonal groups*, preprint, `arXiv:1501.07500`, 2015.

[Art13] J. Arthur, *The endoscopic classification of representations: orthogonal and symplectic groups*, American Mathematical Society Colloquium Publications, vol. 61, American Mathematical Society, Providence, RI, 2013.

[BH05] C. J. Bushnell and G. Henniart, *The essentially tame local Langlands correspondence. II. Totally ramified representations*, Compos. Math. **141** (2005), no. 4, 979–1011.

[GR10] B. H. Gross and M. Reeder, *Arithmetic invariants of discrete Langlands parameters*, Duke Math. J. **154** (2010), no. 3, 431–508.

[HT01] M. Harris and R. Taylor, *The geometry and cohomology of some simple Shimura varieties*, Annals of Mathematics Studies, vol. 151, Princeton University Press, Princeton, NJ, 2001, With an appendix by Vladimir G. Berkovich.

[IT15] ———, *Local Galois representations of Swan conductor one*, preprint, `arXiv:1509.02960`, 2015.

[Mok15] C. P. Mok, *Endoscopic classification of representations of quasi-split unitary groups*, Mem. Amer. Math. Soc. **235** (2015), no. 1108, vi+248.

[Oi16a] M. Oi, *Endoscopic lifting of simple supercuspidal representations of* $\mathrm{SO}(2n+1)$ *to* $\mathrm{GL}(2n)$, preprint, `arXiv:1602.03453`, 2016.

[Oi16b] M. Oi, *Endoscopic lifting of simple supercuspidal representations of* $\mathrm{U}(N)$ *to* $\mathrm{GL}(N)$, preprint, `arXiv:1603.08316`, 2016.

[Pan02] Shu-Yen Pan, *Depth preservation in local theta correspondence*, Duke Math. J. **113** (2002), no. 3, 531–592.

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